Viable Control of the Air Quality in Case of Accidental Pollution

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(Partly based on a joint research with K. Georgiev and S. Margenov)

Velingrad, 21 – 25 October, 2006

Plan of the talk:

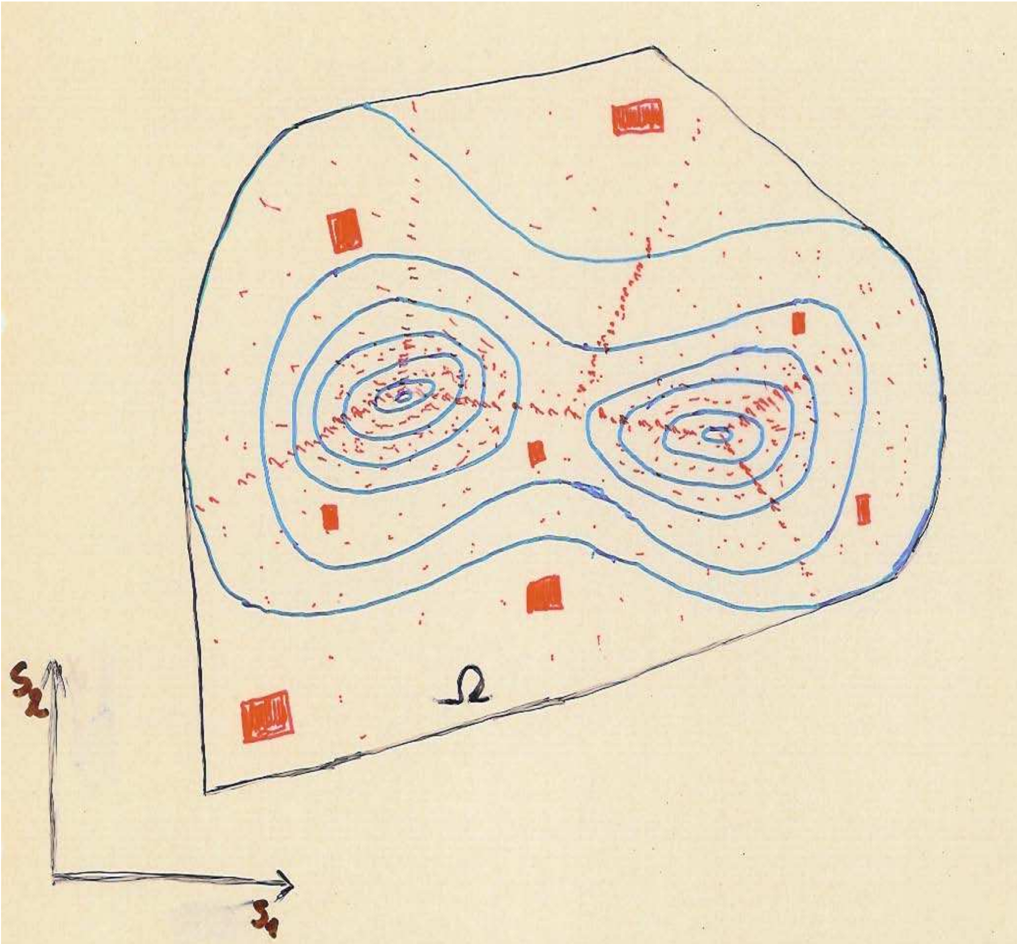
1. An air pollution control problem.

2. Extensions: age/size structured control systems; heterogeneous control systems.

3. Optimality conditions.

4. A numerical solution approach.

5. Comments.



$x(t, s)$ – concentration of pollutant
$f(t, s)$ – emission intensity

The dynamics of the concentration

$$x_t + \langle v(t,s), \nabla x \rangle = f(t,s) - g(t,s,x),$$

$$x(0,s) = \hat{x}_0(s), \qquad s \in \Omega$$

$$x(t,s) = \hat{x}(t,s), \qquad s \in \Gamma_-(t) \subset \partial\Omega,$$

where

$$\Gamma_-(t) = \{s \in \partial\Omega \,:\, v(t,s) \in \mathcal{T}_\Omega(s)\},$$

$\mathcal{T}_\Omega(s)$ – the tangent cone to Ω at s $\qquad (= \Omega - s)$

$f(t,s)$ – emission
$g(t,s,x)$ – deposition

Viability constraints

"Hard" constraint: $x(t,s) \le c(s)$

"Soft" constraint: the violation of $x(t,s) \le c(s)$ is penalized:

Instantaneous *Damage*:

$$\int_\Omega D(s, x(t,s))\,\mathrm{d}s.$$

Example:

$$D(s,x) = \begin{cases} 0 & \text{if} \quad x \le c(s) \\ \alpha(s)\varphi(x) & \text{if} \quad x > c(s), \end{cases}$$

where
$\alpha(s)$ – depends on the density of the population at s
$\varphi(x) \ge 0$ – an increasing function.

Means of control

$$x_t + \langle v(t,s), \nabla x \rangle = f(t,s) - g(t,s,x),$$

Assume

$$f(t,s) = f_0(t,s) + f_1(t,s),$$

f_0 – the uncontrollable emission

f_1 – the controllable emission, $u(t,s) \in [0,1]$ – the control variable:

$$f(t,s) = f_0(t,s) + (1 - u(t,s)) f_1(t,s),$$

Control cost:

$$\int_0^T \int_\Omega C(s, u(t,s)) \, \mathrm{d}s \, \mathrm{d}t.$$

Aims of control:

(i) Keep the "hard" constraints satisfied by minimal costs

$$\min_u \int_0^T \int_\Omega C(s, u(t, s)\, \mathrm{d}s\, \mathrm{d}t$$

or

(ii) Minimize the damage by minimal costs:

$$\min_u \int_0^T \int_\Omega [\alpha C(s, u(t, s)) + \beta D(s, x(t, s)]\, \mathrm{d}s\, \mathrm{d}t \quad \alpha \geq 0,\ \beta > 0.$$

The choice of T?

The overall model:

$$\text{minimize} \int_0^T\int_\Omega L(t,s,x(t,s),u(t,s))\,\mathrm{d}s\,\mathrm{d}t$$

$$x_t + \langle v(t,s), \nabla x\rangle = f_0(t,s) + (1-u)f_1(t,s) - g(t,s,x),$$

$$x(0,s) = \hat{x}_0(s), \qquad x \in \Omega$$

$$x(t,s) = \hat{x}(t,s), \qquad s \in \Gamma_-(t),$$

$$u(t,s) \in [0,1].$$

Here either $L = C$ and there is a state constraint $x(t,s) \le c(s)$, or $L = \alpha C + \beta D$.

Assumptions: All functions involved in the model are nonnegative, continuous, and continuously differentiable with respect to x and u; L is uniformly strongly convex in u.

Point sources

Diffusion

A more general control model

$x(t,s)$, $y(t)$ – state variables, $\quad (t,s) \in [0,T] \times \Omega$

$u(t,s)$, $v(t)$ – control variables

$$
\begin{aligned}
\frac{\partial}{\partial t} x + \frac{\partial}{\partial s}[p(t,s,y(t),v(t))x] &= F(t,s,x(t,s),y(t),u(t,s),v(t)), \\
y(t) &= \int_\Omega q(t,s,x(t,s),u(t,s))\,\mathrm{d}s \\
x(t,s) &= \varphi(t,s,y(t),v(t)) \quad \text{for} \quad s \in \Gamma_-(t) \\
& \quad u(t,s) \in U, \quad v(t) \in V.
\end{aligned}
$$

Optimal control problem: $\min J(u,v)$.

Difficulties:
nonlinear diff. operator: $p(...,y,...)$;
diff. operator depending on the control: $p(...,v)$;
nonlocal dynamics: $F(...,y,...)$;
endogenous and non-local side conditions: $\varphi(...,y,v)$.

Special cases:

– age structured systems ($s \in \mathbf{R}$, $p = 1$): many papers,

... Brokate (1985), G. Tragler + G. Feichtinger + V.V. (2003) ...

– multiple age-structures ($x_t + x_{s_1} + x_{s_2} = ...$): G. Feichtinger + Ts. Tsachev + V.V. (2004)

– single-size-structured (p is diagonal with the same $p(t, s, y, v)$ at the diagonal): O. Tarniceriu + V.V. (??)

– "the air pollution model"

Open questions: terminal constraints, infinite horizon problems, state/mixed constraints, ...

Back to the air pollution problem:

$$\text{minimize} \int_0^T \int_\Omega [\alpha C(s,u(t,s)) + \beta D(s,x(t,s))]\,\mathrm{d}s\,\mathrm{d}t \qquad u(t,s) \in [0,1]$$

$$x_t + \langle v(t,s), \nabla x \rangle = f_0(t,s) + (1-u) f_1(t,s) - g(t,s,x),$$

$$x(0,s) = \hat{x}_0(s), \quad s \in \Omega, \qquad x(t,s) = \hat{x}(t,s), \quad s \in \Gamma_-(t),$$

Adjoint equation:

Denote

$$\Gamma_+(t) = \partial\Omega \setminus \Gamma_-(t).$$

For a fixed reference pair (u,x) define the *adjoint equation*

$$\lambda_t + \operatorname{div}(\lambda v) = \lambda(t,s) g_x(t,s,x(t,s)) + \beta D_x(s,x(t,s)),$$

$$\lambda(T,s) = 0 \quad \text{for} \quad s \in \Omega, \qquad \lambda(t,s) = 0 \quad \text{for} \quad t \in [0,T], \;\; s \in \Gamma_+(t).$$

Necessary optimality conditions (maximum principle)

Theorem 1 *(i) If the admissible pair* (u, x) *is optimal, then*

$$L_u(t, s, x(t, s), u(t, s)) + \lambda(t, s) f_1(t, s) \in -N_{[0,1]}(u(t, s)),$$

where $N_U(u)$ *is the external normal cone to the convex set* U *at* $u \in U$*, that is*

$$-N_{[0,1]}(u) = \begin{cases} [0, \infty) & \text{if } u = -1, \\ 0 & \text{if } u \in (0, 1), \\ (-\infty, 0] & \text{if } u = 1. \end{cases}$$

(ii) If, in addition to the assumptions already made, $D(s, x)$ *is increasing and convex in* x *and* $g(t, s, x)$ *is convex in* x*, then the above condition is sufficient for local (in* L_1*) optimality of* (u, x)*.*

Numerical solution

$$\{u(t,s) \in U\} = \mathcal{U} \ni u \longrightarrow I(u) = \int_0^T \int_\Omega (\alpha C + \beta D)\, \mathrm{d}s\, \mathrm{d}t$$

Gradient projection in the control space. The gradient (in L_2) at a reference pair (u, x) is

$$\frac{\mathrm{d}I}{\mathrm{d}u(\cdot)} = \alpha C_u(s, u(t,s)) + \lambda(t,s) f_1(t,s).$$

$$\begin{array}{ccc}
\text{Continuous problem } (x,u) & \longrightarrow & \text{Discrete problem } (x_h, u_h) \\
\downarrow & & \downarrow \\
\text{Approximation of } \frac{\mathrm{d}I}{\mathrm{d}u(\cdot)} & & \text{Calculation of } \frac{\mathrm{d}I_h}{\mathrm{d}u(\cdot)} \\
\downarrow & & \downarrow \\
\text{Next } (x,u) & \longrightarrow & \text{Next } (x_h, u_h)
\end{array}$$

(i) Runge-Kutta (at least third order local accuracy) discretization along the characteristics

(ii) Projected or conditional gradient direction

(iii) Line search (Armijo)

Error estimate: assume the conditions for sufficiency of the maximum principle, v of bounded variation, and "appropriate" coefficients of the RK scheme. Then

$$\max_{i,j} |u(t_i, x_j) - u_{ij}^N| \leq C \bigvee_0^T v \, h^2 + E^N,$$

where E^N is independent of h.

Scheme of proof: W. Hager+A. Dontchev + V.V. – SIAM J. Numer. Anal. (2000).

On the use of optimal control in environmental problems

— Nominative: policy design

Find an optimal control u^* $\quad\longrightarrow\quad$ implement u^*

— finding most “sensitive points” / best directions of improvement

— the theory of the environmental management: economic/environmental trade-offs, environmental taxes, short versus long run policies, the role of uncertainty, etc. ...

— scenario analysis: test against ”worst” cases

— training

— ... ...

— base for evaluation of decisions and risk-related cost of operation

Risk-related cost of operation

$e(t)$ – abnormal emission, $t \in [0, \tau]$

$v(t, s)$ – wind velocity, $t \in [0, T]$, $x \in \Omega$

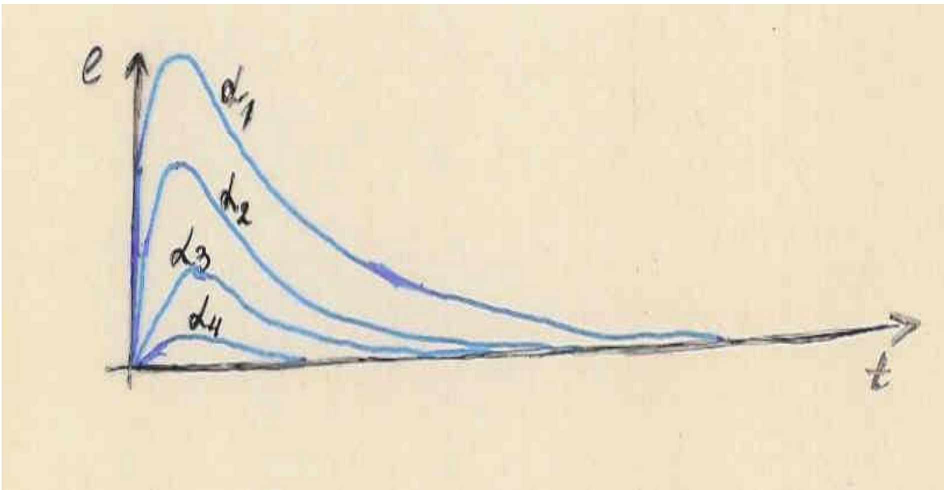
$I^*(e, v)$ – the optimal value

$\mathcal{E}$ – probability space of abnormal emissions

$\mathcal{V}$ – probability space of wind velocities

$$\mathbf{R} = \int_{\mathcal{E}} \int_{\mathcal{V}} \alpha(e)\beta(v) J(e, v)\, \mathrm{d}v\, \mathrm{d}e$$

Approximation:



$(e_1(\cdot), \alpha_1)$, ..., $(e_m(\cdot), \alpha_m)$ $\qquad$ $(v_1(\cdot), \beta_1)$, ..., $(v_n(\cdot), \beta_n)$.

$$\mathbf{R} \approx \sum_{i,j} \alpha_i \beta_j J(e_i, v_j)$$