PLISKA
STUDIA MATHEMATICA BULGARICA

ПЛИСКА
БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

COMPLEX INVERSION FORMULAS FOR THE OBRECHKOFF TRANSFORM

IVAN H. DIMOVSKI, VIRGINIA S. KIRYAKOVA

In 1958 N. Obrechkoff had introduced a generalization of the integral transforms of Laplace and Meijer. One of the authors had considered a modification of the Obrechkoff transform, which can be used as a transform basis for an operational calculus for the general Bessel-type differential operator

$$B=t^{\alpha_0}\frac{d}{dt}t^{\alpha_1}\dots\frac{d}{dt}t^{\alpha_m}$$

with $\beta=m-(\alpha_0+\dots+\alpha_m)>0$.

In this paper some new results for this transform are given. Interesting relations of the modified Obrechkoff transform with the one-dimensional and the m-dimensional Laplace transforms are proved. Three different complex inversion formulas are found.

In [1] N. Obrechkoff had introduced the following generalization of the integral transforms of Laplace and Meijer:

$$F(x)=\int_0^\infty \Phi(xt)f(t)\,dt \tag{1}$$

with the kernel-function

$$\Phi(x)=\int_0^\infty\dots\int_0^\infty u_1^{\alpha_1}\dots u_p^{\alpha_p}\exp\left(-u_1-\dots-u_p-\frac{x}{u_1\dots u_p}\right)du_1\dots du_p$$

with arbitrary real $\alpha_1,\dots,\alpha_p$. In the same paper a real inversion formula of Post-Widder type is found. Now we propose a complex inversion formula in several variants for a modification of the Obrechkoff transform (1). This modification has been introduced in [2].

Definition 1. *Let $\gamma_1\leqq\gamma_2\leqq\dots\leqq\gamma_m$ be an arbitrary sequence of real numbers, $\beta>0$ is arbitrary too, and*

$$K(z)=\underbrace{\int_0^\infty\dots\int_0^\infty}_{m-1}\exp\left(-u_1-\dots-u_{m-1}-\frac{z}{u_1\dots u_{m-1}}\right)\prod_{k=1}^{m-1}u_k^{\gamma_k-\gamma_m-1}du_1\dots du_{m-1}.$$

The integral transform

$$\mathfrak{R}_\gamma\{f(t);\,z\}=\beta\int_0^\infty K[(zt)^\beta]\,t^{\beta(\gamma_m+1)-1}f(t)dt \tag{2}$$

defined for functions on $0 \leqq t < \infty$, *is said to be the modified Obrechkoff transform.*

For the sake of brevity we shall call (2) Obrechkoff transform. It is determined by the non-decreasing sequence $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m)$ and by the exponent $\beta > 0$.

Definition 2. *Ω is the space of locally integrable functions $f(t)$ in $(0, \infty)$ which are $O(t^p)$ with arbitrary $p > \alpha = -\beta(\gamma_1+1)$ for $t \to +0$, and are $O(\exp \lambda t^{\beta/m})$ for $t \to \infty$ with an arbitrary real λ.*

In [2] is shown that each $f(t) \in \Omega$ has well defined Obrechkoff transform $\mathfrak{R}_\gamma\{f(t); z\}$ as an analytic function in the truncated angle domain $D_f = \{z : \operatorname{Re} z > \lambda\} \cap \{z : |\arg z| < \pi m/2\beta\}$. In the same paper the convolution of the Obrechkoff transform (2) is found, and its relation to the m-dimensional Laplace transform is established. For the sake of completeness, we give the corresponding result with its proof.

Theorem 1. *If $f(t) \in \Omega$, then*

$$(3)\qquad \mathfrak{R}_\gamma\{f(t); (z_1 \ldots z_m)^{1/\beta}\} = \left(\prod_{k=1}^{m} z_k^{\gamma_k - \gamma_m}\right)\mathfrak{L}_m\left\{f[(t_1 \ldots t_m)^{1/\beta}] \prod_{k=1}^{m} t_k^{\gamma_k};\ z_1, \ldots, z_m\right\},$$

where by $\mathfrak{L}_m$ is denoted the m-dimensional Laplace transform

$$\mathfrak{L}_m\{\varphi(t_1, \ldots, t_m);\ z_1, \ldots, z_m\} = \int_0^\infty \ldots \int_0^\infty \exp(-z_1 t_1 - \cdots - z_m t_m)\varphi(t_1, \ldots, t_m)\, dt_1 \ldots dt_m.$$

Proof. Relation (3) can be written in the from

$$(4)\qquad \beta \int_0^\infty \Big[\int_0^\infty \ldots \int_0^\infty \exp[-u_1 - \cdots - u_m - (z_1 \ldots z_{m-1} t^\beta)/(u_1 \ldots u_{m-1})]$$

$$\times \prod_{k=1}^{m-1} u_k^{\gamma_k - \gamma_m - 1}\, du_1 \ldots du_{m-1}\Big]\, t^{\beta(\gamma_m+1)-1} f(t)\, dt$$

$$= \prod_{k=1}^{m} z_k^{\gamma_k - \gamma_m} \int_0^\infty \ldots \int_0^\infty \exp(-z_1 t_1 - \cdots - z_m t_m) f[(t_1 \ldots t_m)^{1/\beta}] \prod_{k=1}^{m} t_k^{\gamma_k}\, dt_1 \ldots dt_m.$$

According to the "edge of the wedge theorem" (see [3]), it is sufficient to prove (4) only for real $z_k > 0$, $k = 1, 2, \ldots, m$. In this case we make the substitution $u_k = z_k t_k$, $k = 1, 2, \ldots, m-1$, $t = (t_1 \ldots t_m)^{1/\beta}$ in the left-hand side of (4). Though the corresponding integral is improper, it is easy to see that the standard theorem for a variables change can freely be used. After routine transformations, the left-hand side of (4) can be transformed into the right-hand side.

Now, using the relation (3), we receive a complex inversion formula for the Obrechkoff transform.

Theorem 2. *If $f(t) \in \Omega$ is of the form $f(t) = t^p \tilde{f}(t)$ with $p > \alpha = -\beta(\gamma_1+1)$ and with m-times continuously differentiable function $\tilde{f}$ in $[0, \infty]$, with $\tilde{f}^{(m)}(t) = O(\exp \lambda t^{\beta/m})$ for $t \to \infty$, then*

$$(5)\qquad f(t) = \frac{1}{(2\pi i)^m}\, t^{-\frac{\beta}{m}(\gamma_1 + \cdots + \gamma_m)} \int_{\sigma - i\infty}^{\sigma + i\infty} \ldots \int_{\sigma - i\infty}^{\sigma + i\infty} \exp[t^{\beta/m}(z_1 + \cdots + z_m)]$$

$$\times \prod_{k=1}^{m} z_k^{\gamma_m-\gamma_k} F[(z_1 \dots z_m)^{1/\beta}]dz_1 \dots dz_m,$$

where $F(z)=\mathfrak{K}_\gamma\{f(t);\ z\}$, *and* σ *is a suitably chosen real constant.*

Proof. Under the hypothesis of the theorem, the function $f[(t_1 \dots t_m)^{1/\beta}] \prod_{k=1}^{m} t_k^{\gamma_k}$ satisfies the conditions for validity of the complex inversion formula of the m-dimensional Laplace transformation [4, p. 319], since

$$f[(t_1 \dots t_m)^{1/\beta}] \leq M \exp[\lambda(t_1 \dots t_m)^{1/m}] \leq M \exp\left(\frac{\lambda}{m} t_1 + \dots + \frac{\lambda}{m} t_m\right).$$

iThen

$$f[(t_1 \dots t_m)^{1/\beta}] = (1/2\pi i)^m \prod_{k=1}^{m} t_k^{-\gamma_k} \int_{\sigma-i\infty}^{\sigma+i\infty} \dots \int_{\sigma-i\infty}^{\sigma+i\infty} \exp(z_1 t_1 + \dots + z_m t_m)$$

$$\times \prod_{k=1}^{m} z_k^{\gamma_m-\gamma_k} F[(z_1 \dots z_m)^{1/\beta}] dz_1 \dots dz_m,$$

provided $\sigma > \lambda/m$. We put $t_1 = t_2 = \dots = t_m = t^{\beta/m}$ and get the inversion formula (5).

Corollary. If $\gamma_1 = -\nu/2$, $\gamma_2 = \nu/2$ with $\nu \geq 0$ (the case $m=2$) and $\beta=2$, then the Obrechkoff transform (4) coincides up to a constant with the Meijer transform $\mathfrak{K}_\nu\{f(t);\ z\} = \int_0^\infty K_\nu(zt)(zt)^{1/2} f(t)dt$.

The inversion formula (5) of the Obrechkoff transform is identical with the well-known complex inversion formula [5, p. 81] of the Meijer transform $f(t) = (\pi i)^{-1} \int_{\sigma-i\infty}^{\sigma+i\infty} I_\nu(zt)(zt)^{1/2} \mathfrak{K}_\nu\{f(t);\ z\}\, dz$.

It is especially interesting that there exists a definite relation between Obrechkoff transform (2) and the usual Laplace transform $\mathfrak{L}\{f(t);\ z\} = \int_0^\infty \exp(-zt) f(t)dt$.

Definition 3. *If* $\lambda_k = \gamma_m - \gamma_k + k/m$, $k=1, 2, \dots, (m-1)$, *and* $f(t) \in \Omega$ *then*

$$\varphi f(t) = \left[t^{m(\gamma_m+1)-1} \Big/ \prod_{k=1}^{m-1} \Gamma(\lambda_k)\right] \tag{6}$$

$$\times \int_0^1 \dots \int_0^1 \prod_{k=1}^{m-1} [(1-\tau_k)^{\lambda_k-1} \tau_k^{\gamma_k}] f[t^{m/\beta}(\tau_1 \dots \tau_{m-1})^{1/\beta}] d\tau_1 \dots d\tau_{m-1}.$$

Theorem 3. *If* $f(t) \in \Omega$, *then identically*

$$\mathfrak{K}_\gamma\{f(t);\ (z/m)^{m/\beta}\} = (2\pi)^{(m-1)/2} m^{1/2} \mathfrak{L}\{\varphi f(t);\ z\}. \tag{7}$$

Proof. We should prove the identity

$$\int_0^\infty \exp(-zt) \varphi f(t) dt \tag{8}$$

$$= (2\pi)^{(1-m)/2} m^{-1/2} \beta \int_0^\infty t^{\beta(\gamma_m+1)-1} K[t^\beta (z/m)^m] f(t) dt.$$

To this end, we shall transform the left-hand side of (8) to the right-hand side. Substituting $\varphi f(t)$ with (6), we get

$$\mathfrak{L}\{\varphi f(t);\ z)=\int_0^\infty \exp(-zt)\ (\prod_{k=1}^{m-1} \Gamma^{-1}(\lambda_k))t^{m(\gamma_m+1)-1}dt$$

$$\times\int_0^1 \dots \int_0^1 \prod_{k=1}^{m-1} [(1-\tau_k)^{\lambda_k-1}\tau_k^{\gamma_k}] f[t^{m/\beta}(\tau_1 \dots \tau_{m-1})^{1/\beta}]d\tau_1 \dots d\tau_{m-1}.$$

After the substitution $u_1=t^m\tau_1,\ u_2=t^m\tau_1\tau_2, \dots, u_{m-1}=t^m\tau_1 \dots \tau_{m-1}$, we receive

$$\mathfrak{L}\{\varphi f(t);\ z\}=(1/\prod_{k=1}^{m-1}\Gamma(\lambda_k))\int_0^\infty \exp(-zt)\ t^{m(\gamma_m-\gamma_1-\lambda_1+1)-1}dt$$

$$\times\int_0^{t^m}(t^m-u_1)^{\lambda_1-1}u_1^{\gamma_1-(\gamma_2+\lambda_2)}du_1 \dots \int_0^{u_{m-3}}(u_{m-3}-u_{m-2})^{\lambda_{m-2}-1}u_{m-2}^{\gamma_{m-2}-(\gamma_{m-1}+\lambda_{m-1})}du_{m-2}$$

$$\times\int_0^{u_{m-2}}(u_{m-2}-u_{m-1})^{\lambda_{m-1}-1}u_{m-1}^{\gamma_{m-1}}f[(u_{m-1}^{1/\beta})]du_{m-1}.$$

If we interchange the order of integrations, we get

$$\mathfrak{L}\{\varphi f(t);\ z\}=\int_0^\infty f[u_{m-1}^{1/\beta}]u_{m-1}^{\gamma_{m-1}}du_{m-1}$$

$$\times\Gamma^{-1}(\lambda_{m-1})\int_{u_{m-1}}^\infty (u_{m-2}-u_{m-1})^{\lambda_{m-1}-1}u_{m-2}^{\gamma_{m-2}-(\gamma_{m-1}+\lambda_{m-1})}du_{m-2}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\times\Gamma^{-1}(\lambda_k)\int_{u_k}^\infty (u_{k-1}-u_k)^{\lambda_k-1}u_{k-1}^{\gamma_{k-1}-(\gamma_k+\gamma_k)}du_{k-1}\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\times\Gamma^{-1}(\lambda_2)\int_{u_2}^\infty (u_1-u_2)^{\lambda_1-1}u_1^{\gamma_1-(\gamma_2+\lambda_2)}du_1$$

$$\times\Gamma^{-1}(\lambda_1)\int_{u_1^{1/m}}^\infty (t^m-u_1)^{\lambda_1-1}t^{m[\gamma_m-(\gamma_1+\lambda_1)-\frac{1}{m}+1]}\exp(-zt)dt.$$

The inner integrals here can be calculated in succession using the formula

$$\Gamma^{-1}(\lambda)\int_u^\infty (x-u)^{\lambda-1}G_{p,q}^{m,n}\left[ax\left|\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right.\right]dx$$

$$=u^\lambda G_{p+1,q+1}^{m+1,n}\left[au\left|\begin{matrix}a_1,\dots,a_p,\ 0\\ -\lambda,\ b_1,\dots,b_q\end{matrix}\right.\right],$$

valid for $p+q<2(m+n)$ and $|\arg(au)|<(m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$ (see [7, p. 212, (79)]). For the first of these integrals we get after the substitution $t^m=x$:

$$m^{-1}\Gamma^{-1}(\lambda_1)\int_{u_1}^\infty (x-u_1)^{\lambda_1-1}x^{\gamma_m-(\gamma_1+\lambda_1)}\exp(-zx^{1/m})dx$$

$$=(2\pi)^{(1-m)/2}m^{-1/2}\Gamma^{-1}(\lambda_1)\int_{u_1}^\infty (x-u_1)^{\lambda_1-1}x^{\gamma_m-(\gamma_1+\lambda_1)}G_{0,m}^{m,0}[(\frac{z}{m})^m x$$

$$\left.(j/m)_{j=0}^{m-1}\right]dx=(2\pi)^{(1-m)/2}m^{-1/2}(z/m)^{m[-\gamma_m+(\gamma_1+\lambda_1)]}$$

$$\times\Gamma^{-1}(\lambda_1)\int_{u_1}^{\infty}(x-u_1)^{\lambda_1-1}G_{0,m}^{m,0}\left[\left(\frac{z}{m}\right)^m x\ \Big|\ \left(\frac{j}{m}+\gamma_m-(\gamma_1+\lambda_1)\right)_{j=0}^{m-1}\right]dx=$$

$$(2\pi)^{(1-m)/2}m^{-1/2}(z/m)^{m[-\gamma_m+(\gamma_1+\lambda_1)]}u_1^{\lambda_1}G_{1,m+1}^{m+1,0}\left[(z/m)^m u_1\ \Big|\ {}^{0}-\lambda_1;[j/m+\gamma_m-(\gamma_1+\lambda_1)]_{j=0}^{m-1}\right].$$

Finally, after $(m-1)$ similar steps, we get

$$\mathfrak{L}\{\varphi f(t);\ z\}=(2\pi)^{(1-m)/2}m^{-1/2}(z/m)^{m[-\gamma_m+(\gamma_{m-1}+\lambda_{m-1})]}$$

$$\times\int_0^{\infty}f(u_{m-1}^{1/\beta})u_{m-1}^{\gamma_{m-1}+\lambda_{m-1}}G_{m-1,2m-1}^{2m-1,0}\left[\left(\frac{z}{m}\right)^m u_{m-1}\right.$$

$$\left|\begin{matrix}[\gamma_j+\lambda_j-(\gamma_{m-1}+\lambda_{m-1})]_{j=1}^{m-1}\\ [\gamma_j-(\gamma_{m-1}+\lambda_{m-1})]_{j=m-1}^{1};\ [\frac{j}{m}+\gamma_m-(\gamma_{m-1}+\lambda_{m-1})]_{j=0}^{m-1}\end{matrix}\right]du_{m-1}.$$

After the substitution $u_{m-1}=t^{\beta}$, we get $\mathfrak{L}(\varphi f(t);\ z\}=\int_0^{\infty}f(t)\Lambda(z,t)dt$ with

(9) $$\Lambda(z,t)=(2\pi)^{(1-m)/2}m^{-1/2}(z/m)^{-m(\gamma_m+1-1/\beta)}$$

$$\times G_{m-1,2m-1}^{2m-1,0}\left[\left(\frac{z}{m}\right)^m t^{\beta}\ \middle|\ \begin{matrix}[j/m+\gamma_m+1-1/\beta]_{j=1}^{m-1}\\ [\gamma_j+1-1/\beta]_{j=m-1}^{1},\ [j/m+\gamma_m+1-1/\beta]_{j=1}^{m-1}\end{matrix}\right]$$

$$=(2\pi)^{(1-m)/2}m^{-1/2}(z/m)^{-m(\gamma_m+1-1/\beta)}G_{0,m}^{m,0}\left[\left(\frac{z}{m}\right)^m t^{\beta}\ \Big|\ [\gamma_j+1-1/\beta]_{j=1}^{m}\right].$$

It remains to show that this expression for $\Lambda(z,t)$ is identical with the kernel $t^{\beta(\gamma_m+1)-1}K[(zt)^{\beta}]$ of the Obrechkoff transform, up to the numerical mul tiplier $(2\pi)^{(1-m)/2}m^{-1/2}$. To this end we shall use the Melline transform $\mathfrak{M}\{f(t);\ p\}=\int_0^{\infty}t^{p-1}f(t)dt$ and its complex inversion formula

(10) $$\frac{f(t-0)+f(t+0)}{2}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}t^{-p}\,\mathfrak{M}\{f;\ p\}dp.$$

We have

$$\mathfrak{M}\{t^{\beta(\gamma_m+1)-1}K[(zt)^{\beta}];\ p\}=(1/\beta)z^{-\beta(\gamma_m+1+(p-1)/\beta)m}\prod_{k=1}^{m}\Gamma(\gamma_k+1+(p-1)/\beta).$$

By inversion formula (10) we get

(11) $$t^{\beta(\gamma_m+1)-1}K[(zt)^{\beta}]$$

$$=\frac{1}{2\pi i\beta}z^{-\beta(\gamma_m+1-1/\beta)}\int_{c-i\infty}^{c+i\infty}(zt)^{-p}\prod_{k=1}^{m}\Gamma(\gamma_k+1+(p-1)/\beta)dp$$

$$=z^{\beta(-\gamma_m-1+1/\beta)}G_{m,0}^{0,m}\left[(zt)^{-\beta}\,\Big|\,(-\gamma_k+1/\beta)_{k=1}^{m}\right]=z^{-\beta(\gamma_m+1-1/\beta)}G_{0,m}^{m,0}\left[(zt)^{\beta}\,\Big|\,(1+\gamma_k-1/\beta)_{k=1}^{m}\right].$$

If multiply this expression by $(2\pi)^{(1-m)/2}m^{-1/2}$ and substitute z by $(z/m)^{m/\beta}$, we get (9) (see [6, p. 207, formula (1)]).

Thus the theorem is proved.

Corollary. *If $m=\beta=2$, $\gamma_1=-\nu/2$, $\gamma_2=\nu/2$ with $\nu\geqq 0$ then the identity* (7) *gives the relation between Meijer and Laplace transforms* ([7, p. 122]):

$$\mathfrak{K}_\nu\{f(t);\ z\}=\frac{\sqrt{\pi}\ 2^{-\nu}\ z^{\nu+1/2}}{\Gamma(\nu+1/2)}\ \mathfrak{L}\{\int_0^t (t^2-\tau^2)^{\nu-1/2}\tau^{-\nu+1/2}f(\tau)d\tau;\ z\}.$$

Using the relation (7) between Obrechkoff and Laplace transform, we give one more complex inversion formula for the Obrechkoff transform. By means of the well-known complex inversion formula $f(t)=(2\pi i)^{-1}\int_{\sigma-i\infty}^{\sigma+i\infty}\exp(zt)\,\mathfrak{L}\{f(t);\ z\}dz$ of Laplace transform, (7) leads to

Theorem 4. *If $f(t)\in\Omega$ and is continuous on* $(0,\ \infty)$ *then*

$$f(t)=(2\pi)^{-(m+1)/2}\,m^{-1/2}\,(1/i)\varphi^{-1}\{\int_{\sigma-i\infty}^{\sigma+i\infty}\exp(zt)\,\mathfrak{K}_\gamma\{(\frac{z}{m})^{m/\beta};\ z\}\,dz\}.$$

Proof. As it is shown in [9], transformtation φ, defined by (6) is invertible. There the following inversion formula for φ is found: If $\varphi f(t)=g(t)$, then

$$f[(t_1\ldots t_{m-1})^{1/\beta}]=\prod_{k=1}^{m-1}t_k^{-\gamma_k}\ \frac{\partial^{\lambda_1+\ldots+\lambda_{m-1}}}{\partial t_1^{\lambda_1}\ldots\partial t_{m-1}^{\lambda_{m-1}}}\left\{g[(t_1\ldots t_{m-1})^{1/m}]\prod_{k=1}^{m-1}t_k^{\frac{k+1}{m}-1}\right\},$$

where $\partial^{\lambda_1}/\partial t_1^{\lambda_1}$, $\partial^{\lambda_2}/\partial t_2^{\lambda_2},\ldots,\partial^{\lambda_{m-1}}/\partial t_{m-1}^{\lambda_{m-1}}$ are the operators for fractional partial differentiation of orders $\lambda_1,\ \lambda_2,\ldots,\lambda_{m-1}$ with respect to the variables t_1, $t_2,\ldots,t_{m-1}$ correspondingly. If $\lambda_k>0$ is an integer, then $\partial^{\lambda_k}/\partial t_k^{\lambda_k}$ denotes simply the operator of partial differentiation with respect to t_k of order λ_k, but if $\lambda_k>0$ is not an integer, then

$$\frac{\partial^{\lambda_k}f(t_1,\ldots,t_m)}{\partial t_k^{\lambda_k}}\overset{\text{def}}{=}\frac{\partial^{[\lambda_k]+1}}{\partial t_k^{[\lambda_k]+1}}\int_0^{t_k}\frac{(t_k-\tau_k)^{-\{\lambda_k\}}}{\Gamma(1-\{\lambda_k\})}f(t_1,\ldots,\tau_k,\ldots,t_{m-1})d\tau_k.$$

Then, the proof of the theorem follows immediately from (7).

At last, we give one more complex inversion formula for the Obrechkoff transform.

Theorem 5. *If $f(t)\in\Omega$ and if $f(t)$ has bounded variation in a neighbourhood of a point t_0, then*

$$[f(t_0-0)+f(t_0+0)]/2$$
$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}t_0^{-p}/\prod_{k=1}^{m}\Gamma(\gamma_k+1-p/\beta)\,dp\int_0^\infty\mathfrak{K}_\gamma\{f;\ z\}z^{\beta(\gamma_m+1)-p-1}dz,$$

provided the integrals $\int_0^\infty t^{c-1}f(t)dt$ and $\int_0^\infty z^{\beta(\gamma_m+1)-p-1}\mathfrak{K}_\gamma\{f;z\}dz$ are absolutely convergent for $p=c+iT$, $-\infty<T<\infty$ with a suitably chosen constant $c<-a$ $=\beta(\gamma_1+1)$.

Proof. We integrate the expression $z^{\beta(\gamma_m+1)-p-1}\,\mathfrak{K}_\gamma\{f;\ z\}$ along the real axis from 0 to ∞. According to a theorem of Vallée-Poussin (see Bromwich [10, p. 504]) from the hypothesis it follows that it is allowed to change the order of integration:

$$\beta\int_0^\infty z^{\beta(\gamma_m+1)-p-1}\,dz\int_0^\infty K[(zt)^\beta]\,t^{\beta(\gamma_m+1)-1}f(t)\,dt$$

$$=\beta\int_0^\infty t^{\beta(\gamma_m+1)-1}f(t)dt\int_0^\infty z^{\beta(\gamma_m+1)-p-1}K[(zt)^\beta]dz.$$

Now, using formula (11), derived in the proof of theorem 3, we get for the inner integral:

$$t^{\beta(\gamma_m+1)-1}\int_0^\infty z^{\beta(\gamma_m+1)-p-1}K[(zt)^\beta]dz \tag{12}$$

$$=\int_0^\infty z^{-p}G^{m,0}_{0,m}[(zt)^\beta\,|\,(1+\gamma_k-1/\beta)^m_{k=1}]dz=(t^{p-1}/\beta)\prod_{k=1}^m\Gamma(\gamma_k+1-p/\beta)$$

(see [7, p. 418, formula (3)]). The formula used is valid for $\arg z|<m\pi/2\beta$, which is the case for real $z\in(0,\infty)$.

Now, using (11), it is not difficult to see that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}[t_0^{-p}/\prod_{k=1}^m\Gamma(\gamma_k+1-p/\beta)]dp\int_0^\infty\mathfrak{R}_\gamma\{f;z\}z^{\beta(\gamma_m+1)-p-1}\,dz$$

$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}t_0^{-p}\,dp\int_0^\infty t^{p-1}f(t)\,dt=\frac{f(t_0-0)+f(t_0+0)}{2},$$

according with the Riemann-Mellin inversion formula (10).

Complex inversion formulas for a special case of the Obrechkoff transform are found by E. Krätzel [11].

REFERENCES

1. Н. Обрешков. Върху някои интегрални представяния на реални функции по реалната полуос. *Известия Мат. инст. БАН,* 3, 1958, № 1, 3—28.
2. I. H. Dimovski. A transform approach to operational calculus for the general Bessel-type differential operator. *C. R. Acad. bulg. Sci.,* 27, 1974, 155—158.
3. F. E. Browder. On the "edge of the wedge" theorem. *Canad. Math. J.,* 15, 1963, 125—130.
4. В. А. Диткин, А. П. Прудников. Операционное исчисление. Москва, 1975.
5. В. А. Диткин, А. П. Прудников. Интегральные преобразования и операционное исчисление. Москва, 1974.
6. Bateman Manuscript Project. A. Erdélyi (ed.), Higher transcendental functions, vol. 1. New York, 1953.
7. Bateman Manuscript Project. A. Erdélyi (ed.), Tables of integral transforms, vol. 2. New York, 1954.
8. Y. Luke. The special functions and their approximations, vol. 1. New York, 1969.
9. I. H. Dimovski. Foundations of operational calculi for the Bessel-type differential operators. *Serdica,* 1, 1975, 51—63.
10. T. J. I'a Bromwich. An introduction to the theory of infinite series. London, 1926.
11. E. Krätzel. Eine Verallgemeinerung der Laplace- und Meijer-Transformation. *Wiss. Z. Friedrich-Schiller-Univ. Jena. Math.-Naturwiss. Reihe,* 14, 1965, 369—381.

Centre for Mathematics and Mechanics
1090 Sofia P. O. Box 373

Received 28. 2. 1979