PLISKA
STUDIA MATHEMATICA
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UNIQUENESS ON A RESIDUAL PART OF BEST APPROXIMATIONS IN BANACH SPACES

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The paper presents some general results concerning continuity on a residual part of every semi-continuous multivalued mapping. As an application, some results are proved asserting (under some conditions) that every metric projection is single-valued on a residual part of the space. The results were announced in [6].

Let E be a Banach space and M be its subset. The multivalued mapping $P_M: E \to M$ which takes every x from E into the set $P_M(x) = \{y \in M: \|x-y\| = \inf_{z \in M} \|x-z\|\}$ is called a metric projection. S. B. Stechkin [9] studied the set of points $x \in E$ at which $P_M(x)$ consists of only one point. He proved that in many cases this set is residual in E. (The set $A \subset E$ is called residual iff it is a G_δ dense subset of E or, equivalently, $E \setminus A$ is a countable union of nowhere dense subsets of E.) In particular, if M is boundedly compact (i. e. the intersection of M with any closed ball in E is a compact subset of E) and E is strictly convex Banach space, then $P_M: E \to M$ is single-valued at the points of some residual subset of E. So to say "almost all" points of E have unique best approximation in M. We generalize this result of Stechkin and prove that it is valid not only for boundedly compact $M \subset E$ but also for every M generating an upper semi-continuous metric projection $P_M: E \to M$. On this way we also give a partial answer to a question raised by L. P. Vlasov [10] who asked whether every metric projection defined by a closed subset M of the strictly convex reflexive Banach space E is single-valued on a residual part of E. According to corollary 3 of this paper, this is really so if M is closed with respect to the weak topology in E. To reach this goal we use a general topology approach which enables us to generalize the question under consideration. We consider general Banach spaces (not only strictly convex) and study the set (see Brosowski and Deutsch [2]) $\{x \in E: P_M(x)$ lies in a convex subset of some sphere in $E\}$. If E is strictly convex, this is just the set of points at which P_M is a single-valued mapping. According to theorem 2 and theorem 4 of this paper the above mentioned set is very often residual in E.

Let us now turn to the exact definitions and arguments. First we prove a general and useful lemma concerning real-valued functions defined in $X \times X$, where X is a Čech complete topological spaces (X is a Čech complete space iff it is a G_δ subset of its Čech-Stone compactification βX. Every complete metric space as well as every locally compact space X is Čech complete.) However, for the applications we have in mind, it is quite enough to think that X is a complete metric space.

B a s i c l e m m a. *Let X be a Čech complete topological space and $f(x, y)$ be a non-negative real-valued function defined in $X \times X$. Let further*

a) $x \in X$ *implies* $f(x, x)=0$.

b) *for every triple* x, y, z *in* X $f(x, y) \leq f(x, z)+f(z, y)$,

c) *for every fixed* $x \in X$ $\lim_{y \to x} f(x, y)=0$.

Then the set $\{y \in X: \lim_{x \to y} f(x, y)=0\}$ *is residual in* X.

P r o o f. For $y \in X$ put $\alpha(y)=\inf \{t>0$: there exists an open $V \ni y$ such that $f(x, y) \leq t$ for every $x \in V\}$. Obviously, the lemma will be proved if we show that $\{y \in X: \alpha(y)=0\}$ is a residual subset of X. Thus it is enough to prove that the set $M_n=\{y \in X: \alpha(y)>1/n\}$ is nowhere dense in X, $n=1, 2, \ldots$. Suppose that M_n is dense in some open non-empty $U \subset X$. Put $V_1=U$ and choose $y_1 \in M_n \cap V_1$. According to the definition of M_n, there exists $x_1 \in V_1$ such that $f(x_1, y_1)>1/n$. Then we find an open $V_2 \ni x_1$ such that its closure $\bar{V}_2 \subset V_1$ and $f(x_1, y)<1/2n$ for every y from V_2. This is possible because of c). Since M_n is dense in U the intersection $M_n \cap V_2$ is not empty. Let $y_2 \in M_n \cap V_2$. Repeating the same argument infinitely many times we obtain the sequences $\{x_k\}_{k \geq 1}$, $\{y_k\}_{k \geq 1}$, $\{V_k\}_{k \geq 1}$ satisfying the conditions

1) $f(x_k, y_k)>1/n$,

2) $f(x_k, y)<1/2n$ for each $y \in V_k$,

3) $V_{k+1} \subset \bar{V}_{k+1} \subset V_k$,

4) $y_k \in M_n \cap V_k$

for every $k \geq 1$.

Now we are going to use the Čech completeness of X in order to prove that $\{y_k\}_{k \geq 1}$ has at least one cluster point y_0 in X. The reader who is interested only in the applications given below can think of X as a complete metric space. In such a space the step by step constructed sequence $\{y_k\}_{k \geq 1}$ can be chosen so that the distance between any two consecutive members y_k and y_{k+1} be less or equal to $1/2^k$. Therefore $\{y_k\}_{k \geq 1}$ is a fundamental sequence with $\lim y_k=y_0$. In the general case of an arbitrary Čech complete space X we proceed in a different way. Let $X=\bigcap_{i \geq 1} O_i$ where O_i, $i \geq 1$, are open subsets of βX — the Čech-Stone compactification of X. It is clear that the choice of V_k in the construction can be subordinated to the additional condition

5) $\bar{V}_k^{\beta X} \subset O_k$, $k \geq 1$, where $\bar{V}^{\beta X}$ stands for the closure of V in βX. Since βX is compact, $\{y_k\}_{k \geq 1}$ has at least one cluster point $y_0 \in \beta X$. It follows from 3) and 4) that $y_i \in V_k \subset \bar{V}_k \subset \bar{V}_k^{\beta X}$ for $i \geq k$. Thus y_0 belongs to $\bar{V}_k^{\beta X}$ for every $k \geq 1$. Using 5) we get $y_0 \in \bigcap_{k \geq 1} \bar{V}_k^{\beta X} \subset \bigcap_{k \geq 1} O_k=X$. Therefore y_0 is a cluster point of $\{y_k\}_{k \geq 1}$ in X and $y_0 \in \bar{V}_k$ for $k \geq 1$. On the other hand, it follows by c) that y_0 lies in some open $W \subset X$ such that $f(y_0, y)<1/2n$ for every $y \in W$. Then W contains at least one member y_q of the sequence $\{y_k\}_{k \geq 1}$ and by 1), b) and 2) we obtain the contradiction

$1/n<f(x_q, y_q) \leqq f(x_q, y_0)+f(y_0, y_q)<(1/2n)+(1/2n)=1/n$. The proof of the basic lemma is finished.

Definition. *Let X and Z be topological spaces and $F:X\to Z$ be a set-valued mapping. F is said to be upper (resp. lower) semi-continuous at some point x_0 from X if for every open $U\supset F(x_0)$ (resp. $U\cap F(x_0)\neq\emptyset$) there exists an open $V\ni x_0$ such that $F(x)\subset U$ (resp. $U\cap F(x)\neq\emptyset$) whatever $x\in V_0$. If (Z,d) is a metric space, we say that the multivalued mapping $F:\to(Z,d)$ is metrically uper (resp. lower) semi-continuous at $x_0\in X$ if for every $\varepsilon>0$ there is an open $V\ni x_0$ such that*

$$\sup_{z_1\in F(x)}\ \inf_{z_2\in F(x_0)} d(z_1, z_2)<\varepsilon$$

$$(\text{resp. } \sup_{z_1\in F(x_0)}\ \inf_{z_2\in F(x)} d(z_1, z_2)<\varepsilon)$$

whenever $x\in V$.

We shall write shortly "usc" instead of "upper semi-continuous" and "lsc" instead of "lower semi-continuous". The connection between the usual and metrical semi-continuity is very simple. Every usc mapping F is metrically usc and every metrically lsc mapping is lsc. For multivalued mappings with compact images the usual and metrical semicontinuity coincide.

Theorem 1. *Let X be a Čech complete topological space and $F:X\to(Z,d)$ be a metrically usc (resp. lsc) mapping of X into the metric space (Z,d). Then the set $\{y\in X: F$ is metrically lsc at $y\}$ (resp. $\{y\in X: F$ is metrically usc at $y\}$) is residual in X.*

Proof. First we consider the case when $\sup_{z_1,z_2\in Z} d(z_1,z_2)<\infty$. Denote

$$f(x, y)=\sup_{z_1\in F(y)}\ \inf_{z_2\in F(x)} d(z_1, z_2).$$

This is a non-negative function defined in $X\times X$. It is not difficult to see that $f(x,y)$ satisfies a) and b) from the basic lemma. In addition, F is metrically usc at x iff $\lim_{y\to x} f(x,y)=0$, and metrically lsc iff $\lim_{x\to y} f(x,y)=0$. Thus if F is metrically usc, then c) from the basic lemma is also fulfilled and we get that F is metrically lsc at the points of a residual subset of X. In order to prove the second "(resp. . . .)" part of theorem 1 we should apply the basic lemma to the function $f_1(x,y)=f(y,x)$. The general case ($\sup_{z_1,z_2\in Z} d(z_1,z_2)=\infty$) can be reduced to the particular one we have just considered by a simple change of the metric. The new one is defined by the formula $d'(u,v)=d(u,v)/(1+d(u,v))$. It is a routine matter to verify that F is metrically (upper or lower) semi-continuous at $x_0\in X$ with respect the to d iff it is metrically semi-continuous relative to d'.

Corollary 1. *Let $F:X\to(Z,d)$ be an usc mapping from the Čech complete space X into the metric space Z. Then F is lsc at the points of a residual subset of X.*

In the particular case when X is a complete metric space corollary 1 was proved in [5]. For multivalued mappings with compact images the result expressed in theorem 1 belongs to M. K. Fort [4].

Let us now turn again to the metric projection defined by $M\subset E$, where E is a Banach space. For the real number a and $x\in E$ put $S(x,a)=\{y\in E: \|y-x\|=a\}$ and $d(x,M)=\inf_{z\in M}\|x-z\|$.

Theorem 2. [5] *Suppose that $P_M: E \to M$ is an usc metric projection with non-empty images. Then the set $\{x \in E: P_M(x)$ lies in a convex subset of the sphere $S(x, d(x, M))\}$ is residual in E. If, in addition, E is strictly convex, then $P_M: E \to M$ is single-valued on a residual subset of E.*

Proof. It follows from theorem 1 that P_M is lsc at the points of some residual subset of E. On the other hand, Blatter, Morris and Wulbert [1] (see also Brosowski and Deutsch [2]) have shown that at every point of lower semi-continuity of P_M the set $P_M(x)$ is contained in a convex subset of the sphere $S(x, d(x, M))$. The second part of this theorem concerning strictly convex Banach spaces is now evident because the only convex subsets of any sphere in such a space are the single-point sets.

Corollary 2 (Stechkin [9]). *If M is a boundedly compact subset of the strictly convex Banach space E, then $P_M: E \to M$ is a single-valued mapping on a residual part of E.*

Proof. I. Singer [8] proved a general result implying that every metric projection defined by a boundedly compact set M has to be usc. So it remains to apply theorem 2.

For special Banach spaces the conditions in theorem 2 and corollary 2 can be relaxed. Until now we considered metric projections $P_M: E \to M$ which are usc with respect to the norm topology n in E and M. Now we shall be interested in projections which are usc with respect to the topology n in E and the weak topology w in M. Such a projection will be referred to as n-w-usc.

Theorem 3. *Let M be a subset of the strictly convex Banach space E. Suppose that M generates an n-w-usc metric projection with nonvoid images. Then the set $\{x \in E: x$ has unique best approximation in $M\}$ is residual in E.*

Proof. S. B. Stechkin [9] proved that every metric projection in a strictly convex Banach space is single-valued at the points of some dense subset of E. Consider the functions $f(x) = d(x, M) = d(x, P_M(x))$ and $g(x) = d(x, \mathrm{co}P_M(x))$. Here $\mathrm{co}A$ denotes the closed convex hull of the set A. Since E is strictly convex, $P_M(x)$ consists of only one point if and only if $f(x) = g(x)$. Hence the just mentioned result of Stechkin implies that $f(x) = g(x)$ on a dense subset of E. Since f is a continuous function the equality $f(x) = g(x)$ will hold for every point of continuity of g. Therefore, in order to prove the theorem, we must show that g is continuous on a residual subset of E. To reach this point we establish that g is semi-continuous as a real-valued function and then apply the well-known theorem (see for instance M. K. Fort [3]) that such a function is obliged to be continuous on a residual part of E.

Lemma. *For each real number a the set $\{x \in E: g(x) > a\}$ is open.*

Indeed $g(x) \geq 0$ for every $x \in E$. This means that the interesting case is $a > 0$. Let x_0 is such that $g(x_0) > a > 0$. For some a', $g(x_0) > a' > a$, the closed ball $B[x_0, a'] = \{x \in E: \|x - x_0\| \leq a'\}$ doesn't intersect the closed convex set $\mathrm{co}P_M(x_0)$. Then there exists a closed hyperplane L strictly separating these two sets. $\mathrm{co}P_M(x_0)$ lies into one of the two open halfspaces defined by L.

Since P_M is n-w-usc at x_0, the same open halfspace will contain $P_M(x)$ for all x from some n-open neighbourhood U of x_0. Then $\mathrm{co}P_M(x)$ lies in the closure of that halfspace, and this closed halfspace doesn't intersect $B[x_0, a']$. This means that $g(x)>a$ provided $x \in U \cap B[x_0, a'-a]$. Theorem 3 is proved.

Corollary 3. *Let E be a reflexive strictly convex Banach space and M be its weakly closed subset. Then the metric projection $P_M: E \to M$ is single-valued on a residual subset of E.*

Proof. In view of the previous theorem, we have to show that $P_M: E \to M$ is n-w-usc. Indeed, let U be an w-open subset of E containing $P_M(x_0)$ and suppose that there exists a sequence $\{x_i\}_{i\geq 1}$ such that $x_0 = \lim_i x_i$ (with respect to n) and $P_M(x_i) \cap (E \setminus U) \neq \emptyset$ for $i \geq 1$. Choose $y_i \in P_M(x_i) \cap (E \setminus U)$ $i \geq 1$. Because of the norm continuity of $f(x) = d(x, M)$ the ball $B[x_0, f(x_0)+a]$, $a>0$, contains all y_i, $i \geq i_0$. Therefore $\{y_i\}_{i\geq 1}$ is a bounded subset of the reflexive space E and should have a w-cluster point $y_0 \in B[x_0, f(x_0)+a]$. As $a>0$ was an arbitrary positive number, $y_0 \in B[x_0, f(x_0)]$. On the other hand, $y_0 \in M$ because y_i, $i \geq 1$, belong to the weakly closed set M. Thus $y_0 \in P_M(x_0) \subset U$. But this contradicts the fact that y_0 is a w-cluster point of $\{y_i\}_{i\geq 1} \subset E \setminus U$.

Remark. Corollary 3 answers partially a question by L. P. Vlasov [10] who asked whether every metric projection $P_M: E \to M$ defined by a closed subset M of the strictly convex reflexive space E is single-valued on a residual part of E. As we see this is really so if M is weakly closed.

For separable Banach spaces we have the following generalization of theorem 2 and theorem 3.

Theorem 4 ([7]). *Let E be a separable Banach space and $M \subset E$. Suppose that $P_M: E \to M$ is an n-w-usc metric projection with non-void images. Then the set $\{x \in E : \mathrm{co}\, P_M(x) \subset S(x, d(x, M))\}$ is residual in E.*

The proof is quite similar to that of theorem 2. The only difference is that we cannot apply theorem 1 to the present situation. But all the same $P_M: E \to M$ is $n-n-$lsc at the points of some residual subset of E. This is an immediate consequence of theorem 1 from [7]. Further we proceed as in the proof of theorem 2.

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Received 29. 9. 1976