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EXTREMAL PROBLEMS FOR UNIVALENT FUNCTIONS

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Denote by S the class of functions

$$f(z)=c_0+c_1z+c_2z^2+\cdots, \tag{S}$$

regular and univalent in the unit disc $D:|z|<1$.

Let $L(z_1, z_2)$ be the curve $z=z(s)$, $0\leqq s\leqq \bar{s}$, $z_1=z(0)$, $z_2=z(\bar{s})$, $|z_1|<|z_2|<1$, for which $z'(s)$, and, also $r'(s)=|z(s)|'$ exist and are continuous except for a finite number of values of s. The parameter s denotes the length of the arc.

By $\mathscr{L}(z_1, z_2, f)$ denote the image of $L(z_1, z_2)$ by means of $f(z)\in S$. $\bar{L}(z_1, z_2)$ and $\bar{\mathscr{L}}(z_1, z_2, f)$, denote, the lengths of $L(z_1, z_2)$ and $\mathscr{L}(z_1, z_2, f)$, respectively.

Theorem I. *If $f(z)\in S$ and $|z_1|<|z_2|<1$, then*

$$\frac{1-|z_1|\,|z_2|}{(1+|z_1|)^2(1+|z_2|)^2}\leqq\frac{\bar{\mathscr{L}}(z_1, z_2, f)}{\bar{L}(z_1, z_2)}\leqq\frac{1-|z_1|\,|z_2|}{(1-|z_1|)^2(1-|z_2|)^2}, \tag{1}$$

where the above estimate holds true if $r'(s)\geqq 0$.

For $|z|\leqq r<1$, one obtains

Theorem I*. *If $f(z)\in S$ and $|z_1|<|z_2|\leqq r<1$, then*

$$\frac{1-r}{(1+r)^3}\leqq\frac{\bar{\mathscr{L}}(z_1, z_2, f)}{\bar{L}(z_1, z_2)}\leqq\frac{1+r}{(1-r)^3}, \tag{1*}$$

where the above estimate holds true if $r'(s)\geqq 0$.

As a corollary we get:

Theorem $\bar{\text{I}}$. *If $f(z)\in S$ and $|z_1|<|z_2|\leqq r<1$, then*

$$\frac{1-|z_1|\,|z_2|}{(1+|z_1|)^2(1+|z_2|)^2}\leqq\left|\frac{f(z_1)-f(z_2)}{z_1-z_2}\right|\leqq\frac{1-|z_1|\,|z_2|}{(1-|z_1|)^2(1-|z_2|)^2}, \tag{2}$$

where the left inequality holds if the segment joining the points $f(z_1)$ and $f(z_2)$ lies entirely in the image $f(D)$ of the unit disc by means of $f(z)$, while the right inequality holds if, on the segment joining z_1 with z_2, $|z|$ only increases or only decreases.

Under the same conditions the following inequalities hold:

$$\frac{1-r}{(1+r)^3}\leqq\left|\frac{f(z_1)-f(z_2)}{z_1-z_2}\right|\leqq\frac{1+r}{(1-r)^3}. \tag{2*}$$

These theorems comprise (generalize) the classical Koebe theorem [1]

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Theorem K. *If* $f(z) \in S$ *and* $|z| \leqq r < 1$, *then*

$$\frac{1-r}{(1+r)^3} \leqq |f'(z)| \leqq \frac{1+r}{(1-r)^3}. \tag{3*}$$

The bounds in (3*) *are reached by the function* $f(z) = z/(1-z)^2$.

(2), when $z_1 = 0$, $z_2 = z$, yields the Bieberbach theorem:

Theorem B. *If* $f(z) \in S$ *and* $|z| \leqq r < 1$, *then*

$$\frac{|z|}{(1+|z|)^2} \leqq |f(z)| \leqq \frac{|z|}{(1-|z|)^2} \tag{4}$$

and

$$\frac{1}{(1+r)^2} \leqq \left|\frac{f(z)}{z}\right| \leqq \frac{1}{(1-r)^2}. \tag{4*}$$

The bounds in (4) *and* (4*) *are reached by the function* $f(z) = z/(1-z)^2$.

In the proof of theorems I and I* the Koebe theorem and the integral method of Bieberbach [2] are used.

Proof of theorems I and I*. Let $L(z_1, z_2)$ be a curve $z = z(s)$ $0 \leqq s \leqq \bar{s}$, in the unit disc D, $z(0) = z_1$, $z(\bar{s}) = z_2$, joining the points z_1 and z_2, $|z_1| < |z_2|$. It can be assumed that $z'(s)$ and $|z(s)|'$, $0 \leqq s \leqq \bar{s}$ exist and are continuous except for a finite number of values of s. Here s is the length of the arc along the curve.

Let $\mathscr{L}(z_1, z_2, f)$ be the image of $L(z_1, z_2)$ by means of the function $f(z) \in S$.

The lengths of these curves are denoted by $\bar{L}(z_1, z_2)$ and $\bar{\mathscr{L}}(z_1, z_2, f)$, respectively.

Since $L(z_1, z_2)$ is rectifiable, there exists a positive integer $p \geqq 1$, such that $(p-1)(|z_2| - |z_1|) < \bar{L}(z_1, z_2) \leqq p(|z_2| - |z_1|)$.

Then

$$\bar{\mathscr{L}}(z_1, z_2, f) = \int_0^{\bar{s}} |f'(z)|\,|z'(s)|\,ds = \int_0^{\bar{s}} |f'(z)|\,|dz|. \tag{5}$$

A. Let $\varphi(z) = \varphi(|z|) = \dfrac{1+|z|}{(1-|z|)^3}$. In view of the Koebe theorem

$$\int_0^{\bar{s}} |f'(z)|\,|dz| \leqq \int_0^{\bar{s}} \varphi(|z|)\,|dz|.$$

Let $\zeta_1 = z_1, \zeta_2, \ldots, \zeta_{pn} = z_2$ be $pn+1$ points on $L(z_1, z_2)$ dividing its arc into pn equal parts. Then

$$\int_0^{\bar{s}} \varphi(|z|)|dz| = \lim_{n\to\infty} \sum_{k=1}^{pn} \varphi(|\zeta_k|)\,|\zeta_{k+1} - \zeta_k| = \lim_{n\to\infty} \sum_{k=1}^{pn} \varphi(|\zeta_k|)\,\frac{\bar{L}(z_1, z_2)}{pn}.$$

Assume that $r'(s) = |z(s)|' \geqq 0$. Then the numbers $|\zeta_1| = |z_1| \leqq |\zeta_2| \leqq \cdots \leqq |\zeta_{pn}| \leqq |\zeta_{pn+1}| = |z_2|$ and they are in the unterval $[|z_1|, |z_2|]$ of length $|z_2| - |z_1| > 0$.

Divide the interval $[|z_1|, |z_2|]$ into pn parts, all equal to $\dfrac{|z_2| - |z_1|}{pn}$.

Since $|\zeta_{k+1} - \zeta_k| = \dfrac{\bar{L}(z_1, z_2)}{pn}$ is p times greater than $\dfrac{|z_2| - |z_1|}{pn} = \alpha_n^p$ at most then in every interval consisting of p consecutive intervals of length α_n^p there is at least one of the numbers $|\zeta_k|$. Now divide the interval $[|z_1|, |z_2|]$ into n equal intervals. Each of them represents a group of p intervals of length $\frac{|z_2|-|z_1|}{pn}$ each. All these n groups will be given consecutive numbers. Among the numbers $|\zeta_\nu|$, lying in the ν-th of the groups consisting of p consecutive intervals of length $\frac{|z_2|-|z_1|}{pn}$ each, by $|\zeta_\nu^*|$ denote the number for which $\varphi(|\zeta_k|)$ is the greatest.

Then,

$$\lim_{n\to\infty}\sum_{k=1}^{pn}\varphi(|\zeta_k|)\frac{\overline{L}(z_1, z_2)}{pn}=\frac{\overline{L}(z_1, z_2)}{|z_2|-|z_1|}\lim_{n\to\infty}\sum_{k=1}^{pn}\varphi(|\zeta_k|)\frac{|z_2|-|z_1|}{pn}$$

$$\leq\frac{\overline{L}(z_1, z_2)}{|z_2|-|z_1|}\lim_{n\to\infty}\sum_{\nu=1}^{n}\varphi(|\zeta_k^*|)\frac{|z_2|-|z_1|}{n}=\frac{\overline{L}(z_1, z_2)}{|z_2|-|z_1|}\int_0^{\overline{s}}\varphi(|z|)d|z|,$$

i. e.

$$\overline{\mathfrak{L}}(z_1, z_2, f)\leq\frac{\overline{L}(z_1, z_2)}{|z_2|-|z_1|}\int_{|z_1|}^{|z_2|}\frac{(1+|z|)d|z|}{(1-|z|)^3}=\overline{L}(z_1, z_2)\frac{1-|z_1|\,|z_2|}{(1-|z_1|)^2(1-|z_2|)^2}.$$

Therefore,

$$\frac{\overline{\mathfrak{L}}(z_1, z_2, f)}{\overline{L}(z_1, z_2)}\leq\frac{1-|z_1|\,|z_2|}{(1-|z_1|)^2(1-|z_2|)^2}\leq\frac{1+r}{(1-r)^3},$$

under the condition that $r'(s)\geq 0$ in the interval from $|z_1|$ to $|z_2|$ and $|z_2|\leq r<1$.

B. Let $\psi(z)=\psi(|z|)=\frac{1-|z|}{(1+|z|)^3}$. In view of the Koebe theorem

$$\int_0^{\overline{s}}|f'(z)|\,|dz|\geq\int_0^{\overline{s}}\psi(|z|)\,|dz|.$$

First assume that $r'(s)\geq 0$ in the interval $|z_1|\leq s\leq|z|$. Repeating the considerations and notations from item A, among the numbers $|\zeta_k|$, lying in the ν-th of the groups consisting of p consecutive intervals of length $\frac{|z_2|-|z_1|}{pn}$ each, by $|\zeta^*|$ denote the number for which $\psi(|z_k|)$ is the smallest. Then, by analogy with the mentioned above

$$\overline{\mathfrak{L}}(z_1, z_2, f)\geq\frac{\overline{L}(z_1, z_2)}{|z_2|-|z_1|}\int_{|z_1|}^{|z_2|}\frac{(1-|z|)d|z|}{(1+|z|)^3}=\overline{L}(z_1, z_2)\frac{1-|z_1|\,|z_2|}{(1+|z_1|)^2(1+|z_2|)^2},$$

i. e.

$$\frac{\overline{\mathfrak{L}}(z_1, z_2, f)}{\overline{L}(z_1, z_2)}\geq\frac{1-|z_1|\,|z_2|}{(1+|z_1|)^2\,(1+|z_2|)^2}\geq\frac{1-r}{(1+r)^3},$$

where $|z_2|\leq r<1$.

In this case, the condition $r'(s)\geq 0$ can be easily discharged. Namely, $L=L(z_1, z_2)$ can be represented in the form $L(z_1, z_2)=L_1(z_1, z_2)+L_2(z)$, where $L_1=L_1(z_1, z_2)$ is a curve $z=z(s)$ in D joining z_1 with z_2 and for which $r'(s)\geq 0$, while $L_2=L_2(z)$ is a sum of linear sets.

Then,

$$(L)\int_{|z_1|}^{|z_2|}\psi(|z|)d|z|=(L_1)\int_{|z_1|}^{|z_2|}\psi(|z|)d|z|+(L_2)\int\psi(|z|)d|z|\geq(L_1)\int_{|z_1|}^{|z_2|}\psi(|z|)d|z|.$$

Thus, theorems I and I* are proved.

Note that in view of (5) the inequalities (1) might be written in the form

$$(1') \qquad \frac{1-|z_1|\,|z_2|}{(1+|z_1|)^2(1+|z_2|)^2} \leq \frac{1}{\overline{L}(z_1,\ z_2)} \int\limits_{L(z_1,\,z_2)} |f'(z)|\,|dz| \leq \frac{1-|z_1|\,|z_2|}{(1-|z_1|)^2(1-|z_2|)^2}.$$

If we choose the curve $L(z_1, z_2)$ in such a way that its image $\mathcal{L}(z_1, z_2, f)$ through $f(z) \in S$, to be the segment joining $f(z_1)$ with $f(z_2)$, then

$$\frac{\overline{\mathcal{L}}(z_1,\ z_2,\ f)}{\overline{L}(z_1,\ z_2)} = \frac{|f(z_1)-f(z_2)|}{\overline{L}(z_1,\ z_2)} \leq \left| \frac{f(z_1)-f(z_2)}{z_1-z_2} \right|$$

and thus the left-hand side of (2) is established.

If the curve $L(z_1, z_2)$ is the segment joining z_1 with z_2, then under the conditions of Theorem $\overline{\mathrm{I}}$:

$$\frac{\overline{\mathcal{L}}(z_1,\ z_2,\ f)}{\overline{L}(z_1,\ z_2)} = \frac{\overline{\mathcal{L}}(z_1,\ z_2,\ f)}{|z_1-z_2|} \geq \left| \frac{f(z_1)-f(z_2)}{z_1-z_2} \right|.$$

In this way Theorem $\overline{\mathrm{I}}$ is proved.

Let $f(z) \in S$ map the unit disc D convexly. In this case (see [2], p. 83), if $|z| \leq r < 1$:

$$(3^*) \qquad \frac{1}{(1+|z|)^2} \leq |f'(z)| \leq \frac{1}{(1-|z|)^2}.$$

Using (5) and the approach stated in A. and B. we obtain:

Theorem I_1. *If $f(z) \in S$ is a convex function and $|z_1| < |z_2| < 1$, then*

$$(6) \qquad \frac{1}{(1+|z_1|)(1+|z_2|)} \leq \frac{\overline{\mathcal{L}}(z_1,\ z_2,\ f)}{\overline{L}(z_1,\ z_2)} \leq \frac{1}{(1-|z_1|)(1-|z_2|)},$$

$$(7) \qquad \frac{1}{(1+|z_1|)(1+|z_2|)} \leq \left| \frac{f(z_1)-f(z_2)}{z_1-z_2} \right| \leq \frac{1}{(1-|z_1|)(1-|z_2|)},$$

where the above estimate in (6) *holds true if $r'(s) \geq 0$, while in* (7) *it is true provided $|z|$ only increases or only decreases on the segment joining $|z_1|$ with $|z_2|$.*

Let the function

$$(8) \qquad f_k(z) = z + c_1^{(k)} z^{k+1} + c_2^{(k)} z^{2k+1} + \cdots, \qquad k = 1, 2, \ldots,$$

be k-symmetric and univalent in the disc $|z| < 1$.

Analogously, from (5), A. and B. we obtain the theorems:

Theorem I_2. *If $f_k(z) \in S_k$, then under the conditions of Theorem* I, *we have:*

$$(9) \qquad \frac{1}{|z_2|-|z_1|} \int\limits_{|z_1|}^{|z_2|} \left(\frac{1-r^k}{1+r^k} \right)^3 \frac{dr}{(1-r^k)^{2/k}} \leq \frac{\overline{\mathcal{L}}(z_1,\ z_2,\ f_k)}{\overline{L}(z_1,\ z_2)} \leq \frac{1}{|z_2|-|z_1|} \int\limits_{|z_1|}^{|z_2|} \left(\frac{1+r^k}{1-r^k} \right)^3 \frac{dr}{(1+r^k)^{2/k}}$$

and for the conditions of Theorem $\overline{\mathrm{I}}$:

$$(10) \qquad \frac{1}{|z_2|-|z_1|} \int\limits_{|z_1|}^{|z_2|} \left(\frac{1-r^k}{1+r^k} \right)^3 \frac{dr}{(1-r^k)^{2/k}} \leq \left| \frac{f_k(z_1)-f_k(z_2)}{z_1-z_2} \right| \leq \frac{1}{|z_2|-|z_1|} \int\limits_{|z_1|}^{|z_2|} \left(\frac{1+r^k}{1-r^k} \right)^3 \frac{dr}{(1+r^k)^{2/k}}.$$

Theorem I_3. *If $f_k(z) \in S_k$ is a convex function, then under the conditions of Theorem* I_1 *we have:*

$$\frac{1}{|z_2|-|z_1|}\int_{|z_1|}^{|z_2|}\frac{dr}{(1+r^k)^{2/k}} \leqq \frac{\overline{\mathfrak{L}}(z_1, z_2, f_k)}{\overline{L}(z_1, z_2)} \leqq \frac{1}{|z_2|-|z_1|}\int_{|z_1|}^{|z_2|}\frac{dr}{(1-r^k)^{2/k}} \tag{11}$$

and

$$\frac{1}{|z_2|-|z_1|}\int_{|z_1|}^{|z_2|}\frac{dr}{(1+r^k)^{2/k}} \leqq \left|\frac{f_k(z_1)-f_k(z_2)}{z_1-z_2}\right| \leqq \frac{1}{|z_2|-|z_1|}\int_{|z_1|}^{|z_2|}\frac{dr}{(1-r^k)^{2/k}}. \tag{12}$$

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