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ПЛИСКА

БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

POINCARÉ SERIES OF ALMOST COMPLETE INTERSECTIONS OF EMBEDDING DIMENSION THREE

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A change of rings theorem is proved for the reduction of the Poincaré series of a local Gorenstein ring modulo depth 3 perfect ideals, minimally generated by 4 elements. Together with an earlier result of the author, this theorem establishes the rationality of the Poincaré series of the title.

The conjectured rationality of the Poincaré series $P_A=\Sigma_{i=0}^{\infty}b_iz^i$ ($b_i=\dim_k \mathrm{Tor}_i^A(k, k)$) for the local ring (A, m, k) has been proved now for several types of homomorphic images of regular local rings. Yet there are only very few classes of rings, for which P_A is known to be rational, and which admit an intrinsic ring-theoretical characterization. In order to describe them, we recall that $b_1=\dim_k \mathfrak{m}/\mathfrak{m}^2$ is called the embedding dimension of A (also denoted ε_0), and $d(A)=b_2-\binom{b_1+1}{2}+\dim A=\varepsilon_1-\varepsilon_0+\dim A$ is called the complete intersection defect of A; if one uses Cohen's theorem to represent the $\mathfrak{m}$-adic completion $\hat{A}$ in the form $R/\mathfrak{b}$ with $(R, \mathfrak{n})$ regular local, then another expression for $d(A)$ is $\dim_k(\mathfrak{b}/\mathfrak{n}\mathfrak{b})-(\dim R-\dim A)$.

The rationality of P_A is known in the following cases:

(1) A is a local complete intersection, i. e. $d(A)=0$ (Tate; this includes the case of regular A, due to Eilenberg and Auslander-Buchsbaum);

(2) $\varepsilon_0\leq \operatorname{depth} A+2$ (Scheja);

(3) $\varepsilon_0=\operatorname{depth} A+3$ and A is Gorenstein (Wiebe).

In this note we add to the list the following:

Theorem 1. *Let* $(A, \mathfrak{m}, k)$ *be an almost complete intersection* (*i. e.* $d(A)=1$), *with* $\varepsilon_0=\operatorname{depth} A+3$. *Then* P_A *is a rational function.*

We note that by a theorem of Kunz there is no intersection between the class of rings described in the theorem, and the one in (3) above (cf. [4]); as a matter of fact, this also follows from our explicit formulas for P_A.

We deduce theorem 1 from two more general "change of rings" results. One of them has already been proved in [1], and the second one is:

Theorem 2. *Let* $(A, \mathfrak{m}, k)$ *be a Gorenstein local ring, and let* $\mathfrak{a}$ *be an ideal minimally generated by 4 elements, which is perfect of depth 3 and of type* $t=\dim_k \mathrm{Tor}_3^A(A/\mathfrak{a}, k)$.

If $\mathfrak{a}$ *is not contained in* $\mathfrak{m}^2$, *then* $t=2$ *and*

$$P_A/P_{A/\mathfrak{a}}=(1+z)(1-3z^2-2z^3).$$

PLISKA Studia mathematica bulgarica. Vol. 2, 1981, p. 167—172.

If $\mathfrak{a}\subset\mathfrak{m}^2$, *then* $\mathfrak{a}$ *is small* (cf. [1]), *and*

$$P_A/P_{A/\mathfrak{a}}=\begin{cases}1-4z^2-2z^3+3z^4+2z^5 & \text{if } t=2;\\ 1-4z^2-t\,z^3-(t-3)z^4-z^5-z^6 & \text{if } t \text{ is odd } \geqq 3;\\ 1-4z^2-t\,z^3-(t-3)z^4 & \text{if } t \text{ is even } \geqq 4.\end{cases}$$

In order to establish this theorem we first prove that the "generic" height 3 almost complete intersections of Buchsbaum and Eisenbud (cf. [2, proposition 6.3]), define generically perfect ideals in the sense of Eagon and Northcott [3]. This enables us to apply the techniques of [1], and determine the Poincaré series by making a reduction to explicit examples. An important step is the determination of the algebra structure of $\mathrm{Tor}^A(A/\mathfrak{a}, k)$.

In the most interesting case, when A is regular, theorem 2 has been independently and somewhat earlier obtained by E. S. Golod, who uses a different method (cf. [6]).

Proof of theorem 1. Our assumptions can be written in the form: $\varepsilon_0=\operatorname{depth} A+3$ and $\varepsilon_1=4-(\dim A-\operatorname{depth} A)$. The case, when A is not Cohen-Macaulay is given by [1, corollary 7.3], while the case when $\dim A=\operatorname{depth} A$ is contained in theorem 2.

The proof of theorem 2 occupies most of the paper.

First we note, that if $t=2$, then by [2, corollary 5.4] there exist a depth 2 and type 2 perfect ideal $\mathfrak{a}_1$ and an element $x\in\mathfrak{m}$, which is regular modulo $\mathfrak{a}_1$, such that $\mathfrak{a}=\mathfrak{a}_1+(x)$. It follows that in this case the theorem is a consequence of [1, theorem 7.1], and of the well known "change of rings modulo regular elements" theorems of Tate and Scheja (cf. [1, proposition 4.2]).

From now on we assume $t>2$ and set $s=[t/2]$ (entire part).

Consider the rings, where R is a (commutative) Noetherian ring:

$$S_t=\begin{cases}R[\{Y_{ij}\}_{\substack{1\leqq i\leqq 3,\\ 1\leqq j\leqq t}},\ \{X_{ij}\}_{1\leqq i<j\leqq t}] & \text{if } t \text{ is odd};\\ R[\{Y_{ij}\}_{\substack{i=2,3\\ 2\leqq j\leqq t+1}},\ \{X_{ij}\}_{1\leqq i<j\leqq t+1}] & \text{if } t \text{ is even.}\end{cases}$$

We denote by (Y) the matrix (Y_{ij}) and by (X) the $(2s+1)\times(2s+1)$ alternating matrix with $X_{ii}=0$ and $X_{ji}=-X_{ij} (i<j)$. $|Y|^{i,j,\dots}$ denotes the minor on the columns $i, j, \dots$, while $Pf_{i j,\dots}(X)$ denotes the Pfaffian (cf. [2]) of the alternating matrix obtained from X by deleting rows and columns with indices $i, j, \dots$ We now define ideals in S_t:

$$I_t=\begin{cases}\left(\left\{\sum_{j=1}^{t}(-1)^{j+1}Y_{ij}\,\mathrm{Pf}_j(X)\right\}_{i=1,2,3},\ \sum_{i<j<k}(-1)^{i+j+k+1}|Y|^{i,j,k}\,\mathrm{Pf}_{i,j,k}(X)\right) & \text{if } t \text{ is odd};\\ \left(\mathrm{Pf}_1(X), \left\{\sum_{j=2}^{t+1}(-1)^{j+1}Y_{ij}\,\mathrm{Pf}_j(X)\right\}_{i=2,3},\ \sum_{1<j<k}(-1)^{j+k}|Y|^{j,k}\,\mathrm{Pf}_{1,j,k}(X)\right) & \text{if } t \text{ is even.}\end{cases}$$

Lemma. (a). *I_t is a perfect ideal of depth 3.*

(b) *If $(A,\mathfrak{m})$ is a Gorenstein local ring and $\mathfrak{a}$ is a depth 3 and type t perfect ideal, minimally generated by 4 elements, then there exist x_{ij}, $y_{ij}\in\mathfrak{m}$, such that $\mathfrak{a}=\Phi(I_t)A$ where Φ is the ring homomorphism sending X_{ij} to x_{ij} and Y_{ij} to y_{ij}.*

Proof. (a) By the theory of generically perfect ideals, it is sufficient to prove the lemma under the further assumption that R is either the ring of integers or a field (cf. [3, propositions 4 and 5]). From [3, lemma 6] we see that the first three generators t_1, t_2 and t_3 of $I=I_t$ form an $S=S_t$-regular sequence, hence the Koszul complex E on them is exact; denote its 1-dimensional generators by $T_i : dT_i = t_i$ ($i=1, 2, 3$). Let F denote the generic Buchsbaum-Eisenbud complex, associated to the matrix (X) (cf. [2, section 3] or [1, section 8]); F_1 is the free S-module on $\{A_j\}_{1\le j\le 2s+1}$, with $dA_j=(-1)^{j+1}\mathrm{Pf}_j(X)$, $dF_1=J$. By [1, proposition 8.7 and corollary 8.6], F is exact and $H_0(F)=S/J$. Giving F the multiplication constructed in [2, section 4] and [1, 8.4], define a S-algebra map $f: E\to F$ by setting:

$$f(T_1)=\begin{cases}\Sigma Y_{1j}A_j & \text{if } t \text{ is odd};\\ A_1 & \text{if } t \text{ is even};\end{cases}$$

$$f(T_i)=\Sigma Y_{ij}A_j \qquad \text{for } i=2,\ 3.$$

Denoting $\mathrm{Hom}_S(-, S)$ by $(-)^*$, let G be the mapping cone of the map f^* factored out by the subcomplex $(F_0^* \xrightarrow{(1,\,d_1^*)} E_0^*\oplus F_1^*)$ and suitably renumbered. Localizing at any prime $\mathfrak{P}$ of S, we obtain a regular local ring $S_{\mathfrak{P}}$ and $H_i(G)_{\mathfrak{P}}\simeq H_i(G_{\mathfrak{P}})$. If $\mathfrak{P}$ does not contain J, then $H_i(F_{\mathfrak{P}})=0$ for all i and the mapping cone exact sequence shows that $H_i(G_{\mathfrak{P}})=0$ for $i>0$. If $\mathfrak{P}\supset J$, then the same conclusion holds by [2, proposition 5.1a]. Hence G is acyclic, and $pd_S(S/I)\leqq 3$. As noted at the beginning, $\mathrm{depth}(I, S)\geqq 3$, and the assertion follows.

(b) Let $\mathfrak{a}=(a_1, a_2, a_3, a_4)$. By [2, theorem 5.4. (b)], we can assume that a_1, a_2, a_3 form a regular sequence, and $\mathfrak{a}'=(a_1, a_2, a_3):(a_4)$ is a Gorenstein ideal with $2s+1$ generators. Now [2, theorem 2.1] asserts the existence of a $(2s+1)\times(2s+1)$ matrix (x) with $x_{ij}\in\mathfrak{m}$, such that $\mathfrak{a}'=(\{\mathrm{Pf}_j(x)\}_{1\le j\le 2s+1})$. By [2, proposition 5.1], $\mathfrak{a}=(a_1, a_2, a_3):\mathfrak{a}'$. Considering the resolution of $A/\mathfrak{a}$ constructed according to [2, proposition 5.1 a) from the map of complexes over $A/(a_1, a_2, a_3)\to A/\mathfrak{a}'$, one sees that $\dim_k \mathrm{Im}((a_1, a_2, a_3)\otimes k\to \mathfrak{a}'\otimes k)=0$ or 1 depending on whether t is odd or even. This shows how to choose the $y_{ij}\in\mathfrak{m}$, which finishes the proof of (b).

Corollary. *Under the assumptions of theorem 2, $\mathfrak{a}$ is small and*

$$P_A/P_{A/\mathfrak{a}}=P_{S_t}/P_{S_t/I_t}$$

with $R=k$ in the definition of the ring S_t.

Proof. Immediate from the lemma and [1, theorem 6.2].

Having reduced the problem to the generic situation, we now perform a second reduction, specializing to an explicity constructed ideal in the ring $B=k[X, Y, Z]$. For this purpose define a map of k-algebras $\varphi: S_t\to B$ by setting:

$$\varphi(X_{ij})=\begin{cases} X & \text{if } i \text{ is odd and } j=i+1;\\ Y & \text{if } i \text{ is even and } j=i+1;\\ Z & \text{if } j=2s-j+2;\\ 0 & \text{in all other cases.}\end{cases}$$

$$\varphi(Y_{11})=y=\begin{cases} Y & \text{if } t \text{ is odd;} \\ 1 & \text{if } t \text{ is even;}\end{cases}$$

$$\varphi(Y_{2,s+1})=Z;$$

$$\varphi(Y_{3,2s+1})=X;$$

$$\varphi(Y_{ij})=0 \quad \text{if} \quad (i, j)\neq(1, 1), (2, s+1), (3, 2s+1).$$

(Here we have adopted the convention that $Y_{11}=1$ if t is even.)

We set $\mathfrak{b}=\varphi(I_t)B$ and $L=G\otimes_{S_t}B$, where G is the complex constructed in the proof of the lemma. The complex L has the form:

$$0 \longrightarrow B^{2s+1} \xrightarrow{d_3} B^3\oplus B^{2s+1} \xrightarrow{d_2} B^3\oplus B \xrightarrow{d_1} B$$

and its differentials are given in the canonical bases by the matrices:

$$d_1=(yY^s,\ (-1)^sZg,\ X^{s+1},\ (-1)^sXyZg_1)$$

$$d_2=\begin{pmatrix} 0 & -X^{s+1} & (-1)^sZg & (-1)^{s+1}XZg & \cdots & 0 & \cdots & 0 \\ X^{s+1} & 0 & -yY^s & 0 & \cdots & (-1)^{s+1}Xyg_1 & \cdots & 0 \\ (-1)^{s+1}Zg & yY^s & 0 & 0 & \cdots & 0 & \cdots & (-1)^{s+1}yZg_1 \\ 0 & 0 & 0 & Y^s & \cdots & (-1)^sg & \cdots & X^s \end{pmatrix}$$

(only the columns with indices 1, 2, 3, 4, $s+4$ and $2s+4$ are displayed)

$$d_3=\begin{pmatrix} y & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & Z & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & X \\ & & & & \varphi(X_{ij}) & & & & \end{pmatrix}$$

(the nonzero elements in the first 3 rows are in positions (1, 1), (2, $s+1$) (3, $2s+1$)), where $g=\mathrm{Pf}_{s+1}(\varphi(X))$ and $g_1=\mathrm{Pf}_{1,2s+1}(\varphi(X))$.

In order to show that L is acyclic, it is sufficient, by the lemma and the results in [3], to establish that $\operatorname{depth}(\mathfrak{b}, B)\geq 3$. Since an easy induction gives that $g\equiv\pm Z^s \bmod (XY)$, we see that yY^s, X^{s+1}, Zg form a B-sequence. It is relevant to note at this place that the matrix $\varphi(X)$ we have used in the definition of $\mathfrak{b}$ has already been exploited by Buchsbaum and Eisenbud in order to construct height 3 Gorenstein ideals in B, with exactly $2s+1$ generators: [2, proposition 6.2].

Applying once again [1, theorem 6.2], we get:

$$P_{S_t}/P_{S_t/I_t}=P_B/P_{B/\mathfrak{b}}.$$

We now apply [1, corollary 3.3] and obtain a final reduction of our problem

$$P_B/P_{B/\mathfrak{b}}=(P_{\mathrm{Tor}^B(B/\mathfrak{b},\,k)})^{-1}.$$

In order to compute this last Poincaré series we construct partially the strictly skew-commutative DG algebra structure on L, whose existence is established by [2, proposition 1.3]. To this effect we denote by V_1, V_3, V_3, M_1, $M_2, \ldots, M_{2s+1}$ the elements of the canonical basis of L_2, by U_1, U_2, U_3, N those of L_1, and set:

$$(*)\qquad \begin{aligned} &U_1U_2=-V_3, \quad U_2U_3=-V_1, \quad U_3U_1=-V_2,\\ &U_1N=yM_1, \quad U_2N=ZM_{s+1}, \quad U_3N=XM_{2s+1}.\end{aligned}$$

The Leibnitz rule is immediate from the matrix description of d. Further we note that both d_1 and d_2 have entries in $(X, Y, Z)^2$, hence $d_3(L_1L_2)\subset(d_1L_1)L_2+L_1(d_2L_2)\subset(X, Y, Z)^2L_2$. Using the fact that B is graded and comparing degrees, we easily conclude that $L_1L_2\subset(X, Y, Z)L_3$, which implies:

$$(**)\qquad \Lambda_1\Lambda_2=0, \quad \Lambda=\mathrm{Tor}^B(B/\mathfrak{b}, k).$$

We denote by bars the images of elements of L in $L\otimes_B k$ and use the same letters for cycles and the homology classes they define. The exterior algebra functor is denoted by $E(-)$ and square brackets are the symbol for trivial extensions. The cases of odd and even t require now separate consideration.

If t is odd, formulas $(*)$ show that $\bar U_i\bar N=0$ and that the subalgebra $\Gamma\subset\Lambda$ generated by $\bar U_1$, $\bar U_2$, $\bar U_3$ is isomorphic to $E(k\bar U_1\oplus k\bar U_2\oplus k\bar U_3)/(\bar U_1\bar U_2\bar U_3)$. From this and $(**)$ one sees that $\Lambda\simeq\Gamma[\Lambda/\Gamma]$, where the quotient vector space is given the trivial Γ-action via the augmentation $\varepsilon:\Gamma\to\Gamma_0=k$. Using Gulliksen's computation of the Poincaré series of trivial extensions (cf, [1, proposition 9.1]), we get:

$$P_\Lambda^{-1}=P_\Gamma^{-1}(1-zP_\Gamma^{\Lambda/\Gamma})=P_\Gamma^{-1}(1-zP_\Gamma P_k^{\Lambda/\Gamma})=P_\Gamma^{-1}-z(z+tz^2+tz^3).$$

To finish with this case we substitute in the above formula the expression $P_\Gamma^{-1}=1-3z^2+3z^4-z^5-z^6$, obtained in [1, proposition 9.2].

In the case of even t, denote by Γ_1 (resp. Γ_2) the algebra $k[k\bar N\oplus k\bar U_2\oplus k\bar U_3]$ (resp. $k[k\bar U_1]$), and let $\Gamma=\Gamma_1\otimes\Gamma_2$. Formulas $(*)$ and $(**)$ in this case establish an isomorphism of k-algebras $\Lambda\simeq\Gamma[\Lambda/\Gamma]$. Reasoning as above and invoking once more Gulliksen's theorem in order to obtain P_{Γ_i} $(i=1, 2)$, we get:

$$P_\Lambda^{-1}=P_\Gamma^{-1}-z(tz^2+tz^3)=P_{\Gamma_1}^{-1}\cdot P_{\Gamma_2}^{-1}-z(tz^2+tz^3)$$

$$=(1-3z^2)(1-z^2)-tz^3-tz^4=1-4z^2-tz^3-(t-3)z^4.$$

The proof of theorem 2 is now complete.

Before proceeding to the last result of this paper, we pause to make a few remarks.

(1) If A is a Gorenstein local ring of depth at least 3, and X, Y, Z is an A-regular sequence, take the ring S_t defined over the integers and construct a ring homomorphism $\varphi:S_t\to A$ as in the proof of the theorem. It is immediately checked, that $\varphi(S_t)A$ is a depth 3 and type t perfect ideal, minimally generated by 4 elements. Hence all the possible values of $P_A/P_{A/\mathfrak{a}}$, given by theorem 2, are indeed achieved by suitable ideals of A.

(2) When $t=3$, the ideals of Theorem 2 are included among those considered in theorem 6.10 of [1]: take $n=3$ in the last theorem. It is easy to see that the Poincaré series in both cases are the same — as they should be!

(3) One of the by-products of the proof of theorem 2 is the fact that for $\Lambda=\mathrm{Tor}^A(A/\mathfrak{a}, k)$ and $\mathfrak{a}$ as in the theorem, one always has the equality

$\dim_k \Lambda_1^2=3$. Taking into account the classical result that $\Sigma_{i=0}^{3}(-1)^i \dim_k \Lambda_i=0$, this is seen to be equivalent to the relation $\dim_k(\Lambda_2/\Lambda_1^2)=t$. In this form the result can be generalized to embedding dimensions higher than three. To this end we recall that when A is regular and $\mathfrak{a}$ is in the square of the maximal ideal, Λ is identified with the homology of the Koszul complex K of $A/\mathfrak{a}$. Then $\varepsilon_2=\dim_k H_2(K)/H_1^2(K)$ is an important invariant of $A/\mathfrak{a}$, called the second deviation, and the dimension of the last non-vanishing homology group of K is called the type of $A/\mathfrak{a}$.

Proposition. *If A is a Cohen-Macaulay almost complete intersection, then its second deviation equals its type.*

Proof. Assume, as we can, $A=R/(a_1,\ldots, a_d, a_{d+1})$, with $(R, \mathfrak{n})$ regular local, $a_i\in\mathfrak{n}^2$ for $1\le i\le d+1$ and $a_1,\ldots, a_d$ forming a regular sequence. Denote by E (resp. F) a minimal R-free resolution of $R/(a_1,\ldots, a_d)$(resp. $R/(a_1,\ldots,a_d)$: (a_{d+1})). If $f:E\to F$ denotes an R-linear map of complexes over the projection $R/(a_1,\ldots, a_d)\to R/(a_1,\ldots, a_d):(a_{d+1})$, then by [2, proposition 5.1a] the mapping cone of f^*, truncated by a suitable subcomplex as in the proof of the lemma above, gives an R-free resolution of A. Hence we have:

$$\mathrm{Tor}_d^R(A, k)\simeq \mathrm{Ker}\,(f_1^*\otimes k).$$

On the other hand, the commutative diagram:

$$\begin{array}{ccc} E_1\otimes k & \xrightarrow{f_1\otimes k} & F_1\otimes k \\ \downarrow\wr & & \downarrow\wr \\ (a_1,\ldots, a_d)\otimes k & \longrightarrow & ((a_1,\ldots, a_d):(a_{d+1}))\otimes k \end{array}$$

shows that $\mathrm{Coker}\,(f_1\otimes k)\simeq[((a_1,\ldots, a_d):(a_{d+1}))/(a_1,\ldots, a_d)]\otimes k$. We conclude from the chain of equalities, the last of which is a result of Wiebe [5, corollary 3 to theorem 5]:

$$\text{type of } A=\dim_k H_d(K)=\dim_k \mathrm{Ker}\,(f_1^*\otimes k)=\dim_k \mathrm{Coker}\,(f_1\otimes k)$$

$$=\dim_k[((a_1,\ldots, a_d):(a_{d+1}))/(a_1,\ldots,a_d)]\otimes k=\varepsilon_2.$$

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