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ПЛИСКА

БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

CHARACTERIZATIONS OF EXPONENTIAL AND GEOMETRIC DISTRIBUTIONS BASED ON ORDER STATISTICS

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This paper concerns the exponential and geometric distributions. Some characterization theorems for these two distributions are proved. Many characterization properties of the exponential distribution are contained in the monograph of Galambos and Kotz.

In this work we shall need the following theorem proved by Ka-Sing Lau and Rao ([1], Theorem 3.1):

Theorem 1_0. *Let f be a positive locally integrable solution of the equation*

$$\int_0^\infty f(x+t)d\mu(t)=f(x), \tag{1}$$

where μ is a regular positive Borel measure on $[0, \infty)$ *and not degenerated at* 0. *Then $f(x)=p(x)e^{\lambda x}$ a. e. where $\lambda\in(-\infty, \infty)$ is determined uniquely by $\int_0^\infty e^{\lambda y}d\mu(y)=1$, p is a positive periodic function with every $z\in \operatorname{supp}\mu$ as a period.*

Remark. If $f(x)=1-F(x)$, where $F(x)=\sum_{k<x} \mathsf{P}(\xi=k)$ is a discrete d. f. and μ is allowed to take values only in all integral points, then F is geometrically distributed.

The proof of this theorem is quite long. We shall give here another version of it looking for a solution of (1) in a special class of functions f. The probability d. f. $F(x)$, $x\geqq 0$, belongs to the so-called class DMRL if the function $\bar{F}(x)/\int_x^\infty \bar{F}(u)du$ is nondecreasing. The class DMRL contains the class of p. d. f. with increasing failure rate. These classes occur in renewal theory.

Lemma. *Let $F(x)$, $x\geqq 0$, be a distribution function, for which $F(0)=0$ and*

$$\int_0^\infty [1-F(x)]dx<\infty, \quad \varphi(x)=\int_x^\infty f(u)du\neq 0, \quad f(u)=1-F(u).$$

If the d. f. F belongs to the class DMRL and $f(x)$ satisfies the equation (1) *with* $\operatorname{supp}\mu=(a, \infty)$, $0\leqq a<\infty$, *then* $F(x)=1-e^{-\lambda x}$, $x\geqq 0$.

Proof. Without restrictions we can suppose that

$$\int_0^\infty f(x)dx=1. \tag{2}$$

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We can easily verify that the quotient $h(x, t)=\varphi(x+t)/\varphi(t)$ increases with respect to t, for fixed x. It can be seen that the function φ satisfies the equation (1), i. e.

$$\int_0^\infty \varphi(x+t)d\mu(t)=\varphi(x), \quad \varphi(0)=1, \tag{3}$$

which implies that $J(0, \infty)=\int_0^\infty \varphi(t)d\mu(t)=1$. Because $h(x, t)$ is increasing, then for arbitrary $\delta>0$ we have

$$\varphi(x)=\int_0^\delta h(x, t)\varphi(t)d\mu(t)+\int_\delta^\infty h(x, t)\varphi(t)d\mu(t)\geqq h(x, 0)J(0, \delta)+h(x, \delta)J(\delta, \infty), \tag{4}$$

or $\varphi(x)J(\delta, \infty)\geqq h(x, \delta)J(\delta, \infty)$.

Notice that $J(\delta, \infty)\neq 0$, since $|\operatorname{supp} \mu|=\infty$, and from (4) we have $\varphi(x)\geqq h(x, \delta)$. On the other hand, since h is increasing $h(x, \delta)\geqq h(x, 0)=\varphi(x)$. Therefore $h(x, \delta)=\varphi(x)$ or $\varphi(x+t)=\varphi(x)\varphi(t)$. The last equality yields $\varphi(x)=e^{-ax}$, $a>0$. Using that $\varphi(x)=\int_x^\infty f(u)du$ we find $f(x)=ae^{-ax}$. But $f(0)=1$, and so $f(x)=e^{-x}$. The general case, without the restriction (2), is reduced to the solution $\lambda^{-1}x=e^{-x}$ by setting $y=\lambda^{-1}x$.

Theorem 1. *Let $F(x)$, $x\geqq 0$, be a continuous d. f. and $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with common d.f. $F(x)$. Then $F(x)=1-e^{-\lambda x}$ if and only if for some integer $k\geqq 1$, the conditional expectation*

$$\mathrm{E}\Big[\frac{1}{n-1}\sum_{j=1}^{n}(X_j-y)^k \,\Big|\, \min_{1\leqq j\leqq n} X_j=y\Big]=\text{const} \tag{5}$$

a.s. with respect to F and for all n.

Proof. We first shall show that (5) is valid, if $F=1-e^{-\lambda x}$. Denote by $X_{1:n}=\min X_j$. By a direct calculation we get that

$$\mathrm{P}(X_j<x \mid X_{1:n}=y)=\frac{1}{n}+\frac{n-1}{n}\frac{F(x)-F(y)}{f(y)}, \quad x>y \text{ and } 0,\ x\leqq y, \tag{6}$$

where $f(y)=1-F(y)$.

Using (6) we find the conditional expectation (5):

$$\mathrm{E}\Big[\frac{1}{n-1}\sum_{j=1}^{n}(X_j-y)^k \mid X_{1:n}=y\Big]=\frac{1}{f(y)}\int_y^\infty (x-y)^k dF(x). \tag{7}$$

If $F(x)$ is the exponential distribution, then

$$\frac{1}{f(y)}\int_y^\infty (x-y)^k\, dF(x)=\int_0^\infty x^k d(1-e^{-\lambda x})=k!\ \lambda^{-k}$$

and (7) is really a constant.

Let now (5) is assumed to be true. For $y=0$ we get from (7) that the assumed constant is the k-th moment of $F(x)$, say a_k. Hence $\int_y^\infty (x-y)^k dF(x)=a_k f(y)$ or

$$\int_0^\infty t^k df(y+t)=-a_k f(y). \tag{8}$$

Thus, if we denote $\mu(t)=t^k/a_k$ then (8) can be written as

$$\int_0^\infty f(y+t)d\mu(t)=f(y)$$

and, according to the proved Lemma, $f(x)=e^{-\lambda x}$. λ is obtained from $\int_0^\infty e^{-\lambda t}d\mu(t)=1$ and we get $\lambda^k=k!\ a_k^{-1}$. This completes the proof of Theorem 1.

Theorem 2. *Under the same conditions as in Theorem 1, $F(x)$ is the exponential distribution if and only if*

$$E\left[\frac{1}{n-r}\sum_{j=r}^{n}(X_{j:n}-y)^k \mid X_{r:n}=y\right]=\text{const},$$

where $X_{j:n}$ is the j-th order statistic.

The proof of Theorem 2 is exactly the same as that of Theorem 1. It is used instead of (6), that for $j>r$

$$P(X_{j:n}<x \mid X_{r:n}=y)=\frac{1}{n-r+1}+\frac{n-r}{n-r+1}\frac{F(x)-F(y)}{f(y)}$$

if $x\geqq y$, and equals to 0 otherwise. After that, following the same manner of proving as in Theorem 1, the Lemma could be applied again.

Theorem 3. *Let $p_k=P(X=k)$, $p_k\neq 0$, $k=0, 1, 2, \ldots$, be a discrete distribution and $X_1, X_2, \ldots, X_n$ be independent r.v. with common d.f. $F(\nu)=\Sigma_{k=0}^{\nu-1}p_k$. The condition*

(9) $$E\left[\frac{1}{n-1}\sum_{j=1}^{n}(X_j-\nu) \mid \min_{1\leqq j\leqq n} X_j=\nu\right]=\text{const}$$

is a necessary and sufficient condition for X to be geometrically distributed.

Proof. A direct calculation gives

(10) $$P(X_j=k \mid X_{1:n}=\nu)=\begin{cases}\frac{1}{n}+\frac{n-1}{n}\frac{p_\nu}{f(\nu)}, & k=\nu\\ \frac{n-1}{n}\frac{p_k}{f(\nu)}, & k\geqq\nu+1\\ 0 & k<\nu,\end{cases}$$

where $f(\nu)=\Sigma_{i=\nu}^{\infty}p_i$. From (10) we get

$$E(X_j \mid X_{1:n}=\nu)=\frac{\nu}{n}+\frac{n-1}{nf(\nu)}\sum_{k\geqq\nu}kp_k$$

and therefore

(11) $$E\left[\frac{1}{n-1}\sum_{j=1}^{n}(X_j-\nu) \mid X_{1:n}=\nu\right]=-\nu+\frac{1}{f(\nu)}\sum_{k\geqq\nu}kp_k.$$

Let us suppose $p_k=pq^k$, $k\geqq 0$. Then $f(\nu)=q^\nu$ and $\Sigma_{k\geqq\nu}kp_k=\nu q^\nu+q^{\nu+1}p^{-1}$. Hence (9) is a constant.

Assume that (9) is fulfilled. Taking into consideration (11), we have

(12) $$\sum_{k\geqq\nu}(k-\nu)p_k=cf(\nu).$$

Let $\mu(t)$ be the measure with increments $\Delta\mu(t)=1/c$ only in the points $t=1,2,3,\ldots$. Then (12) can be written in the form

$$\int_0^\infty f(\nu+t)d\mu(t)=f(\nu), \quad \nu=0,\ 1,\ 2,\ldots.$$

According to corollary of the lemma we get $f(\nu)=q^\nu$, $q=c/(1+c)$.

Theorem 4. *Let X be a discrete distributed r. v. with a distribution $p_k=\mathsf{P}(X=k)$, $p_k\neq 0$, $k=0,\ 1,\ 2,\ldots$, and*

(13) $$Y_\nu(k)=(X-\nu)(X-\nu-1)\ldots(X-\nu-k+1), \quad Y_\nu(0)=1.$$

Then $p_j=pq^j$, $j=0,\ 1,\ 2,\ldots$, if and only if for some integer $k\geqq 1$,

(14) $$\mathsf{E}(Y_\nu(k)\mid X\geqq\nu)=\text{const}$$

with respect to ν.

Proof. If $p_k=pq^k$, then it is not difficult to find that $\mathsf{E}(Y_\nu(k)\mid X\geqq\nu)=k!(q/p)^k$, i. e. (14) depends only on k.

Let now (13) take place and c is the constant from the right hand side of (14). Since the conditional distribution $\mathsf{P}(X=i\mid X\geqq\nu)$ is

$$\mathsf{P}(X=i\mid X\geqq\nu)=\begin{cases}p_i/f(\nu), & i\geqq\nu,\\ 0, & i<\nu,\quad f(\nu)=\Sigma_{k\geqq\nu}p_k.\end{cases}$$

Then $\mathsf{E}(Y_\nu(k)\mid X\geqq\nu)=\sum_{i>\nu+k-1}(i-\nu)(i-\nu-1)\ldots(i-\nu-k+1)p_i/f(\nu)$

and because of (13)

(15) $$\sum_{m>k-1}m(m-1)\ldots(m-k+1)p_{m+\nu}=cf(\nu).$$

If we denote by $\mu(t)$ the measure for which, with in integer t,

$$\Delta\mu(t)=\mu(t+0)-\mu(t)=\begin{cases}c^{-1}(t-1)(t-2)\ldots(t-k+1)k, & t\geqq k,\\ 0, \text{ otherwise},\end{cases}$$

then the equality (15) may be written in the following integral form

$$\int_0^\infty f(\nu+t)d\mu(t)=f(\nu), \quad \nu=0,\ 1,\ 2,\ldots.$$

Therefore $f(\nu)=q^\nu$, $0<q<1$, $\nu=0,\ 1,\ldots$, which was to be proved.

Theorem 5. *Let us keep the notations of Theorem 4 and let X_1, $X_2,\ldots,X_n$ be independent r. v. distributed as X. Then*

(16) $$\mathsf{E}\Big(\frac{1}{n-1}\sum_{j=1}^{n}Y_\nu^j(k)\mid\min_{1\leqq i\leqq n}X_i=\nu\Big)=\text{const}$$

for some integer $k\geqq 1$ and all $\nu\geqq 0$, is a necessary and sufficient condition for $p_i=pq^i$, $i=0,\ 1,\ 2,\ldots$. In (16) $Y_\nu^j(k)$ are the same as in (13) for X_j.

Theorem 5 can be proved in a similar way as Theorem 3, so we omit the proof.

The above results are close to those obtained in papers [2, 3, 4, 5], and are some extention of them. Several of these theorems, however, can be obtained in the same manner as in the present paper, when based on the proved Lemma.

REFERENCES

1. Ka-Sing Lau, C. R. Rao. Integrated Cauchy Functional Equation and Characterization of the Exponential Law. University of Pittsburgh, Technical Report 1981, No 81-10.
2. Т. А. Азларов, А. А. Джамирзаев, М. М. Султанова. Характеризационные свойства экспоненциального распределения и их устойчивость. — В: Случайные процессы и статистические выводы. Вып. 2. Ташкент, 1972, 10—19.
3. О. М. Сахобов, А. А. Гешев. Характеристические свойства экспоненциального распределения. *Природа*, Пловд. унив., 7, 1974, 1, 25—28.
4. Л. Б. Клебанов. Некоторые результаты, связанные с характеризацией экспоненциального распределения. *Теория вероятн. примен.*, **24**, 1979, 628—632.
5. J. Galambos, S. Kotz. Characterizations of Probability Distributions. Berlin, 1978.

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