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ПЛИСКА
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DUAL SUPERNILPOTENT RADICAL CLOSURES OF RIGHT IDEALS

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The concept radical closure is introduced for the first time by V. A. Andrunakievic and Yu. M. Riabuchin [3]. The introduction of that concept enables us to find some properties of right ideals of a fixed algebra, which cannot be proved with the concept radical. In this paper we shall give a description of one class radical closures which are analogous to dual special radicals, introduced by Andrunakievic [1].

1. Preliminary remarks. Let R is an associative algebra (not necessarily with an unit) over a commutative ring Φ with an unit $(1\in\Phi)$. We shall note with $\mathfrak{P}(R)$ the set of the right ideals of R and with $\mathfrak{A}(R)$ — the class of all pairs (A, B), where $A, B\in\mathfrak{P}(R)$ and $A\supset B$. If $A\supset C\supset B$ are right ideals of R then we shall call the pairs (C, B) and (A, B) subpair and factor-pair of the pair (A, B).

It is clear that with every pair $(A, B)\in\mathfrak{A}(R)$ it can be connected a R-module A/B. We shall call a pair (A, B) respectively: zero, simple, subdirectly irreducible pair when the R-module A/B is zero, simple, subdirectly irreducible. When the pair (A, B) is subdirectly irreducible it is clear that the intersection $H=\cap Q_{\alpha}$ of all right ideals Q_{α} $(\alpha\in I)$ of R for which $A\supset Q_{\alpha}\supset B$, $B\neq Q_{\alpha}$ is different from B. In that case we shall call the pair (H, B) (which is a simple one) a heart of the pair (A, B).

The mapping $\varrho:\mathfrak{A}(R)\rightarrow\mathfrak{P}(R)$ will be called a radical closure if the following conditions hold:

ϱ.1. For every pair $(A, B)\in\mathfrak{A}(R): B\subset\varrho(A, B)\subset A$.

ϱ.2. For every pair $(A, B)\in\mathfrak{A}(R)$:

$$\varrho(\varrho(A, B), B)=\varrho(A, \varrho(A, B))=\varrho(A, B).$$

ϱ.3. If $A\supset C\supset B$ are right ideals of R, then

$$\varrho(A, C)\supset\varrho(A, B)\supset\varrho(C, B).$$

If the radical closure ϱ satisfies the condition

ϱ.4. If $A\supset C\supset B$ are right ideals of R, then $\varrho(C, B)=\varrho(A, B)\cap C$, then it will be called a hereditary radical closure.

If ϱ_1 and ϱ_2 are radical closures, then we note $\varrho_1\leqq\varrho_2$ if for every pair $(A, B)\in\mathfrak{A}(R)$ is held: $\varrho_1(A, B)\subset\varrho_2(A, B)$.

Let ϱ be a radical closure. The pair (A, B) will be called a ϱ-semi-simple one, if $\varrho(A, B)=B$. The pair (A, B) will be called a ϱ-radical one, if $\varrho(A, B)=A$.

With every radical closure ϱ it can be connected two classes of pairs: the class $\mathcal{R}(\varrho)$ of all ϱ-radical pairs and the class $\mathcal{S}(\varrho)$ of all ϱ-semi-simple pairs.

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A class of pairs $\mathcal{S}$ (further we shall suppose, that every subclass of the class $\mathfrak{A}(R)$ contains all zero pairs) will be called a semi-simple one if there exists a radical closure ϱ such that $\mathcal{S}=\mathcal{S}(\varrho)$. It is proved [3] that the class of pairs $\mathcal{S}$ is a semi-simple one if and only if the following conditions hold:

s.1. If $(A, B)\in\mathcal{S}$, then for every right ideal C of R: $(A\cap C, B\cap C)\in\mathcal{S}$.

s.2. If for every non zero subpair (C, B) of the pair (A, B) there exists a non-zero factor-pair $(C, D)\in\mathcal{S}$, then $(A, B)\in\mathcal{S}$.

Let now $\mathcal{C}$ be a class of pairs. We note with $\mathcal{C}_1$ the class of all pairs $(A\cap C, B\cap C)$, where $(A, B)\in\mathcal{C}$ and C is a right ideal of R and with $\mathcal{C}_2$ — the class of all pairs (A, B) such that for every non-zero subpair (C, B) of the pair (A, B), there exists a non-zero factor-pair $(C, D)\in\mathcal{C}_1$. It is not difficult to prove, that the conditions s.1. and s.2. are held for the class $\mathcal{C}_2$ and therefore there exists a radical closure ϱ^t, such that $\mathcal{C}_2=\mathcal{S}(\varrho^t)$. At that if η is a radical closure, for which $\mathcal{C}\subset\mathcal{S}(\eta)$, then $\eta\leqq\varrho^t$. The radical closure ϱ^t will be called an upper radical closure, determined by the class $\mathcal{C}$.

It is proved [4] that if the class $\mathcal{C}$ has a condition s.1, then the conditions (i) ϱ^t to be a hereditary radical closure and (ii) $\varrho^t(A, B)=\cap T_\alpha$, where $\{T_\alpha \mid \alpha\in I\}$ is the set of all right ideals of R for which $B\subset T_\alpha\subset A$ and $(A, T_\alpha)\in\mathcal{C}$ are equivalent to the following condition:

s.3. If for a non-zero subpair (C, B) of the pair (A, B) there exists a non-zero factor-pair $(C, D)\in\mathcal{C}$, then there exists a factor pair (A, T) of the pair (A, B), such that $(A, T)\in\mathcal{C}$ and T does not $\supset C$.

If ϱ is a radical closure, then the radical closure ϱ^* will be called supplementary to ϱ, if (i) for every pair $(A, B)\in\mathfrak{A}(R)$: $\varrho(A, B)\cap\varrho^*(A, B)=B$ and (ii) for every radical closure η, for which $\eta(A, B)\cap\varrho(A, B)=B$, is held: $\eta\leqq\varrho^*$. The radical closure ϱ will be called a dual one, if there exists the radical closures ϱ^* and $\varrho^{**}=(\varrho^*)^*$ and $\varrho=\varrho^{**}$.

According to [3] and [4] for every radical closure ϱ there exists a supplementary radical closure ϱ^*, which is hereditary and dual one. If ϱ is a hereditary radical closure at that, then the radical closure ϱ^* is equal to the upper radical closure ϱ^t, determined by the class of all subdirectly irreducible pairs with a ϱ-radical heart. And what is more, the radical closure ϱ^* is equal to the upper radical closure, determined by the class of all subdirectly irreducible pairs (R, P) with a ϱ-radical heart. In that case $\varrho^*(A, B)=A$ iff for every right ideal D of R is held $\varrho(A+D, B+D)=B+D$.

A pair $(A, B)\in\mathfrak{A}(R)$ will be called a nilpotent one if there exists an integer $n\geqq 2$, such that $A^n\subset B$ and will be called almost nilpotent, if there exists a right ideal C of R such that $C^n\subset B$ and $B+C=A$. The radical closure ϱ will be called a supernilpotent one, if it is hereditary and if every nilpotent pair is a ϱ-radical one. It is clear that the radical closure ϱ is supernilpotent one if and only if it is hereditary and every almost nilpotent pair is a ϱ-radical one.

A pair $(A, B)\in\mathfrak{A}(R)$ will be called a weak regular one if for every right ideal T of R for which $T\subset A$, is held: $T\subset T^2+B$. It is not difficult to be proved that the following five conditions are equivalent:

(a) (A, B) is a weak regular pair.

(b) For every element $x\in A$ there exists an element $x'\in(x)$ such that $x-xx'\in B$.

(c) For every element $x\in A$ is held: $x\in x(x)+B$.

(d) For every element $x\in A$ is held: $x\in(x)^2+B$.

(e) For every element $x \in A$ is held: $|x) \subset |x)^2 + B$.

Remark. We note here with (x) the two-sided ideal of R, generated by x, and with $|x)$ — the right ideal of R, generated by x, i. e.

$$(x) = \{\alpha x + x r_1 + r_2 x + s_1 x s_2 \mid \alpha \in \Phi,\ r_1,\ r_2,\ s_1,\ s_2 \in R\},$$

$$|x) = \{\alpha x + x r \mid \alpha \in \Phi,\ r \in R\}.$$

A pair (A, B) will be called a hereditary idempotent one if for every right ideal C of R the pair $(A+C, B+C)$ is a weak regular pair.

The radical closure ϱ will be called an underidempotent one if it is a hereditary one and if every ϱ-radical pair is a hereditary idempotent one. The following theorem is proved in [4].

If ϱ is a supernilpotent radical closure, then supplementary to it radical closure ϱ^* is a dual underidempotent one. The radical closure ϱ^{**} is a dual supernilpotent one and if η is a dual supernilpotent radical closure for which $\varrho \leqq \eta$, then $\varrho^{**} \leqq \eta$.

2. Special radical closures. Let $\mathfrak{M}$ be a (right) R-module. If S is a subset of $\mathfrak{M}$, we note with $\operatorname{ann} S$ the annihilator of S, i. e.

$$\operatorname{ann} S = \{x \in R \mid Sx = (0)\}.$$

It is clear that $\operatorname{ann} S$ is a right ideal of R. If S is a submodule of $\mathfrak{M}$ at that, then $\operatorname{ann} S$ is a two-sided ideal of R. In case $\operatorname{ann} \mathfrak{M} = (0)$ we call the module $\mathfrak{M}$ a faithful one.

The module $\mathfrak{M}$ will be called a prime one if $\mathfrak{M}R \neq (0)$ and if for any non-zero submodule $\mathfrak{N}$ of $\mathfrak{M}$ is held: $\operatorname{ann} \mathfrak{N} = \operatorname{ann} \mathfrak{M}$. It is easy to see that every submodule of a prime module is prime too.

A pair (A, B) will be called a prime one if there exists a right ideal P of R, such that R/P is a prime module, $\operatorname{ann}(R/P) \subset P$ and $P \cap A = B$.

Lemma 1. *If (A, B) is a prime pair, then for every right ideal C of R for which $A \cap C \neq B \cap C$, the pair $(A \cap C, B \cap C)$ is prime too.*

Proof. Let (A, B) is a prime pair. Then there exists a right ideal P of R such that R/P is a prime module, $\operatorname{ann}(R/P) \subset P$ and $P \cap A = B$. From here we have $P \cap A \cap C = B \cap C$ and therefore $(A \cap C, B \cap C)$ is a prime pair too.

Lemma 2. *The pair $(A, B) \in \mathfrak{A}(R)$ is a prime one iff A/B is a prime module and* $\operatorname{ann}(A/B) \cap A \subset B$.

Proof. Let (A, B) is a prime pair. Then there exists a right ideal P of R such that R/P is a prime module, $\operatorname{ann}(R/P) \subset P$ and $P \cap A = B$. But

$$A/B = A/P \cap A \cong A + P/P \subset R/P$$

and therefore A/B is a prime module.

On the other hand,

$$\operatorname{ann}(A/B) = \operatorname{ann}(A+P/P) = \operatorname{ann}(R/P).$$

Thus, $\operatorname{ann}(A/B) \cap A \subset P \cap A = B$.

Contrary let A/B is a prime module and $\operatorname{ann}(A/B) \cap A \subset B$. We note with $\mathfrak{S}$ the set of all right ideals T of R for which $B + \operatorname{ann}(A/B) \subset T$ and $T \cap A = B$. It is clear that $B + \operatorname{ann}(A/B) \in \mathfrak{S}$, because if $x \in (B + \operatorname{ann}(A/B)) \cap A$ then $x = b_1 + b_2$, $b_1 \in B$, $b_2 \in \operatorname{ann}(A/B)$ and therefore $b_2 = x - b_1 \in A \cap (\operatorname{ann}(A/B) \subset B$, i. e. $x \in B$. From here we receive that the set $\mathfrak{S}$ is not empty and by Zorn's lemma there exists a maximal element P in $\mathfrak{S}$.

We have first that if $x \in \operatorname{ann}(R/P)$, then $Rx \subset P$ and $Ax \subset P \cap A = B$, i. e. $x \in \operatorname{ann}(A/B) \subset P$. Therefore, $\operatorname{ann}(R/P) \subset P$.

On the other hand, it is clear that R^2 does not $\subset P$. (If $R^2 \subset P$, then $AR \subset P \cap A = B$, which is not true.)

Let now C/P is a non-zero submodule of R/P. Then $\operatorname{ann}(C/P) \supset \operatorname{ann}(R/P)$. But $C \neq P$. Therefore, $C \cap A \neq B$, i. e. $C \cap A/B$ is a non-zero submodule of A/B and

$$C \cap A/B = C \cap A/P \cap A \cong C \cap A + P/P.$$

From here we have

$$\operatorname{ann}(C \cap A + P/P) = \operatorname{ann}(C \cap A/B) = \operatorname{ann}(A/B) \subset P$$

because A/B is a prime module and $C \cap A/B$ is its non-zero submodule. At that $\operatorname{ann}(C \cap A + P/P)$ is a two-sided ideal of R and thus

$$R \operatorname{ann}(C \cap A + P/P) \subset \operatorname{ann}(C \cap A + P/P) \subset P.$$

Therefore,

$$\operatorname{ann}(R/P) \supset \operatorname{ann}(C \cap A + P/P) \supset \operatorname{ann}(C/P) \supset \operatorname{ann}(R/P)$$

and $\operatorname{ann}(R/P) = \operatorname{ann}(C/P)$, i. e. R/P is a prime module.

The lemma is proved.

Corollary 1. *If (A, B) is a subdirectly irreducible and prime pair, then there exists a subdirectly irreducible and prime pair (R, P) such that $P \cap A = B$. At that if (H, B) is the heart of (A, B), then $(H+P, P)$ is the heart of (R, P).*

Proof. If (A, B) is a prime pair, then there exists a right ideal P of R, such that (R, P) is a prime pair and $P \cap A = B$. We can choose P — maximal to the condition $P \cap A = B$. Thus, if Q is a right ideal of R and $Q \supset P$, $Q \neq P$, then $Q \cap A \supset B$, $Q \cap A \neq B$ and $Q \supset H + P$. On the other hand,

$$H + P/P \cong H/P \cap H = H/B$$

and, therefore, (R, P) is a subdirectly irreducible pair with a heart $(H+P, P)$.

The class Σ of pairs will be called a special one, if the following conditions hold:

$\sigma.1.$ If $(A, B) \in \Sigma$, then (A, B) is a prime pair.

$\sigma.2.$ If $(A, B) \in \Sigma$, then for every right ideal C of R: $(A \cap C, B \cap C) \in \Sigma$.

$\sigma.3.$ If (C, B) is a non-zero subpair of the pair (A, B) and $(C, B) \in \Sigma$, then there exists a non-zero factor-pair (A, T) of (A, B) such that T does not $\supset C$ and $(A, T) \in \Sigma$.

It is clear that every special class Σ determines an upper radical closure ϱ^s. The radical closure ϱ will be called a special one, if there exists a special class of pairs Σ, such that $\varrho = \varrho^s$. By condition $\sigma.3$ we have, that every special radical closure ϱ^s is a hereditary one and for every pair (A, B): $\varrho^s(A, B) = \cap T_\alpha$, $(\alpha \in I)$ where $\{T_\alpha \mid \alpha \in I\}$ is the set of all right ideals of R, such that $B \subset T_\alpha \subset A$ and $(A, T_\alpha) \in \Sigma$.

It is obvious that if ϱ is a special radical closure determined by the class Σ and (A, B) is a subdirectly irreducible pair, then $\varrho(A, B) = B$ iff $(A, B) \in \Sigma$.

Lemma 3. *If $\{\varrho_\alpha \mid \alpha \in I\}$ is a family of special radical closures, then there exists a special radical closure $\varrho = \Lambda \varrho_\alpha$, such that $\varrho \leq \varrho_\alpha$ for every α and if η is a radical closure for which $\eta \leq \varrho_\alpha$, then $\eta \leq \varrho$.*

Proof. Let the radical closure ϱ_α is determined by the class Σ_α. Then it is not difficult to be seen, that $\Sigma = \cup \Sigma_\alpha (\alpha \in I)$ is a special class. To note with ϱ the special radical closure, determined by Σ. Then for every $\alpha \in I$ we have $\Sigma_\alpha \subset \Sigma$ and, therefore, $\varrho \leqq \varrho_\alpha$. On the other hand, if η is a radical closure, for which $\eta \leqq \varrho_\alpha$, then $\Sigma_\alpha \subset \mathcal{S}(\varrho_\alpha)$ for every $\alpha \in I$. Therefore, $\Sigma \subset \mathcal{S}(\eta)$ and $\eta \leqq \varrho$.

The lemma is proved.

Proposition 1. *Every special radical closure ϱ is a supernilpotent one.*

Proof. It is clear that ϱ is a hereditary radical closure. Let (A, B) be a nilpotent pair. Suppose that $\varrho(A, B) \neq A$. It follows from σ.2. and σ.3. that there exists a non-zero factor pair (A, T) of the pair (A, B), such that $(A, T) \in \Sigma$. (We note here with Σ the special class, which determines ϱ.) Then by σ.1. we have that (A, T) is a prime pair. On the other hand (A, B) is a nilpotent pair and, therefore (A, T) is nilpotent too. Let $n \geqq 2$ is such an integer for which $A^n \subset T$ and A^{n-1} does not $\subset T$. Thus, $A^n = A \,.\, A^{n-1} \subset T$ and therefore $A^{n-1} \subset \operatorname{ann}(A/T) \cap A \subset T$, which is a contradiction.

The proposition is proved.

Proposition 2. *If ϱ is a suppernilpotent radical closure, then there exists a special radical closure ϱ_1, such that $\varrho \leqq \varrho_1$ and for every special radical closure η for which $\varrho \leqq \eta$, is held: $\varrho_1 \leqq \eta$ too.*

Proof. Let $\mathfrak{M}$ is the class of all prime pairs (R, P), which are ϱ-semisimple, i. e. $\mathfrak{M} = \{(R, P) \mid (R, P)$ is a prime pair and $(R, P) \in \mathcal{S}(\varrho)\}$ and let $\mathfrak{M}^0$ is the class of all pairs $(A, P \cap A)$, where $A \in \mathfrak{P}(R)$ and $(R, P) \in \mathfrak{M}$.

First we shall prove, that $\mathfrak{M}^0$ is a special class of pairs. It is clear that the conditions σ.1. and σ.2. are true for the class $\mathfrak{M}^0$. Let $(C, B) \in \mathfrak{M}^0$ is a nonzero subpair of the pair (A, B). There exists a pair $(R, P) \in \mathfrak{M}$ such that $P \cap C = B$. Then $(A, P \cap A) \in \mathfrak{M}^0$ and $P \cap A$ does not $\supset C$ since P does not $\supset C$. Therefore, the class $\mathfrak{M}^0$ has the condition σ.3, and it is a special class of pairs.

We denote with ϱ^m the special radical closure determined by $\mathfrak{M}^0$. Since $\mathfrak{M} \subset \mathcal{S}(\varrho)$, then $\mathfrak{M}^0 \subset \mathcal{S}(\varrho)$, i. e. $\varrho \leqq \varrho^m$ and, therefore, the class $\mathcal{L}$ of all special radical closures, which are $\geqq \varrho$ is not empty. By lemma 3 it follows that the radical closure $\varrho_1 = \Lambda \varrho_\alpha$, $(\varrho_\alpha \in \mathcal{L})$ is special too. By that it is clear that the conditions of proposition are true for ϱ_1.

In the end we shall prove that $\varrho_1 = \varrho^m$. We have $\varrho_1 \leqq \varrho^m$. Suppose that $\varrho_1 \neq \varrho^m$. Then there exists a pair (A, B), such that $\varrho_1(A, B) = B$ but $\varrho^m(A, B) = A$. If Σ is the special class, determined by ϱ_1, then

$$B = \cap \{T \in \mathfrak{P}(R) \mid B \subset T \subset A \quad \text{and} \quad (A, T) \in \Sigma\}.$$

From here we have that there exists a pair $(A, T) \in \Sigma$ for which $\varrho^m(A, T) = A$. On the other hand (A, T) is a subpair of the pair (R, T) and, therefore, there exists a right ideal $P \in \mathfrak{P}(R)$, such that $(R, P) \in \Sigma$, $P \supset T$ and P does not $\supset T$. But it follows from $\varrho \leqq \varrho_1$ that $\Sigma \subset \mathcal{S}(\varrho)$ and then $(R, P) \in \mathfrak{M}$ and $(A, P \cap A) \in \mathfrak{M}^0$, $A \supset P \cap A \supset T$, $A \neq P \cap A$. This contradicts with $\varrho^m(A, T) = A$. Therefore, $\varrho^m = \varrho_1$.

The proposition is proved.

3. Classes of subdirectly irreducible pairs.

Lemma 4. *Let (A, B) is a subdirectly irreducible pair with a heart (H, B). Then (A, B) is a prime pair iff (H, B) is a weak regular pair.*

Proof. Let (A, B) be a prime pair. Then by lemma 1. $(H, B) = (A \cap H, B \cap H)$ is prime too. Let $x \in H$. If $x \in B$, then $x \in x(x) + B$. Let $x \notin B$. Suppose that $x(x) \subset B$.

Then $H(x)=(|x)+B)(x)\subset B$ and $(x)\subset \mathrm{ann}(H/B)$. It follows from here that $x\in \mathrm{ann}(H/B)\cap H\subset B$, which is a contradiction. Therefore, $x(x)$ does not $\subset B$; $x(x)+B=H$ and $x\in x(x)+B$, i. e. (H, B) is a weak regular pair.

Contrary, let (H, B) be a weak regular pair. Then H^2 does not $\subset B$ and $(H/B)R\neq(0)$. The condition for anihilator is held, since H/B is a simple module. Therefore, H/B is a prime module.

Let now $x\in \mathrm{ann}(H/B)\cap H$. If we suppose that $x\notin B$, then $|x)+B=H$ and $H.|x)\subset B$. We receive $|x)^2\subset B$ which is a contradiction with a weak regularity of (H, B). So we have, that (H, B) is a prime pair. It follows from here that there exists a right ideal P of R, such that (R, P) is a prime pair and $P\cap H=B$. Thus $(A, P\cap A)$ is also a prime pair. By that if $P\cap A\neq B$, then $P\cap A\supset H$ and $P\cap H=H$, which is not true. Therefore, $(A, B)=(A, P\cap A)$.

The lemma is proved.

We note now with $B(R)$ the class of all subdirectly irreducible pairs (R, P).

L e m m a 5. *Every pair (R, P) of $B(R)$ has either a weak regular heart or an almost nilpotent heart.*

P r o o f. Let (H, P) is a heart of the pair (R, P). If (H, P) is not a weak regular pair, there exists a right ideal T of R, such that T does not $\subset T^2+P$ and $T\subset H$. But since (H, P) is a simple pair and $T^2\subset H$, then either $T^2+P=H$ or $T^2\subset P$. Therefore, $T^2\subset P$ and $P+T=H$, i. e. the pair (H, P) is an almost nilpotent pair.

C o r o l l a r y 2. *Every pair (R, P) of $B(R)$ has either a hereditary idempotent heart, or an almost nilpotent heart.*

P r o o f. Let (H, P) is a heart of $(R, P)\in B(R)$. If (H, P) is not almost nilpotent pair, then (H, P) is a weak regular one. But it is clear, that for every right ideal C of R is held: either $H+C=P+C$, or $(H+C, P+C)=(H, P)$. Therefore (H, P) is a hereditary idempotent pair.

C o r o l l a r y 3. *If $(R, P)\in B(R)$, then either (R, P) is a prime pair, or (R, P) has an almost nilpotent heart.*

L e m m a 6. *Let $\mathfrak{M}$ is a subclass of the class $B(R)$. Then the class $\mathfrak{M}^0$ of all pairs $(A, P\cap A)$, where $(R, P)\in\mathfrak{M}$ has the condition s. 3.*

P r o o f. Let $(C, D)\in\mathfrak{M}^0$ is a non-zero factor pair of the pair (C, B), which is a subpair of the pair (A, B). Then $D=P\cap C$, where $(R, P)\in\mathfrak{M}$. It follows from here that $(A, P\cap A)\in\mathfrak{M}^0$. By that $P\cap A$ does not $\supset C$ since $P\cap C\neq C$, i. e. P does not $\supset C$.

The lemma is proved.

C o r o l l a r y 4. *The upper radical closure ϱ^m, determined by the class $\mathfrak{M}$ is a hereditary one and for every pair $(A, B)\in\mathfrak{A}(R)$ is held:*

$$\varrho^m(A, B)=A\cap(\cap\{P\in\mathfrak{P}(R)\,|\,(R, P)\in\mathfrak{M} \text{ and } P\supset B\}).$$

We recieve from here that the pair (R, P) is in $\mathfrak{M}$ iff the heart of (R, P) is a ϱ^m-semi-simple one.

L e m m a 7. *Let $\mathfrak{M}$ is a subclass of the class $B(R)$, such that every pair of $\mathfrak{M}$ has a hereditary idempotent heart. Then the class $\mathfrak{M}^0$ of all pairs $(A, P\cap A)$, where $(R, P)\in\mathfrak{M}$ is a special class of pairs.*

P r o o f. It follows by corollary 3., that all pairs of $\mathfrak{M}$ and, therefore, all pairs of $\mathfrak{M}^0$ are prime pairs. Similarly if $(A, P\cap A)\in\mathfrak{M}^0$ and $C\in\mathfrak{P}(R)$, then $(A\cap C, P\cap A\cap C)\in\mathfrak{M}^0$. At the end it follows by lemma 6 that the condition σ.3. is held for $\mathfrak{M}^0$.

So we have, that the class $\mathfrak{M}^0$ has the conditions σ.1., σ.2., σ3., i. e. $\mathfrak{M}^0$ is a special class of pairs.

The lemma is proved.

Corollary 5. *The upper radical closure ϱ^m, determined by the class $\mathfrak{M}$ is a special one.*

Proof. It follows by the results in [3] and [4] that ϱ^m is equal to the upper radical closure, determined by the class $\mathfrak{M}^0$. Therefore, ϱ^m is a special radical closure.

Corollary 6. *If ϱ is an underidempotent radical closure, then a supplementary to it radical closure ϱ^* is a special one.*

Proof. The radical closure ϱ^* is equal to the upper radical closure, determined by the class $\mathcal{C}(\varrho)$ of all subdirectly irreducible pairs (R, P) with a ϱ-radical heart. It follows from the underidempotentity of ϱ that all pairs of $\mathcal{C}(\varrho)$ have a hereditary idempotent heart. Therefore, ϱ^* is a special radical closure.

Corollary 7. *Every dual supernilpotent radical closure is a special radical closure.*

Proof. If ϱ is a dual supernilpotent radical closure, then ϱ^* is an underidempotent radical closure and, therefore, $\varrho=\varrho^{**}$ is a dual special radical closure.

Theorem (Duality theorem). *Let $\mathfrak{M}$ be a subclass of $B(R)$, such that all pairs of $\mathfrak{M}$ have a hereditary idempotent heart and let $\overline{\mathfrak{M}}$ is the class of all the rest of the pairs of $B(R)$. Then the upper radical closure ϱ^m, determined by the class $\mathfrak{M}$ is a dual special radical closure; the upper radical closure $\overline{\varrho^m}$, determined by the class $\overline{\mathfrak{M}}$ is a dual underidempotent radical closure; ϱ^m and $\overline{\varrho^m}$ supplement each other, i. e. $(\varrho^m)^*=\overline{\varrho^m}$ and $(\overline{\varrho^m})^*=\varrho^m$. This way we can receive every dual supernilpotent and every dual underidempotent radical closure.*

Proof. It follows from lemmas 6 and 7, that ϱ^m is a special radical closure and $\overline{\mathfrak{M}}$ is equal to the class of all subdirectly irreducible pairs (R, P) with a ϱ^m-radical heart. Therefore, $(\varrho^m)^*=\overline{\varrho^m}$ and $\overline{\varrho^m}$ is a dual underidempotent radical closure. Again from lemma 6 we have that $\mathfrak{M}$ is equal to the class of all subdirectly irreducible pairs with a $\overline{\varrho^m}$-radical heart, i. e. $(\overline{\varrho^m})^*=\varrho^m$ and ϱ^m is a dual special radical closure.

Let now ϱ is a dual supernilpotent radical closure. Then ϱ^* is an underidempotent radical closure and, therefore, $\varrho=\varrho^{**}$ is equal to the upper radical closure, determined by the class $\mathcal{C}(\varrho^*)$ of all subdirectly irreducible pairs (R, P) with a ϱ^*-radical heart. But it is clear that all pairs of $\mathcal{C}(\varrho^*)$ have hereditary idempotent hearts.

At the end let η be a dual underidempotent radical closure and let $\overline{\mathfrak{M}}$ be the class of all subdirectly irreducible pairs (R, P) with ϱ-semi-simple hearts. It is clear that the class $\mathfrak{M}$ of all the rest of pairs of $B(R)$ determines the radical closure η^*, which is a special one. Therefore, $\eta=\eta^{**}$ is equal to the upper radical closure, determined by $\overline{\mathfrak{M}}$.

The theorem is proved.

REFERENCES

1. В. А. Андрунакиевич. Радикалы ассоциативных колец I. *Мат. сб.*, **44**, 1958, 179—212.
2. В. А. Андрунакиевич. Первичные модули и радикал Бэра. *Сиб. мат. ж.*, **11**, 1961, 801—806.
3. В. А. Андрунакиевич, Ю. М. Рябухин. Радикальные замыкания правых идеалов ассоциативных колец и алгебр. *Мат. исследования*, вып. 38, Кольца и модули, 1976, 3—70.
4. L. Iv. Davidov. Supernilpotent and underidempotent radical closures of right ideals. *Serdica*, **2**, 1976, 315—324.

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