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BEST APPROXIMATION OF LINEAR FUNCTIONALS IN W_p^r

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The best method of approximation in a minimal supnorm sense of a given linear functional in $W_p^r[a, b]$ $(1<p\leq 2)$ on the basis of the values of f at a fixed finite set of points is exact for a class of weighted natural splines. We use this property to obtain a characteristic of the optimal quadrature formula with fixed multiplicities of the nodes.

1. Introduction. As usually $W_p^r[a, b]$ denotes the Sobolev space

$$W_p^r[a, b]:=\{f\in C^{r-1}[a, b]: f^{(r-1)} \text{ abs. cont.}; f^{(r)}\in L_p[a, b]\}.$$

Throughout this paper we shall assume that $L(f)$ is a functional of the form

$$L(f)=\int_a^b \sum_{i=0}^{r-1} a_i(x) f^{(i)}(x)\,dx+\sum_{j=0}^{r-1}\sum_{i=1}^{\rho_j} b_{ij} f^{(j)}(\xi_{ij}), \tag{1}$$

where the functions $a_i(x)$ are piecewise continuous over $[a, b]$ and the abscissas ξ_{ij} are in $[a, b]$. Let

$$\bar{x}=\begin{pmatrix} x_1,\ldots, x_n \\ \nu_1,\ldots, \nu_n \end{pmatrix} \tag{2}$$

be a given system of nodes $(x_k)_1^n$ with multiplicities $(\nu_k)_1^n$ respectively, satisfying $a\leq x_1<\ldots<x_n\leq b$, $1\leq \nu_k\leq r$ $(k=1,\ldots, n)$. We shall consider the problem of approximation of the functional $L(f)$ in the class $W_p^r[a, b]$ $(1<p<\infty$, $r=1, 2,\ldots)$ on the basis of the information $T(\bar{x}; f):=\{f^{(\lambda)}(x_k), k=1,\ldots, n; \lambda=0,\ldots, \nu_k-1\}$. By a method of approximation S on the basis of the information $T(\bar{x}; f)$ we mean any transformation $S:\{T(\bar{x}; f): f\in W_p^r[a, b]\}\to \mathbf{R}$ of the set of all vectors $(f(x_1),\ldots, f^{(\nu_1-1)}(x_1),\ldots, f(x_n),\ldots, f^{(\nu_n-1)}(x_n))$ into the real Euclidean space $\mathbf{R}$.

Denote by $S(f)$ the approximate value of $L(f)$ given by the method S. The quantity

$$R_S:=\sup\{|L(f)-S(f)|: f\in W_p^r[a, b], \|f^{(r)}\|_p\leq 1\}$$

is said to be the error of the method S in the class $W_p^r[a, b]$. The method S^* for which $R_{S^*}=\inf\{R_S: S\}$ is called best. Note that in the case $p=2$

the above definition coincides with the definition of the best method in the sense of Sard [1].

It follows from a general result of S. A. Smoljak [2] (see also [3]) that there exists a linear best method of approximation of L. On account of this we shall consider only linear methods, i. e., where

$$S(f)=S(a;f)=\sum_{k=0}^{n}\sum_{\lambda=0}^{\nu_k-1} a_{k\lambda} f^{(\lambda)}(x_k).$$

In the case $p=2$ and $\nu_1=\ldots=\nu_n=1$ I. J. Schoenberg [4] developed a beautiful theory of best approximation of an arbitrary linear functional (1) showing that Sard's best approximation to $L(f)$, of degree $r-1$, is obtained by operating with L on both sides of the interpolation formula $f(x)\approx S_f(x)$, where S_f is the unique natural spline which interpolates $f(x)$ at the nodes x. Later on several analogous results have been proved including multiple nodes and boundary constraints (see J. H. Ahlberg and E. N. Nilson [5], I. J. Schoenberg [6]).

Our purpose in this paper is to extend the result of I. J. Schoenberg to the classes $W_p^r[a,b]$ $(1<p<2)$. We introduce natural spline functions with weight and prove that the approximation of $L(f)$ by the value of L at the unique interpolation natural spline with an appropriate weight is the best method of approximation in the sense prescribed above.

2. Natural spline functions with weight. Suppose $\omega(t)$ is a given integrable function which is positive almost everywhere in $[a,b]$. Given the nodes x of the form (2) we denote by $B(\bar{x})$ the class of all spline functions

(3)
$$g(t)=\sum_{k=1}^{n}\sum_{\lambda=0}^{\nu_k-1} c_{k\lambda}\frac{(x_k-t)_+^{r-\lambda-1}}{(r-\lambda-1)!}$$

such that $g(t)=0$ for $t\leq x_1$. Here $x_+=\max(x,0)$. It can be easily verified that $g\in B(\bar{x})$ iff the coefficients $\bar{c}=\{c_{k\lambda}\}$ satisfy for every $Q\in\pi_{r-1}$ the relations

(4)
$$\sum_{k=1}^{n}\sum_{\lambda=0}^{\nu_k-1} c_{k\lambda}Q^{(\lambda)}(x_k)=0.$$

Here, as elsewhere in this paper, π_{r-1} denotes the class of polynomials of degree $\leq r-1$.

We call a function $\varphi(t)$ a natural spline of degree $2r-1$ with weight $\omega(t)$ and nodes x if it has a representation of the form

(5)
$$\varphi(t)=P(t)+\frac{1}{(r-1)!}\int_a^b (t-\tau)_+^{r-1}\,\omega(\tau)\,g(\tau)\,d\tau,$$

where $g\in B(\bar{x})$ and $P\in\pi_{r-1}$. Let us denote by $N_{2r-1}(\omega;x)$ the set of all such splines. In the case $\omega(t)\equiv 1$ we shall use the abbreviation $N_{2r-1}(x)$. Remark that I. J. Schoenberg introduced the notation natural splines for the functions $f\in N_{2r-1}(x)$.

Following the exposition of T. N. E. Greville [7] we prove

Theorem 1. *Let $N=\Sigma_{k=1}^{n}\nu_k\geqq r$. Then, given a function $f\in C^{r-1}[a,b]$ there exists a unique spline $\varphi_f\in N_{2r-1}(\omega;x)$ satisfying the interpolation conditions*

$$\varphi_f^{(\lambda)}(x_k)=f^{(\lambda)}(x_k)\quad (k=1,\ldots,n;\ \lambda=0,\ldots,\nu_k-1). \tag{6}$$

Proof. Since the spline φ_f is supposed to be of the form (5) thus the relations (4) for $Q(t)=t^m$, $m=0,\ldots,r-1$ and (6) form a linear system of $N+r$ equations with $N+r$ unknown quantities: the coefficients $\{c_{k\lambda}\}$ of the spline $g(t)$ and the coefficients of the polynomial $P(t)$. So we need only to prove that the overall system is nonsingular which is equivalent to show that the corresponding homogeneous system has only the trivial solution. To prove this let $\varphi\in N_{2r-1}(\omega;x)$ be a spline satisfying the homogeneous system. Consider the integral $\sigma(g)=\int_a^b\omega(t)[g(t)]^2dt$, where $g(t)$ is the corresponding function from the representation (5) of φ. Evidently $\varphi^{(r)}(t)=\omega(t)g(t)$. Put $s^{(r)}(t)=g(t)$. Then $\sigma(g)=\int_a^b\varphi^{(r)}(t)\,s^{(r)}(t)dt$. Observe that $s\in N_{2r-1}(x)$ and $s^{(\mu)}(a)=s^{(\mu)}(b)=0$ for $\mu=r,\ r+1,\ldots,\ 2r-1$. Now taking into account the assumption that $\varphi^{(\lambda)}(x_k)=0$, $k=1,\ldots,n$, $\lambda=0,\ldots,\nu_{k-1}$ we get, by successive integration by parts, that $\sigma(g)=0$. Consequently $\varphi^{(r)}(t)\equiv 0$. But φ has N zeros and $N\geqq r$. Thus $\varphi(t)\equiv 0$. Since $\omega(t)$ is positive almost everywhere we conclude from (3) and (5) that $P(t)\equiv 0$ and $c_{k\lambda}=0$ $(k=1,\ldots,n;\ \lambda=0,\ldots,\nu_k-1)$. The theorem is proved.

3. **Main results.** We assume in this section that $N\geqq r$. It is easily seen that the best method of approximation of the functional L in the class $W_p^r[a,b]$ on the basis of the information $T(x;f)$ must be exact for every polynomial $P\in\pi_{r-1}$. Indeed, suppose that there exists a polynomial $P\in\pi_{r-1}$ for which $L(P)-S(a;P)=\varepsilon>0$. Then $L(\alpha P)-S(a;\alpha P)=\alpha\,\varepsilon$ for every number α. As far as $\alpha P\in W_p^r[a,b]$ and $\|P^{(r)}\|_p=0$ we conclude that the error of the method S in the class $W_p^r[a,b]$ is greater than any positive number. On the other hand, Smoljak's lemma (see [2], [3]) asserts that the error of the best method is equal to

$$\sup\{L(f): f\in W_p^r[a,b],\ \|f^{(r)}\|_p\leqq 1,\ f^{(\lambda)}(x_k)=0,\ k=1,\ldots,n,\ \lambda=0,\ldots,\nu_k-1\},$$

which is a bounded quantity because of the assumption $N\geqq r$. This proves our assertion. So, searching for the best method we may restrict ourself to the linear methods satisfying for every $P\in\pi_{r-1}$ the equality

$$L(P)=S(\bar a;P). \tag{7}$$

Evidently this relation is equivalent to the following one

$$\Sigma_{k=1}^{n}\Sigma_{\lambda=0}^{\nu_k-1}a_{k\lambda}P_j^{(\lambda)}(x_k)=L(P_j),\ P_j(t)=t^j;\ j=0,\ldots,r-1. \tag{8}$$

Given a system $\bar a=\{a_{k\lambda}\}$ satisfying (8) we denote $R(\bar a;f)=L(f)-S(\bar a;f)$, $f\in W_p^r[a,b]$. It is seen that $R(\bar a;\cdot)$ is a linear functional of the form (1). According to the known Peano theorem (see [7]) the error functional $R(\bar a;f)$ can be written in the form

(9) $$R(\bar{a}; f) = \int_a^b M(L, a; t) f^{(r)}(t)\, dt,$$

where

$$M(L, \bar{a}; t) = \frac{1}{(r-1)!} L_x[(x-t)_+^{r-1}] - \sum_{k=1}^{n} \sum_{\lambda=0}^{\nu_k-1} a_{k\lambda} \frac{(x_k-t)_+^{r-\lambda-1}}{(r-\lambda-1)!}$$

and the subscript "x" attached to the functional L indicates that we are to perform the operation L with respect to the variable x.

It follows immediately from (9) that

(10) $$R(a) := \sup\{R(a; f) : f \in W_p^r[a, b],\ \|f^{(r)}\|_p \leq 1\} = \|M(L, a; .)\|_q,$$

where $1/p + 1/q = 1$.

Theorem 2. *There exists a unique linear best method of approximation of the functional L in the class $W_p^r[a, b]$ $(1 < p < \infty,\ r = 1, 2, \ldots)$ on the basis of the information $T(x; f)$. The two conditions (7) and that there exists a polynomial $Q \in \pi_{r-1}$ such that*

(11) $$\frac{1}{(r-\lambda-1)!} \int_a^b M(L, a; t)\, |M(L, \bar{a}; t)|^{q-2} (x_k - t)_+^{r-\lambda-1} dt = Q^{(\lambda)}(x_k)$$

for $k = 1, \ldots, n$, $\lambda = 0, \ldots, \nu_k - 1$ are necessary and sufficient conditions for the method $S : L(f) \approx S(\bar{a}; f)$ to be best.

Proof. First we prove the necessity. Suppose that the coefficients $\bar{a}^*$ defined the best method. By the definition $R(a^*) = \inf\{R(a) : a\}$, where inf is extended over all systems a satisfying (8). It follows from (10) that a^* is a solution of the extremal problem

$$\min\{\int_a^b |M(L, a; t)|^q\, dt : \text{over all } a \text{ satisfying } (8)\}.$$

By virtue of a known theorem from classical analysis it is seen that the coefficients a^* must satisfy the first necessary conditions

$$\frac{\partial}{\partial a_{k\lambda}} \int_a^b |M(L, a; t)|^q\, dt - \sum_{j=0}^{r-1} \mu_j Q_{(j)}^{(\lambda)}(x_k) = 0$$

for $k = 1, \ldots, n$, $\lambda = 0, \ldots, \nu_k - 1$ where $Q_j(t) = t^j$ and $(\mu_j)_0^{r-1}$ are Lagrange multipliers. Differentiating the above relations we get (11). The necessity is shown.

Now let us suppose that the coefficients a satisfy (8) and (11). Let b be an arbitrary system which defines a method exact for π_{r-1}. Put for simplicity $M(t) = M(L, a; t)$, $M_1(t) = M(L, b; t)$. We have

(12) $$\int_a^b |M(t)|^q\, dt = \int_a^b |M(t)|^{q-1} M(t) \operatorname{sign} M(t)\, dt$$

$$= \int_a^b |M(t)|^{q-1} [M(t) - M_1(t) + M_1(t)] \operatorname{sign} M(t)\, dt$$

$$= \int_a^b |M(t)|^{q-1} M_1(t) \operatorname{sign} M(t)\, dt \leq \int_a^b |M(t)|^{q-1} |M_1(t)|\, dt.$$

Here we made use of the assumptions (11) and (7) to get

$$\int_a^b |M(t)|^{q-1} [M(t) - M_1(t)] \operatorname{sign} M(t)\, dt = \int_a^b |M(t)|^{q-2} M(t) [M(t) - M_1(t)]\, dt$$

$$=\int_a^b |M(t)|^{q-2} M(t) \sum_{k=1}^{n} \sum_{\lambda=0}^{\nu_k-1} (a_{k\lambda}-b_{k\lambda}) \frac{(x_k-t)_+^{r-\lambda-1}}{(r-\lambda-1)!}\, dt$$

$$=\sum_{k=1}^{n} \sum_{\lambda=0}^{\nu_k-1} (a_{k\lambda}-b_{k\lambda}) Q^{(\lambda)}(x_k) = L(Q)-L(Q)=0.$$

Now applying Hölder's inequality to (12) we obtain

$$\int_a^b |M(t)|^q dt \leqq \{\int_a^b |M(t)|^{(q-1)p} dt\}^{1/p} \cdot \|M_1\|_q, \tag{13}$$

which yields $\|M\|_q \leqq \|M_1\|_q$. Therefore the coefficients $\bar{a}$ define the best method. The uniqueness of the best method follows immediately from Hölder's inequality. Indeed, the equality in (13) holds only for $M_1(t) \equiv M(t)$. Another way to show the uniqueness is to observe that the determinant of the system (8), (11) is the same as the determinant considered in Theorem 1. We proved there that it is nonzero. So the system has a unique solution. The theorem is proved.

Theorem 3. *The best method* $S: L(f) \approx S(\bar{a}; f)$ *of approximation of the functional* L *in the class* $W_p^r[a, b]$ $(1<p\leqq 2,\ r=1, 2, \dots)$ *on the basis of the information* $T(x; f)$ *is uniquely determined by the property that* $L(\varphi)=(\bar{a}; \varphi)$ for every $\varphi \in N_{2r-1}(|M(L, \bar{a}; t)|^{q-2}; x)$. *Here* $1/q+1/p=1$.

Proof. First we prove that the best method S is precise for the splines from $N_{2r-1}(|M(L, \bar{a}; t)|^{q-2}; x)$. Indeed, let φ be of the form (5) with $\omega(t)=|M(L, \bar{a}; t)|^{q-2}$. Then

$$L(\varphi)=S(\bar{a}; \varphi)-R(\bar{a}; \varphi)=S(\bar{a}; \varphi)-R[\int_a^b \frac{(x-t)_+^{r-1}}{(r-1)!} |M(L, \bar{a}; t)|^{q-2} g(t)dt]$$

$$=S(\bar{a}; \varphi)-\int_a^b M(L, \bar{a}; t) |M(L, \bar{a}; t)|^{q-2} g(t)\, dt$$

$$=S(\bar{a}; \varphi)-\sum_{k=1}^{n} \sum_{\lambda=0}^{\nu_k-1} \frac{c_{k\lambda}}{(r-\lambda-1)!} \int_a^b M(L, \bar{a}; t) |M(L, \bar{a}; t)|^{q-2} (x_k-t)_+^{\nu-\lambda-1}\, dt.$$

Next using the assumption that S is the best method, i. e., the relation (11) and the relations (4) between the coefficients $\{c_{k\lambda}\}$ of the spline $g(t)$ we get

$$L(\varphi)=S(\bar{a}; \varphi)-\sum_{k=1}^{n} \sum_{\lambda=0}^{\nu_k-1} c_{k\lambda} Q^{(\lambda)}(x_k)=S(\bar{a}; \varphi).$$

Our assertion is proved.

Now suppose that the method S with coefficients $\bar{a}$ is exact for the class $N_{2r-1}(|M(L, \bar{a}; t)|^{q-2}; x)$. We shall prove that S is the best method. Indeed, let $\bar{b}$ be the coefficients of an arbitrary method of approximation of L which is exact for π_{r-1}. Using the abbreviations $M(t)$ and $M_1(t)$ instead of $M(L, \bar{a}; t)$ and $M(L, \bar{b}; t)$ we get as in (12)

$$(14)\qquad \int_a^b |M(t)|^q dt \leq \int_a^b |M(t)|^{q-1} |M_1(t)| dt + \int_a^b |M(t)|^{q-2} M(t) g_1(t) dt,$$

where $\quad g_1(t) = M(t) - M_1(t) = \Sigma_{k=1}^n \Sigma_{\lambda=0}^{\nu_k-1} (a_{k\lambda} - b_{k\lambda})[(x_k - t)_+^{r-\lambda-1}/(r-\lambda-1)!]$.
It follows from (7) that $\Sigma_{k=1}^n \Sigma_{\lambda=0}^{\nu_k-1} (a_{k\lambda} - b_{k\lambda}) Q^{(\lambda)}(x_k) = L(Q) - L(Q) = 0$ for $Q \in \pi_{r-1}$. Therefore the coefficients $c_{k\lambda} = a_{k\lambda} - b_{k\lambda}$ satisfy the relations (4). Thus $|M(t)|^{q-2} g_1(t)$ is the r-th derivative of a function f_1 from the class $N_{2r-1}(|M|^{q-2}; \bar{x})$. On the basis of the assumption that S is exact for this class we conclude that $R(\bar{a}; f_1) = \int_a^b M(t) |M(t)|^{q-2} g_1(t) dt = 0$.

Then applying Hölder's inequality to (14) we get $\|M(L, \bar{a}; .)\|_q \leq \|M(L, \bar{b}; .)\|_q$ with equality only for $\bar{a} = \bar{b}$. The theorem is proved.

As a simple consequence of the previous theorems we get the

Corollary (I. J. Schoenberg). *The best approximation in the sense of Sard of order $r-1$ to a linear functional $L(f)$ is obtained by operating with L on the unique spline function $\varphi_f \in N_{2r-1}(\bar{x})$ which interpolates $f(t)$ at the nodes $\bar{x}$.*

Remark 1. Note that the weight $|M(L, \bar{a}; t)|^{q-2}$ depends on the functional L. Consequently the classes $N_{2r-1}(|M(L, \bar{a}; .)|^{q-2}; \bar{x})$ are different for different approximated functionals. It is just so for $p \neq 2$. In the case $p = 2$ the weight is equal to 1 for any functional. Then the class of exactness is the same for any approximated functional L. This is the reason for the surprising simplicity and beauty of Schoenberg's result.

4. Optimal quadrature formulas. In this section we apply the results from the previous sections to obtain a characterization of the optimal quadrature formulas with multiple nodes in the classes $W_p^r[a, b]$ for $1 < p \leq 2$. First we give some definitions.

Let the multiplicities $(\nu_k)_1^n$ be fixed satisfying

$$(15)\qquad 1 \leq \nu_k \leq r \quad (k = 1, \ldots, n).$$

For every system

$$(16)\qquad \bar{x} = \begin{pmatrix} x_1, \ldots, x_n \\ \nu_1, \ldots, \nu_n \end{pmatrix}$$

of nodes $(x_k)_1^n$, $a \leq x_1 < \ldots < x_n \leq b$ with multiplicities $(\nu_k)_1^n$ respectively we denote by $R(\bar{x})$ the error of the best method of approximation of the functional $I(f) := \int_a^b f(t) dt$ in the class $W_p^r[a, b]$ on the basis of the information $T(\bar{x}; f)$. Let

$$(17)\qquad R(\nu_1, \ldots, \nu_n) = \inf\{R(\bar{x}) : \bar{x}\},$$

where the infimum is extended over all systems $\bar{x}$ from (16). The nodes $\bar{x}^*\begin{pmatrix} x_1^*, \ldots, x_n^* \\ \nu_1, \ldots, \nu_n \end{pmatrix}$ for which $R(\bar{x}^*) = R(\nu_1, \ldots, \nu_n)$ are said to be optimal of the type $(\nu_1, \ldots, \nu_n)$. The corresponding best quadrature formula with nodes $\bar{x}^*$

(i. e. the best linear method of approximation of the functional $I(f)$ on the basis of the information $T(x^*; f)$) is called optimal of the type $(\nu_1, \ldots, \nu_n)$. Let us remark that the question of the existence of the optimal nodes is not a trivial one. It is conceivable that in calculating (17) certain of the nodes $x_1, \ldots, x_n$ may coalesce, thus the minimum in (17) will be not achieved for a system $(x_1, \ldots, x_n)$ with n distinct nodes having the presigned multiplicities $\nu_1, \ldots, \nu_n$ respectively, i. e., the optimal nodes of the type $(\nu_1, \ldots, \nu_n)$ will not exist. We showed in [8] that for any system $(\nu_k)_1^n$ satisfying (15) the optimal nodes in the class $W_\infty^r[a, b]$ exist and they lie in the open interval (a, b). Now, applying Theorem 3, we shall prove a characterizing property of the optimal quadrature in $W_p^r[a, b]$ for $1<p\leq 2$. We begin by formulating several auxiliary lemmas.

Let $(t_k)_1^N$ be a given non-decreasing sequence of points in $[a, b]$ such that $t_i<t_{i+r}$ for $i=1, \ldots, N-r$. It is well-known (see [9]) that for $f\in W_1^r[a, b]$ the divided difference $f[t_i, \ldots, t_{i+r}]$ of f at the points $t_i, \ldots, t_{i+r}$ can be represented in the form

$$f[t_i, \ldots, t_{i+r}]=\int_a^b \psi_i(t)f^{(r)}(t)\,dt,$$

where $\psi_i(t)$ is the divided difference of the function $(\cdot-t)_+^{r-1}/(r-1)!$ at the points $t_i, \ldots, t_{i+r}$. The functions $\psi_i(t)$ are known as B-splines (see [10]) of degree $r-1$ with knots $t_i, \ldots, t_{i+r}$ respectively.

L e m m a 1. *Every function $f\in \operatorname{span}(\psi_1, \ldots, \psi_{N-r})$ has $N-r-1$ sign changes at most.*

The proof of this known lemma is based on the variation diminishing property of totally positive matrices (see [11]).

We prove the next lemma for an arbitrary fixed weight function $\omega(t)$ which is strictly positive almost everywhere in $[a, b]$.

L e m m a 2. *Let the nodes*

$$\bar{x}=\begin{pmatrix} x_1, \ldots, x_k, \ldots, x_n \\ \nu_1, \ldots, \nu_k, \ldots, \nu_n \end{pmatrix}$$

be such that $\nu_k<r$, $a\leq x_1<\ldots<x_n\leq b$ and $1\leq\nu_j\leq r$ $(j=1, \ldots, n)$, ν_j an even number if $\nu_j<r$. Then the unique natural spline $\varphi\in N_{2r-1}(\omega; \bar{x})$ satisfying the interpolation conditions

$$\begin{aligned} &\varphi^{(\nu_k)}(x_k)=1,\ \varphi^{(\lambda)}(x_k)=0,\ \lambda=0, \ldots, \nu_k-1, \\ &\varphi^{(\lambda)}(x_j)=0 \text{ for } j=1, \ldots, n,\ \lambda=0, \ldots, \nu_j-1, \end{aligned} \tag{18}$$

does not change its sign in $[a, b]$ and $\int_a^b\varphi(t)dt>0$.

P r o o f. Let the node $x_m\in(x_1, \ldots, x_n)$ have a multiplicity $\nu_m=r$ and $x_m<x_k$. Then $\varphi(t)=0$ for $t\leq x_m$. Indeed, there is a unique spline $\varphi_0(t)\in N_{2r-1}(\omega; \bar{y})$ satisfying the conditions $\varphi_0^{(\nu_k)}(x_k)=1$, $\varphi^{(\lambda)}(x_k)=0$, $\lambda=0, \ldots, \nu_k-1$, $\varphi_0^{(\lambda)}(x_j)=0$ for $j=m, m+1, \ldots, n$, $\lambda=0, \ldots, \nu_j-1$, where

$$\bar{y}=\begin{pmatrix} x_m, \ldots, x_n \\ \nu_m, \ldots, \nu_n \end{pmatrix}.$$

Moreover, it follows from (5) that $\varphi(t)$ coincides with a polynomial $P(t) \in \pi_{r-1}$ in the interval $[a, x_m]$. Then the condition $\varphi_0^{(\lambda)}(x_m)=0$ for $\lambda=0,\ldots, r-1$ implies that $P(t) \equiv 0$. But φ_0 belongs to the class $N_{2r-1}(\omega; \bar{x})$ also and evidently satisfies (18). Thus, it follows from Theorem 1 that $\varphi(t) \equiv \varphi_0(t)$. Similarly one can show that $\varphi(t)=0$ for $t \geqq x_m$ if $x_m > x_k$ and $\nu_m = r$.

Now, let us denote

$$\xi = \max\{\tau: a \leqq \tau < x_k, \ \varphi(t)=0 \text{ for all } t \text{ in } [a, \tau]\},$$

$$\eta = \min\{\tau: x_k < \tau \leqq b, \ \varphi(t)=0 \text{ for all } t \text{ in } [\tau, b]\}.$$

We set $\xi=a$ in the case the first set of points τ is empty and $\eta=b$ if the second one is empty. Since ν_k is an even number and $\varphi^{(\nu_k)}(t)>0$ in a neighborhood of x_k, we have $\xi<x_k<\eta$. Obviously the points ξ and η belong to the set $(a, x_1, \ldots, x_n, b)$.

We remark that there is no interval $[\alpha, \beta]$ such that $\xi<\alpha<\beta<\eta$ and $\varphi(t)=0$ for all $t \in [\alpha, \beta]$. Indeed, otherwise, assuming without loss of generality that $\beta<x_k$ we define a new spline $\tilde{\varphi}$ by

$$\tilde{\varphi}(t) = \begin{cases} -\varphi(t) & \text{for } t \in [a, (a+\beta)/2], \\ \varphi(t) & \text{for } t \in [(a+\beta)/2, b]. \end{cases}$$

Evidently $\tilde{\varphi} \in N_{2r-1}(\omega; x)$, $\tilde{\varphi} \neq r$ and $\tilde{\varphi}$ fulfils the conditions (18). On the other hand, there exists only one function from $N_{2r-1}(\omega; x)$ satisfying (18), by virtue of Theorem 1. We obtained a contradiction. Thus $\varphi(t)$ has only isolated zeros in (ξ, η).

Let us set

$$M = \begin{cases} \nu_i + \ldots + \nu_j & \text{if } \xi = x_i, \eta = x_j, \\ \nu_1 + \ldots + \nu_j & \text{if } \xi = a, \eta = x_j, \\ \nu_i + \ldots + \nu_n & \text{if } \xi = x_i, \ \eta = b, \\ \nu_1 + \ldots + \nu_n & \text{if } \xi = a, \ \eta = b. \end{cases}$$

Now suppose that the lemma is not true, i. e., the function $\varphi(t)$ changes its sign. According to the remark above, there exists a point $t_0 \in (\xi, \eta)$ such that $\varphi(t)$ has a zero with odd multiplicity at t_0. Since all nodes $x_j \in (\xi, \eta)$ are zeros of φ with multiplicities at least ν_j, and ν_j are even numbers it follows that φ has at least one new zero in (ξ, η). Therefore φ has $M+1$ zeros at least in $[\xi, \eta]$, counting multiplicities. Applying Rolle's theorem r times we establish that $\varphi^{(r)}(t)$ has $M+1-r$ sign changes in (ξ, η). Let $g(t)$ be the slpline corresponding to $\varphi(t)$ in the representation (5). Since $\varphi^{(r)}(t)=\omega(t)g(t)$ thus $g(t)$ must have $M+1-r$ changes also. But $g(t)$ can be written as a linear combination of $M+1-r$ B-splines [10]. Then, it follows from Lemma 1 that $g(t)$ has $M-r$ sign changes at most. We obtain a contradiction. Therefore the function $\varphi(t)$ does not change its sign. Now observe that $\varphi(t)$ is not zero identically, because $\varphi^{(\nu_k)}(x_k)=1$. Moreover, $\varphi^{(\nu_k)}(t)$ is strictly positive in a neighborhood of x_k since $\varphi^{(\nu_k)}(t)$ is a continuous function as far as $\nu_k<r$. Then, making use of the assumption that ν_k is an even number we see that $\varphi(t)$ is non-negative in a neig hbourhood of x_k, and cosequently in the whole interval $[a, b]$. Therefore $\int_a^b \varphi(t)dt>0$. The lemma is proved.

Theorem 4. *Let* $N=\Sigma_{k=1}^{n}\nu_k\geq r$ *and the multiplicities* $(\nu_k)_1^n$ *satisfy* $1\leq\nu_k\leq r$ $(k=1,\ldots,n)$; ν_k *is even if* $\nu_k<r$. *Suppose that the quadrature formula*

$$I(f)\approx\sum_{k=1}^{n}\sum_{\lambda=0}^{\nu_k-1}a_{k\lambda}f^{(\lambda)}(x_k)$$

is optimal of the type $(\nu_1,\ldots,\nu_n)$ *in the class* $W_p^r[a,b]$ $(1<p\leq 2)$ *and the optimal nodes* $(x_k)_1^n$ *satisfy the inequalities* $a<x_1<\ldots<x_n<b$. *Then* $a_{k,\nu_k-1}=0$, $a_{k,\nu_k-2}>0$ *if* ν_k *is even*; $a_{k,\nu_k-1}>0$ *if* ν_k *is odd.*

Proof. It is not difficult to see that there exists only one system a of best coefficients, i. e., only one linear best method of approximation of the functional $I(f)$ on the basis of the information $T(x\,;\,f)$ in the class $W_p^r[a,b]$. Indeed, it follows from Theorem 2 that every system a of best coefficients satisfies the relations

$$(19)\qquad \frac{1}{(r-\lambda-1)!}\int_a^b|M(I,\bar a\,;\,t)|^{q-1}.\operatorname{sign}M(I,\bar a\,;\,t).(x_k-t)_+^{r-\lambda-1}dt+Q^{(\lambda)}(x_k)=0$$

for $k=1,\ldots,n$, $\lambda=0,\ldots,\nu_k-1$ and some $Q\in\pi_{r-1}$. These are the necessary and sufficient conditions for a to be a solution of the extremal problem

$$\inf\{\|M(I,\bar b\,;\cdot)\|_q:\bar b\}$$

under the constraints $\Sigma_{k=1}^{n}\Sigma_{\lambda=0}^{\nu_k-1}b_{k\lambda}P^{(\lambda)}(x_k)=0$, for every $P\in\pi_{r-1}$. But this problem has a unique solution since the norm in L_q is strictly convex for $1<q<\infty$. Hence the uniqueness of a is proved.

Denote by $f(\bar a\,;\,x\,;\,t)$ the function

$$cQ(t)+\frac{c}{(r-1)!}\int_a^b(t-\tau)_+^{r-1}\,|M(I,\bar a\,;\,\tau)|^{q-1}.\operatorname{sign}M(I,\bar a\,;\,\tau)d\tau,$$

where $c=\{\|M(I,a\,;\cdot)\|_q\}^{-q/p}$. It is easy to verify that $R(x)=\int_a^b f(\bar a,x;t)dt$ and $\|f^{(r)}(\bar a,\bar x\,;\cdot)\|_p=1$. Moreover, (19) implies

$$(20)\qquad f^{(\lambda)}(\bar a,\bar x\,;\,x_k)=0\quad\text{for } k=1,\ldots,n,\ \lambda=0,\ldots,\nu_k-1.$$

Next we shall prove that

$$(21)\qquad f^{(\nu_k)}(\bar a,\bar x\,;\,x_k)\neq 0\ \text{ for } \nu_k<r \text{ and}$$

$$(22)\qquad \operatorname{sign}f^{(\nu_k)}(\bar a,\bar x\,;\,x_k-0).\operatorname{sign}f^{(\nu_k)}(\bar a,\bar x\,;\,x_k+0)\geq 0$$

when $\nu_k=r$ and r is an even number.

To show this, let us assume contrary to (21) that $f^{(\nu_k)}(\bar a,\bar x\,;\,x_k)=0$ and $\nu_k<r$. Then this equality together with (20) yields on the basis of Theorem 2 that the coefficients

$$\begin{array}{l} a_{1,0},\ldots,\ a_{1,\nu_1-1} \\ \ldots\ldots\ldots\ldots \\ a_{k,0},\ldots,\ a_{k,\nu_k-1},\ a_{k,\nu_k} \\ \ldots\ldots\ldots\ldots \\ a_{n,0},\ldots,\ a_{n,\nu_n-1} \end{array}$$

with $a_{k,\nu_k}=0$ are best for the nodes

$$y=\begin{pmatrix} x_1,\ldots,\ x_{k-1},\ x_k,\ x_{k+1},\ldots,\ x_n \\ \nu_1,\ldots,\ \nu_{k-1},\ \nu_k+1,\ \nu_{k+1},\ldots,\ \nu_n \end{pmatrix}.$$

On the other hand, by Theorem 3 $a_{k,\nu_k}=\int_a^b\varphi(t)dt$, where φ is the unique natural spline from the class $N_{2r-1}\ (M(I,a;\cdot)^{q-2};\ y)$ satisfying the interpolation conditions (18). We proved in Lemma 2 that $\varphi(t)$ does not change its sign in $[a,b]$ and $\int_a^b\varphi(t)dt>0$. This contradicts to the condition that $a_{k,\nu_k}=0$. The relation (21) is proved.

We now turn to (22). On assuming the contrary, it follows that $\bar{f}^{(r)}(a,x;\ t)$ changes its sign at the point x_k. Since r is an even number the equalities (20) imply that $f(a,x;\ t)$ changes its sign at x_k too. Let $\xi_1,\ldots,\xi_j$ be all points in $[a,b]$ at which $f(a,x;t)$ changes sign. It is not difficult to see that $\xi_1,\ldots,\ \xi_j\in(x_1,\ldots,\ x_n)$. Indeed, suppose that ξ_i does not belong to the set $(x_k)_1^n$. Consider the best quadrature formula with nodes

$$\begin{pmatrix} x_1,\ldots,\ x_n,\ \xi_i \\ \nu_1,\ldots,\ \nu_n,\ 1 \end{pmatrix}.$$

The condition $f(a,x;\ \xi_i)=0$ and Theorem 2 yield that the coefficient A before the value $f(\xi_i)$ in the best quadrature must be zero. On the other hand, it follows from Lemma 2 that $\int_a^b\varphi(\xi_i;\ t)dt>0$, where $\varphi(\xi_i;\ t)$ is the corresponding weighted natural spline satisfying the interpolation conditions $\varphi(\xi_i;\ \xi_i)=1,\ \ \varphi^{(\lambda)}(\xi_i;\ x_k)=0\ \ (k=1,\ldots,\ n;\ \lambda=0,\ldots,\ \nu_k-1)$. Therefore $A>0$. This contradicts the condition $A=0$ established before. Hence $\xi_i\in(x_k)_1^n$. In view of (21) the function $f(a,x;t)$ does not change its sign at x_k if $\nu_k<r$ since x_k is a zero with an even multiplicity in this case. Thus all nodes $\xi_1,\ldots,\ \xi_j$ must have multiplicities equal to r. Let us consider the function $\tilde{f}(t)=f(\bar{a},x;t)$. Obviously $\tilde{f}\neq f(a,x;\cdot)$, $\|\tilde{f}^{(r)}\|_p=1$ and $\tilde{f}^{(\lambda)}(x_k)=0$ for $k=1,\ldots,\ n,\ \lambda=0,\ldots,\ \nu_k-1$. Consequently

$$R(x)=\int_a^b f(a,\ x;\ t)\,dt<\int_a^b\tilde{f}(t)dt\leqq R(x),$$

which is impossible. This completes the proof of (22).

Now let us use the assumption that the nodes $\bar{x}$ are optimal of the type $(\nu_1,\ldots,\ \nu_n)$. Evidently they satisfy the necessary conditions

$$(23)\qquad \frac{\partial}{\partial x_k}\{\int_a^b M(I,\bar a;t)^q dt+\sum_{j=0}^{r-1}\mu_j(\sum_{k=1}^{n}\sum_{\lambda=0}^{\nu_k-1}a_{k\lambda}P_j^{(\lambda)}(x_k)-I(P_j))\}=0$$

for $k=1,\ldots,n$, where $(\mu_j)_0^{r-1}$ are Lagrange multipliers and $P_j(t)=t^j$, $j=0,\ldots,r-1$. Dividing the integral in (23) into two parts as follows

$$\int_a^b M(I,a;t)^q dt=\int_a^{x_k} M(I,\bar a;t)^q dt+\int_{x_k}^b M(I,a;t)^q dt$$

and differentiating we rewrite (23) in the form

$$(24)\qquad \sum_{\lambda=0}^{\min(\nu_k-1,\,r-2)} q a_{k\lambda}\int_a^{x_k} M(I,a,t)^{q-1}\frac{(x_k-t)_+^{r-\lambda-2}}{(r-\lambda-2)!}\cdot\operatorname{sign} M(I,a;t)dt$$

$$+\ M(I,\bar a;x_k-0)^q=M(I,\bar a;x_k+0)^q+\sum_{j=0}^{r-1}\mu_j(\sum_{\lambda=0}^{\nu_k}a_{k\lambda}P_j^{(\lambda+1)}(x_k))=0.$$

In addition, the best coefficients $\{a_{k\lambda}\}$ for the optimal nodes $\bar x$ satisfy

$$(25)\qquad q\int_a^b\frac{(x_k-t)_+^{r-\lambda-1}}{(r-\lambda-1)!}\cdot M(I,\bar a;t)|^{q-1}.\operatorname{sign} M(I,\bar a;t)\,dt+\sum_{j=0}^{r-1}\mu_j P_j^{(\lambda)}(x_k)=0.$$

By using the definition of $f(a,x;t)$ with $Q(t)=\sum_{j=0}^{r-1}\mu_j P_j(t)$ the relations (24) and (25) take the more interesting form

$$\sum_{\lambda=0}^{\min(\nu_k-1,\,r-2)} a_{k\lambda}f^{(\lambda+1)}(\bar a,x;x_k)+|M(I,a;x_k-0)^q-M(I,\bar a;x_k+0)^q=0,$$

$f^{(\lambda)}(\bar a,x;x_k)=0$ $(k=1,\ldots,n,\ \lambda=0,\ldots,\nu_k-1)$. Now observe that $M(I,\bar a;t)$ is continuous at x_k if $\nu_k<r$. Therefore the above equalities give $a_{k,\nu_k-1}f^{(\nu_k)}(a,x;x_k)=0$ for $\nu_k<r$, $M(I,a;x_k-0)=M(I,a;x_k+0)$ for $\nu_k=r$. Taking into account the properties (21) and (22) we conclude $a_{k,\nu_k-1}=0$ for $\nu_k<r$, i. e., for even ν_k:

$$(26)\qquad M(I,a;x_k-0)=M(I,a;x_k+0)\quad\text{for even } r.$$

It is easy to verify by using only the definition of $M(I,\bar a;t)$ that $a_{k,\nu_k-1}=M(I,a;x_k-0)-M(I,\bar a;x_k+0)$ for $\nu_k=r$. Hence (26) gives $a_{k,\nu_k}=0$ for $\nu_k=r$ and r even.

To prove the positivity of the coefficients a_{k,ν_k-2} for even ν_k and a_{k,ν_k-1} for ν_k odd we need only to see that the weighed natural spline φ satisfying the corresponding interpolation conditions like (18) is non-negative in $[a,b]$. This follows from Lemma 2. The theorem is proved.

Remark 2. The author was informed by A. A. Žensykbaev that he has proved (a paper sent to *Matem. Zametki*) the following assertion:

The optimal quadrature formula of the type $(\nu,\nu,\ldots,\nu)$ *in the class* $W_p^r[a,b]$ $(1<p<\infty)$ *coincides with the optimal formula of the type* $(\nu-1,\nu-1,\ldots,\nu-1)$ *if* ν *is an even number.*

Evidently Žensykbaev's result follows from Theorem 4 (for $1 < p \leqq 2$) putting all ν_k equal to an even number $\nu \leqq r$. Moreover, Theorem 4 asserts that the coefficients before the highest derivatives appearing in the optimal quadrature formula are positive.

REFERENCES

1. A. Sard. Linear approximation. Providence, 1963.
2. С. А. Смоляк. Об оптимальном восстановлений функций и функционалов от них. Диссертация, МГУ, 1965.
3. Н. С. Бахвалов. Об оптимальности линейных методов приближения операторов на выпуклых классах функций. *Ж. вычисл. мат. и мат. физ.*, **11**, 1971, № 4, 1014 — 1018.
4. I. J. Schoenberg. On best approximation of linear operators. *Nederl. Acad. Wetensch. Proc. Ser. A*, **67**, 1964, 155 — 163.
5. J. H. Ahlberg, E. N. Nilson. The approximation of linear functionals. *SIAM J. Numer. Anal.*, **3**, 1966, 173 — 182.
6. I. J. Schoenberg. On the Ahlberg-Nilson extension of spline interpolation: The *g*-splines and their optimal properties. *J. Math. Anal. Appl.*, **21**, 1968, 207 — 231.
7. T. N. E. Greville. Introductions to spline functions, in T. N. E. Greville (Ed.), Theory and applications of spline functions. New York, 1969, 1 — 36.
8. Б. Д. Боянов. Характеристика и существование оптимальных квадратурных формул для одного класса дифференцируемых функций. *Доклады АН ССРР*, **232**, 1977, 1233—1236.
9. L. Tschakaloff. Eine Integraldarstellung des Newton'schen Differenzenquotienten und ihre Anwendungen. *Ann. Univ. Sofia, Fac. Math.*, **34**, 1938, 354 — 405.
10. H. B. Curry, I. J. Schoenberg. On Polya frequency functions. IV. The fundamental spline functions and their limits. *J. Analyse Math.*, **17**, 1966, 71 — 107.
11. S. Karlin. Total positivity, vol. I. Stanford, 1968.

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