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THE MAXIMAL SUBGROUPS OF THE TITS SIMPLE GROUP

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The main result of this paper determines the subgroup structure of the finite simple group of Tits ${}^2F_4(2)'$.

1. Introduction. Let F denote the Ree group ${}^2F_4(2)$ and G be the commutator subgroup of F. It was shown by J. Tits [12] that G is a simple group of order $17\ 971\ 200=2^{11}.3^3.5^2.13$ and index 2 in F. The aim of this paper is to determine the subgroup structure of G. The main result is as follows[1].

Theorem 1. *The Tits simple group has exactly eight conjugacy classes of maximal subgroups with representatives:*

1) Q_1 *of order* $2^{11}.5$ (2-local, "parabolic" subgroup);
2) Q_2 *of order* $2^{11}.3$ (2-local, "parabolic" subgroup);
3) $E_{25}.(Z_4\setminus A_4)$ (5-local);
4), 5) $\mathrm{Aut}(A_6)\cong S_6.Z_2$ (two classes fused in F);
6), 7) $\mathrm{Aut}(PSL_3(3))\cong PSL_3(3).Z_2$ (two classes fused in F);
8) $PSL_2(25)$.

This result was announced in [11].

In the above, Q_1 and Q_2 are the intersections with G of the representatives of the two conjugacy classes of maximal parabolic subgroups of F. Next, $X.Y$ ($X\setminus Y$) denotes a split (non-split) extension of X by Y. Other group-theoretic notation used in the paper is standard. Thus Z_n, E_n, and D_n are respectively the cyclic, elementary Abelian, and dihedral group of order n; A_n and S_n are the alternating and symmetric group of degree n. If X is a subset of G, $N(X)$ and $C(X)$ always stand for the normalizer and the centralizer of X in G. If $x, y\in F$ then $x^y=y^{-1}xy$ and $[x, y]=x^{-1}y^{-1}xy$. We use the symbol $<$ to emphasize proper subgroup inclusion. Finally, character means irreducible complex character. Some additional notation specific for this paper will be introduced in Section 2.

The general theory of the Lie type groups is developed in [2]. Those of type (${}^2F_4$) are described in Ree's original paper [8]. Useful information on individual groups which appear as subgroups of G can be found in [2, 7]. Throughout the paper essential use is made of the character table of G built up in [5]. Repeated application is made of the so-called Brauer trick [6, p. 70] in order to construct various simple subgroups of G. Techniques of Finkelstein and Rudvalis [3] are used in determining the conjugacy classes of simple subgroups of G.

Although some subgroups of the Tits group (as local subgroups and the presence of $PSL_2(25)$) seem to be known, in general proofs are not available

[1] The author has just been informed (February 10, 1983) by Professor W. Feit that this result had been independently proved by R. A. Wilson.

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in the literature. So we give complete proofs of all the assertions. Verifications which are straightforward or computational are omitted.

The standard technology of determining the maximal subgroups of a finite simple group depends upon the remark that any maximal subgroup is the normalizer either of an elementary Abelian subgroup or of a direct product of isomorphic non-Abelian simple subgroups. The two kinds of normalizers are treated in Sections 3 and 4. In Section 5, the resulting list of subgroups readily implies Theorem 1.

2. Preliminaries. In this section we collect a number of facts which we shall need in the proof of Theorem 1. The following lemma is deduced from [3, 5, 8, 9, 10, 12] and will be freely used in the next sections.

Lemma 2.1. (i) *$F=\langle U, W\rangle$ where U, W are subgroups to be described below. G is a normal simple subgroup of index 2 in F.*

(ii) *U is a Sylow 2-subgroup of F. Every element u of U is uniquely written in the form*

$$u=\prod_{i=1}^{12} \alpha_i(t_i), \quad t_i \in GF(2),$$

where $\alpha_i(t_i)$ are those from [8, p. 407]. *Further, $u \in G$ if and only if an even number of t_1, t_4, t_5, t_6 are* 0.

(iii) *Set $\alpha_i=\alpha_i(1)$ ($\alpha_i(0)=1$), $1 \leq i \leq 12$. All the non-trivial relations of the α_i's are as follows*[2].

$$\alpha_1^2=\alpha_2,\ \alpha_4^2=\alpha_3,\ \alpha_5^2=\alpha_{12},\ \alpha_6^2=\alpha_{11},\quad \alpha_i^2=1 \quad \text{for} \quad i\neq 1, 4, 5, 6;$$

$$\begin{array}{ll}
[\alpha_1, \alpha_3]=\alpha_4\alpha_5\alpha_7\alpha_{11}\alpha_{12}, & [\alpha_3, \alpha_5]=\alpha_8, \\
[\alpha_1, \alpha_4]=\alpha_5\alpha_6\alpha_7\alpha_9\alpha_{10}\alpha_{11}\alpha_{12}, & [\alpha_3, \alpha_6]=\alpha_8\alpha_9\alpha_{12}, \\
[\alpha_1, \alpha_6]=\alpha_7, & [\alpha_3, \alpha_7]=\alpha_9\alpha_{10}, \\
[\alpha_1, \alpha_8]=\alpha_9\alpha_{11}\alpha_{12}, & [\alpha_3, \alpha_{11}]=\alpha_{12}, \\
[\alpha_1, \alpha_9]=\alpha_{10}\alpha_{11}\alpha_{12}, & [\alpha_4, \alpha_5]=\alpha_9, \\
[\alpha_1, \alpha_{10}]=\alpha_{11}, & [\alpha_4, \alpha_7]=\alpha_{10}\alpha_{11}\alpha_{12}, \\
[\alpha_2, \alpha_3]=\alpha_5\alpha_6\alpha_7\alpha_8\alpha_9, & [\alpha_4, \alpha_{10}]=\alpha_{12}, \\
[\alpha_2, \alpha_4]=\alpha_7\alpha_{11}\alpha_{12}, & [\alpha_5, \alpha_6]=\alpha_{10}, \\
[\alpha_2, \alpha_8]=\alpha_{10}\alpha_{11}\alpha_{12}, & [\alpha_5, \alpha_7]=\alpha_{11}, \\
[\alpha_2, \alpha_9]=\alpha_{11}, & [\alpha_6, \alpha_9]=\alpha_{12}, \\
 & [\alpha_7, \alpha_8]=\alpha_{12}.
\end{array}$$

(iv) *$W=\langle w_1, w_2\rangle \cong D_{16}$ and $W \leq G$. Here $w_1^2=w_2^2=(w_1w_2)^8=1$ and $w_1=w(1, -1, \infty)$, $w_2=w(1, 1, 1)$ in the notation of* [8, p. 403].

(v) *The action of W on the α_i's is given in Table* 1. *The elements α_{-i}($\alpha_i'(1)$ in* [8, p. 409]) *are as follows: $\alpha_{-1}=\alpha_1 w_1 \alpha_1$, $\alpha_{-2}=\alpha_1^{-1} w_1 \alpha_1$, $\alpha_{-3}=\alpha_3 w_2 \alpha_3$.*

[2] These relations are given, for instance, in [10, p. 81—82] (where, in the expression of $[\alpha_2, \alpha_8]$ the last term α_{11} must be deleted) and are checked by the present author using [8].

Table 1

α_i	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}
$\alpha_i^{w_1}$	α_{-1}	α_{-2}	α_7	α_6	α_5	α_4	α_3	α_{11}	α_{10}	α_9	α_8	α_{12}
$\alpha_i^{w_2}$	α_4	α_8	α_{-3}	α_1	α_6	α_5	α_9	α_2	α_7	α_{10}	α_{12}	α_{11}

(vi) *The order of any element of* $F—G$ *is a multiple of* 4.

(vii) *The conjugacy classes and centralizers of the elements of* G *are given in Table 2 below. Because of* (vi), *this provides sufficient knowledge of centralizers in* F. *In Table 2,* (n_k) *(or simply* (n)*) is the* k*-th class (the unique class) of elements of order* n, *and* $(\widetilde{n_k})$, *respectively* $(\widetilde{n})$ *denotes a pair of classes each consisting of the inverses, respectively the fifth powers of elements in the other class.*

Table 2

Class	Representative	Centralizer	Square
(1)		$2^{11}.3^3.5^2.13$	
(2_1)	α_{12}	$2^{11}.5$	
(2_2)	α_{10}	$2^9.3$	
(4_1)	$\alpha_7\alpha_8$	2^7	(2_1)
(4_2)	$\alpha_5\alpha_6\alpha_8$	2^6	(2_2)
$(\widetilde{4_3})$	$\alpha_5\alpha_6$	$2^6.3$	(2_2)
$(\widetilde{8_1})$	$\alpha_2\alpha_4\alpha_5$	2^5	(4_1)
(8_2)	$\alpha_2\alpha_3$	2^4	(4_2)
(8_3)	$\alpha_2\alpha_3\alpha_8$	2^4	(4_2)
$(\widetilde{16_1})$	$\alpha_1\alpha_3\alpha_5$	2^4	$(\widetilde{8_1})$
$(\widetilde{16_2})$	$\alpha_1\alpha_3\alpha_5\alpha_8$	2^4	$(\widetilde{8_1})$
(3)	$\alpha_3 w_2$	$2^2.3^3$	
(5)	$\alpha_2 w_1$	2.5^2	
(6)		$2^2.3$	
(10)		2.5	
$(\widetilde{12})$		$2^2.3$	
$(\widetilde{13})$		13	

(viii) *Let* $\gamma \in (n_k)$ *and* $\Delta(l_i, m_j, \gamma)$ *be the set of all pairs* (α, β) *with* $\alpha \in (l_i)$, $\beta \in (m_j)$, *and* $\alpha\beta = \gamma$. *The cardinality of this set, denoted* $\#(l_i, m_j, n_k)$ *(as it is a class function), is computed in a routine way from the character table of* G. *If* $(\alpha, \beta) \in \Delta(l_i, m_j, \gamma)$ *for some* $\gamma \in (n_k)$ *we say that the group* $\langle\alpha, \beta\rangle$ *is of type* (l_i, m_j, n_k). *Thus any subgroup of* G *of type* $(2_i, 3, 3)$ *is isomorphic to* A_4 *and any subgroup of type* $(2_i, 3, 5)$ *is isomorphic to* A_5. *Further, in the obvious action of* $C(\gamma)$ *on* $\Delta(l_i, m_j, \gamma)$ *by conjugation, the stabilizer of an* (α, β) *is* $C(\langle\alpha, \beta\rangle)$. *If* $C(\gamma)$ *has* r *orbits on* $\Delta(l_i, m_j, \gamma)$ *then* G *has at most* r *conjugacy classes of subgroups of type* (l_i, m_j, n_k).

3. Local subgroups of G. In this section we deal with the p-local subgroups of G ($p=2, 3, 5$ and 13). A part of this analysis will be made use of in Section 4.

The group F has two conjugacy classes of (proper) maximal parabolic subgroups, with representatives $P_1=\langle U, w_1\rangle$ and $P_2=\langle U, w_2\rangle$ of respective orders $2^{12}.5$ and $2^{12}.3$. P_1 is the centralizer of an involution ($\alpha_{12} \in G$) and P_2 is the normalizer of a four-group ($\langle\alpha_{11}, \alpha_{12}\rangle \leq G$). By a result of Borel and

Tits [1] every 2-local subgroup of F is contained in a maximal parabolic subgroup. It follows that any 2-local subgroup of G lies in a conjugate (in F) of either $Q_1 = P_1 \cap G$ or $Q_2 = P_2 \cap G$. Since $|P_i : Q_i| = 2$, in particular $Q_i \lhd P_i$ $(i = 1, 2)$, it is immediate that Q_1^F and Q_2^F are conjugacy classes of subgroups in G. Thus we have:

Lemma 3.1. *G has two conjugacy classes of maximal 2-local subgroups, with representatives Q_1 of order $2^{11}.5$ and Q_2 of order $2^{11}.3$.*

A Sylow 3-subgroup T of G is the non-Abelian group of order 27 and exponent 3. All the elements of T^* are conjugate in G. Set $Z(T) = \langle\omega\rangle (\cong Z_3)$. Then $|C(\omega)| = 4.27$ and hence $|N(\langle\omega\rangle)| = 8.27$. $C(\omega)$ has cyclic Sylow 2-subgroups (as G has elements of order 12) and it follows (by a well-known theorem of Burnside) that $T \lhd C(\omega)$. So $N(\langle\omega\rangle) \leqq N(T)$ and then clearly $N(\langle\omega\rangle) = N(T) = T . D$, where $|D| = 8$. Let $D \cap C(\omega) = \langle u\rangle$. Then $u\omega$ has order 12 and is inverted in G by some involution (as $\#(2_2, 2_2, \tilde{1}2) \neq 0$) which, therefore, inverts both u and ω. So there is an involution in $N(\langle\omega\rangle)$ inverting u and hence $D \cong D_8$.

We next turn to the elementary Abelian subgroups of order 9. There are exactly four such subgroups in T. Any two of them intersect in $Z(T)$ and (being normal in T) are conjugate in G if and only if they are conjugate in $N(T)$. If E is any E_9 subgroup in T then $E^u \neq E$ as otherwise $\langle u\rangle$ (which centralizes $Z(T)$) must act regularly on $E/Z(T) \cong Z_3$, an impossibility. Thus G has at most two classes of E_9 subgroups.

Now direct computations yield the following. Let $\tau = (\alpha_3 w_2)^{\alpha_3\alpha_9\alpha_{10}w_2w_1w_2}$ and $\sigma = \tau^{\alpha_3}$. Then $E = \langle\tau, \sigma\rangle \cong E_9$. Further, $t = \alpha_7\alpha_{10}w_1 \in (8_2) \cup (8_3)$, $v = \alpha_7 \in (2_2)$, and $t^v = t^3$ Hence $Q = \langle t, v\rangle$ is a quasidihedral group of order 16. We note for later use that all the six elements of order 4 in Q are from (4_2). Now $\tau^t = \tau\sigma$, $\sigma^t = \tau^{-1}\sigma$ and $\tau^v = \tau$, $\sigma^v = \sigma^{-1}$. Thus $Q \leqq N(E)$ and hence $|N(E)| \geqq 2^4 . 3^3$. On the other hand, as $C(E) = E$, $N(E)/E$ is a subgroup of Aut $(E) \cong GL_2(3)$. So $|N(E)| \leqq 3^2 . 2^4 . 3 = 2^4 . 3^3$. It follows that $N(E)/E \cong GL_2(3)$ whence (by a well-known theorem of Gaschütz) $N(E) \cong E_9 . GL_2(3)$, the holomorph of E_9. As $C_F(E) = E$, this implies $N_F(E) = N(E)$. Hence if $f \in F - G$ then E and E^f are not conjugate in G. Together with the above paragraph, this implies that G has exactly two classes of E_9 subgroups, with representatives E, E^f and hence with isomorphic normalizers. We have proved:

Lemma 3.2. *G has three conjugacy classes of 3-local subgroups: one class of Sylow 3-normalizers isomorphic to $T . D_8$ and two classes* (fused in F) *of subgroups isomorphic to $E_9 . GL_2(3)$.*

Further, let P be a Sylow 5-subgroup of G. P is elementary Abelian of order 25. All the elements of P^* are conjugate in G and hence in $N(P)$. Note if $\rho \in P^*$ then $|C_F(\rho)| = 100$ ($\alpha_5 \in F - G$ centralizes $\alpha_2 w_1 \in (5)$) so $P \lhd C_F(\rho)$. Now $N_F(P)$ acts transitively (by conjugation) on P^*. A point stabilizer in this action has order 100 (from the above), hence $|N_F(P)| = 100 . 24 = 2^5 . 3 . 5^2$. But $C_F(P) = P$ so $N_F(P)/P$ is a $5'$-subgroup in Aut $(P) \cong GL_2(5)$, of order $2^5 . 3 . 5$. Thus $X = N_F(P)/P$ is a 5-complement in $GL_2(5)$. Then X contains $\langle z\rangle = Z(GL_2(5)) (\cong Z_4)$ and $X/\langle z\rangle$ is a subgroup of order 24 in $PGL_2(5) \cong S_5$. This forces $X/\langle z\rangle \cong S_4$. Now $Y = N(P)/P$ is a subgroup of index 2 in X hence $Y \geqq \langle z\rangle$, as z is a square in $GL_2(5)$. It follows that $Y/\langle z\rangle \cong A_4$. Thus $N(P)/P \cong Z_4 \diagdown A_4$ (the extension being non-split as $GL_2(5)$ has no A_4 subgroups) and, consequently, $N(P) \cong E_{25} . (Z_4 \diagdown A_4)$.

If $\rho \in P^*$ then $P \lhd N(\langle\rho\rangle)$ by the above paragraph, hence $N(\langle\rho\rangle) < N(P)$. In fact, it is easily checked that $\langle\alpha_1\alpha_5, \alpha_{12}\rangle \cong Z_4 \times Z_2$ normalizes $\langle\alpha_2 w_1\rangle \cong Z_5$ so that the normalizer of a subgroup of order 5 in G is isomorphic to $E_{25}.(Z_4\times Z_2)$.

The following has been proved:

L e m m a 3.3. *G has one conjugacy class of maximal 5-local subgroups: Sylow 5-normalizers isomorphic to* $E_{25}.(Z_4\diagdown A_4)$.

Finally, from Table 2 we obviously have:

L e m m a 3.4. *G has one conjugacy class of 13-local subgroups: Frobenius groups of order* 6.13.

4. Normalizers of simple subgroups of G. In this section we determine all conjugacy classes of (characteristically) simple subgroups of G and their normalizers.

L e m m a 4.1. *Let H be a proper non-Abelian characteristically simple subgroup of G. Then $H\cong A_5$, A_6, $PSL_3(3)$, or $PSL_2(25)$.*

P r o o f. H is the direct product of isomorphic non-Abelian simple groups. From the centralizers of the elements of G it follows that H is simple. Since every non-principal character degree of G is at least 26, clearly $|H|<10^6$. Now [4] implies $H\cong A_5, A_6, PSL_3(3), PSL_2(25)$, or $PSU_3(4^2)$. The group $PSU_3(4^2)$ is ruled out by the fact that it has elements of order 15 while G has none. It will be shown below that G in fact contains subgroups isomorphic to each of the remaining four groups.

L e m m a 4.2. *G has two conjugacy classes (fused in F) of subgroups isomorphic to A_5. The normalizer of any A_5 subgroup is isomorphic to S_5 and is not a maximal subgroup of G.*

P r o o f. G has no A_5 subgroups of type $(2_1, 3, 5)$ as $\#(2_1, 3, 5)=0$. Let $\gamma\in(5)$ and $\Delta(\gamma)=\Delta(2_2, 3, \gamma)$. Then $|\Delta(\gamma)|=100.C(\gamma)$ (of order 50) acts on $\Delta(\gamma)$ by conjugation and the stabilizer of an (α, β) in this action is $C(\langle\alpha, \beta\rangle)=1$ by Table 2, as $\langle\alpha, \beta\rangle\cong A_5$. Thus $\Delta(\gamma)$ splits into two orbits under $C(\gamma)$ and hence there are at most two classes of A_5 subgroups in G. On the other hand, if $A_5\cong A\leq G$ and $f\in F-G$ then A and A^f are not conjugate in G. For, otherwise $A=A^{fg}$ with some $g\in G$ and then $x=fg\in F-G$ normalizes A. Since $C_F(A)=1$, $S=\langle A, x\rangle\cong S_5$. Let i be an involution in $S-A\,(S_5-A_5)$. As $S\cap G=A, i\in F-G$ which is impossible by Lemma 2.1. This contradiction proves that A and A^f are not conjugate in G. Thus G has exactly two classes of A_5 subgroups and these classes fuse in F.

The second statement of the lemma will follow from the proof of the next lemma.

L e m m a 4.3. *G has subgroups isomorphic to A_6.*

P r o o f. We first claim that every A_4 subgroup of G of type $(2_2, 3, 3)$ is centralized by an involution. Indeed, the elements $\alpha_9\alpha_{10}\alpha_{11}\in(2_2)$ and $\gamma=\alpha_3 w_2\in(3)$ generate an A_4 subgroup (because their product has order 3) which is centralized by $\alpha_{10}\in(2_2)$. Next, set $\Delta(\gamma)=\Delta(2_2, 3, \gamma)$; then $|\Delta(\gamma)|=108$. For any $(\alpha, \beta)\in\Delta(\gamma)\,|C(\langle\alpha, \beta\rangle)|$ is prime to 3 as $\langle\alpha. \beta\rangle\cong A_4$ while the centralizer of an element of order 3 has cyclic Sylow 2-subgroups. Thus $|C(\langle\alpha, \beta\rangle)|=1, 2$, or 4 (from $|C(\alpha)|$ and $|C(\beta)|$). Suppose $C(\langle\alpha, \beta\rangle)=1$ for some $(\alpha, \beta)\in\Delta(\gamma)$. Then this (α, β) is in an orbit of length 108 under the action of $C(\gamma)$ on $\Delta(\gamma)$. This means there is a single orbit ($\Delta(\gamma)$ itself) and hence G has a single class of A_4 subgroups of type $(2_2, 3, 3)$, with trivial centralizers. This is impossible, as shown above, and thus proves that any A_4 of type $(2_2, 3, 3)$ in G is centralized by an involution.

Now let A be an arbitrary A_5 subgroup of G (recall its involutions are from (2_2)) and $A_4\cong K\leq A$. We have seen that there is an involution $j\in C(K)$. Hence $K\leq A\cap A^j$, and $A^j\neq A$ as otherwise $\langle A, j\rangle\cong S_5$ (recall $C(A)=1$) whereas an A_4 subgroup in S_5 has trivial centralizer. It follows that $K=A\cap A^j$. Now G has a character χ_4 of degree 27. Computations by the character table yield $(\chi_4\,|\,A, 1_A)=(\chi_4\,|\,A^j, 1_{A^j})=2$ and $(\chi_4\,|\,K, 1_K)=3$. As $A\cap A^j=K$ and $(\chi_4\,|\,A, 1_A)+(\chi_4\,|\,A^j, 1_{A^j})>(\chi_4\,|\,K, 1_K)$, the Brauer trick implies that $B=\langle A, A^j\rangle$ is a proper subgroup of G. The results of Section 3 show that all local subgroups of G are solvable (which is, in fact, well-known). As B is non-solvable, a minimal non-trivial normal subgroup N of B is simple. Then $N\cong A_5$, A_6, $PSL_3(3)$, or $PSL_2(25)$ by Lemma 4.1, and $C(N)=1$. If, say, $N\cap A=1$ then A is contained in $N(N)/N\leq \mathrm{Out}(N)$. This is, however, impossible as $\mathrm{Out}(N)$ is solvable. So N contains A, A^j as they are simple groups. Thus $B=N$ is simple, and obviously B cannot be A_5 or $PSL_3(3)$. As j interchanges A and A^j, j normalizes B. But $j\notin B$ because in each of the groups A_6 and $PSL_2(25)$ any A_4 subgroup has trivial centralizer. It is shown in Lemma 4.6 that any $PSL_2(25)$ subgroup of G is selfnormalizing. It follows that B cannot be $PSL_2(25)$. This forces $B\cong A_6$, thus proving the lemma.

Moreover, the group A_6 has two classes of elements of order 3 and two classes of A_5 subgroups which are distinguished by their elements of order 3. Hence two non-conjugate A_5's of A_6 cannot intersect in an A_4. Thus there is an element $b\in B$ such that $A^j=A^b$. Now $bj\in N(A)$ but $bj\notin A$ as otherwise $j\in B$, contradicting the above paragraph. Thus we have proved: any A_5 subgroup A of G lies in an A_6 subgroup B, $N(A)>A$ and hence $N(A)\cong S_5$, and also that $N(A)$ is properly contained in $N(B)$ so it is not maximal in G. This completes also the proof of Lemma 4.2.

Lemma 4.4. *An A_5 subgroup of G is contained in exactly one A_6 subgroup.*

Proof. We have just seen that any A_5 subgroup lies in some A_6 subgroup. Assume now that $A\cong A_5$ is contained in two distinct subgroups $B, C\cong A_6$, so $A=B\cap C$. We apply the Brauer trick this time to derive a contradiction.

Any A_6 subgroup in G contains 45 involutions, which invert elements of order 3 and hence are in (2_2) as $\#(2_1, 2_1, 3)=0$, 80 elements of order 3, 90 elements of order 4, which are all either in (4_2) or in (4_3) as elements in (4_1) have their squares in (2_1), and 144 elements of order 5. G has a character χ_6 of degree 78 which takes the same value on both (4_2) and (4_3). Now one computes $(\chi_6\,|\,B, 1_B)=(\chi_6\,|\,C, 1_C)=1$ and $(\chi_6\,|\,A, 1_A)=1$. It follows that $D=\langle B, C\rangle$ is a proper subgroup of G. Furthermore, the same argument as in the proof of Lemma 4.3 implies that a minimal normal subgroup $N\neq 1$ of D is simple and that N contains both B and C, whence $N=D$. Thus D is a proper simple subgroup of G. However, none of the groups listed in Lemma 4.1 contains A_6 properly. This contradiction proves the lemma.

Lemma 4.5. *G has two conjugacy classes (fused in F) of subgroups isomorphic to A_6. If B is a member of either class, then $N(B)\cong \mathrm{Aut}(A_6)\cong S_6\cdot Z_2$.*

Proof. Let B be any A_6 subgroup of G and $A_5\cong A\leq B$. By Lemma 4.2 $N(A)\cong S_5$, so choose an involution $j\in N(A)-A$. As $A\leqq B\cap B^j$, $B^j=B$ by Lemma 4.4. Thus $j\in N(B)$ but $j\notin B$ as $N_B(A)=A$. Hence B is a subgroup of index 2 in $R=\langle B, j\rangle$ and $R\leq \mathrm{Aut}(A_6)\cong \mathrm{Aut}(S_6)$. Now, there are exactly three subgroups in $\mathrm{Aut}(A_6)$ containing the A_6 with index 2: S_6, $PGL_2(9)$, and M_{10} (a point stabilizer in the Mathieu group M_{11}). But $PGL_2(9)$ has no S_5 subgroups as it has two classes of elements of order 5 while S_5 has only one class. M_{10} has no S_5 subgroups, too, because there are involutions in S_5-A_5 whereas all the involutions of M_{10} are already in A_6. We therefore have $R\cong S_6$.

Thus, any $B\cong A_6$ is contained in exactly one $R\cong S_6$. If $f\in F-G$, R and R^f are not conjugate in G, again because there are involutions in $\mathrm{Aut}(S_6)-S_6$ but not in $F-G$. This says that G has at least two classes of A_6 subgroups. On the other hand, G has two classes of A_5 subgroups and each A_5 lies in exactly one A_6, from which it follows immediately that G has at most two classes of A_6 subgroups. Eventually, G has exactly two classes of A_6 subgroups, with representatives B and B^f where $f\in F-G$.

Moreover, these two classes are distinguished by their A_5 subgroups, as A and A^f are representatives of the two classes of A_5's in G. This means that any two A_5 subgroups of B are conjugate in G. Hence the same is true of their normalizers, that is, any two S_5 subgroups of R are conjugate in G. Now the group $R\cong S_6$ has two classes of S_5's, with representatives, say, R_1 and R_2. By the above there is an element $g\in G-R$ with $R_1=R_2^g$.Hence $R_1\leq R\cap R^g$ and consequently $R^g=R$, as clearly an S_5 subgroup of G lies in a unique S_6. Thus $g\in N(B)-R$ which implies $N(B)\cong\mathrm{Aut}(A_6)\cong S_6 . Z_2$. All parts of the lemma are proved.

L e m m a 4.6. *G has one conjugacy class of subgroups isomorphic to $PSL_2(25)$. Any such subgroup is self-normalizing.*

P r o o f. Take a subgroup $S\cong S_5$ in G. We first claim that the involutions of S are from (2_2) and the elements of order 4 in S are from (4_3). For, we know that S is contained in a subgroup $R\cong S_6$. Each involution in R either centralizes or inverts an element of order 3, hence it is in (2_2). So R has no elements from (4_1). Then, by restricting the character χ_4 of G of degree 27 to R, one sees that $R\cap(4_2)$ and $R\cap(4_3)$ are the two classes of elements of order 4 in R. One of these classes lies in the A_6 subgroup B of R while the other class appears in the S_5 subgroups of R. The elements of order 4 in $B\cong A_6$ normalize E_9 subgroups hence they are in (4_2), from the discussion preceding Lemma 3.2 and the remark that the classes (4_2) and (4_3) do not fuse in F. It follows that $R\cap(4_2)\subset B$ and then precisely $R\cap(4_3)$ appears in the S_5 subgroups of R. This proves the claim.

Now let $\rho\in S\cap(5)$. Then $K=N_S(\langle\rho\rangle)$ is a Frobenius group of order 20. As $N(\langle\rho\rangle)$ has a normal Sylow 5-subgroup, K lies in a subgroup $V\cong E_{25} . Z_4$ of $N(\langle\rho\rangle)$. Thus $K=S\cap V$. The non-identity elements in a Z_4 subgroup of V are from (4_3) and (2_2) hence they cannot centralize non-identity elements of the E_{25} subgroup. This determines the class structure of V, and that of S also follows from the above paragraph. Now computations yield $(\chi_4\,|\,S, 1_S)=2$, $(\chi_4\,|\,V, 1_V)=3$, and $(\chi_4\,|\,K, 1_K)=4$. The Brauer trick implies that $L=\langle S, V\rangle$ is a proper subgroup of G.

A minimal normal subgroup $N\neq 1$ of L is simple (as L is non-solvable), and $L\leq\mathrm{Aut}(N)$. Thus $|\mathrm{Aut}(N)|$ is a multiple of 5^3 whence necessarily $N\cong PSL_2(25)$ (by Lemma 4.1). Moreover, $L=N$. Indeed, $|\mathrm{Aut}(N):N|=4$ and $L>N$ would imply that a Sylow 13-normalizer in L has order 4.13 or 8.13, contradicting Lemma 3.4. (Note the same argument proves that any $PSL_2(25)$ subgroup of G coincides with its normalizer.) Thus $L\cong PSL_2(25)$.

Further, let $\gamma\in(\widetilde{13})\cap L$. Then $|\Delta(2_2, 3, \gamma)|=104$. Now, $(2_2)\cap L$ and $(3)\cap L$ are conjugacy classes in L and $|\Delta((2_2)\cap L, (3)\cap L, \gamma)|=26$. Every pair of $\alpha\in(2_2)\cap L$, $\beta\in(3)\cap L$ with $\alpha\beta=\gamma$ generates the whole L (from the subgroup structure of $PSL_2(25)$). An easy count shows that γ belongs to $|N(\langle\gamma\rangle)$: $N_L(\langle\gamma\rangle)|=3$ distinct conjugates of L. All the above put together imply that each conjugacy class of $PSL_2(25)$'s contributes 3.26=78 elements to $\Delta(2_2, 3, \gamma)$.

As $|\Delta(2_2, 3, \gamma)|<2.78$, there is a single class of $PSL_2(25)$ subgroups in G. This completes the proof of the lemma.

Lemma 4.7. *G has two conjugacy classes (fused in F) of subgroups isomorphic to $PSL_3(3)$. If M is a representative of either class, $N(M) \cong \mathrm{Aut}(PSL_3(3)) \cong PSL_3(3).Z_2$.*

Proof. We first apply the Brauer trick with the rational-valued character χ_9 of G of degree $351=3^3.13$ to construct a subgroup of G isomorphic to Aut $(PSL_3(3))$. Note χ_9 (of course) vanishes on all elements of order divisible by 3. Thus, in determining the class structure of the subgroups considered below, we can limit ourselves on 2-elements only.

Now let E be an E_9 subgroup of G. We have seen that $N(E) \cong E_9 . GL_2(3)$. From the structure of this group and the information in Section 3, one verifies that $N(E)$ contains 45 involutions which centralize elements of order 3 and hence lie in (2_2), 54 elements from (4_2), and 108 elements from $(8_2) \cup (8_3)$. This produces $(\chi_9 | N(E), 1_{N(E)})=3$. Next, let $T \leqq N(E)$ be a Sylow 3-subgroup of G and $V=N_{N(E)}(T)$. Then V is a split extension of T by E_4 and V contains 27 involutions from (2_2). This yields $(\chi_9 | V, 1_V)=7$. Lastly, from the known structure of $N(T)$ one checks that $N(T)$ has 45 involutions from (2_2) (as they normalize $Z(T)$), and 18 elements from (4_3) (as they centralize $Z(T)$). This implies $(\chi_9 | N(T), 1_{N(T)})=5$.

Now, as $N(E) \cap N(T)=V$ and $3+5>7$, $N=\langle N(E), N(T)\rangle$ is a proper subgroup of G. Since no local subgroup of G contains both $N(E)$ and $N(T)$, a minimal normal subgroup $M \neq 1$ of N is simple. Since $N \leqq \mathrm{Aut}(M)$, $|\mathrm{Aut}(M)|$ is divisible by 3^3 and then $M \cong PSL_3(3)$ (by Lemma 4.1). Moreover, $N>M$ as a Sylow 3-normalizer has order 8.27 in N and only 4.27 in $PSL_3(3)$. This forces $N=N(M) \cong \mathrm{Aut}(PSL_3(3)) \cong PSL_3(3).Z_2$.

Finally, let $\gamma \in (\widetilde{13})$. As shown in Lemma 4.6, $|\Delta(2_2, 3, \gamma)|=104$ and 78 elements of this set arise from $PSL_2(25)$ subgroups of G. However, each $PSL_3(3)$ subgroup containing γ contributes 13 to $|\Delta(2_2, 3, \gamma)|$. As $104-78 <3.13$, γ belongs to at most two $PSL_3(3)$'s. Hence there are at most two classes of $PSL_3(3)$ subgroups in G. But if $PSL_3(3) \cong M \leqq G$ and $f \in F-G$ then M and M^f are not conjugate in G, again because there are involutions in Aut(M) $-M$ but not in $F-G$. This proves that G has exactly two classes of $PSL_3(3)$'s, which fuse in F and so they have isomorphic normalizers in G. The lemma is proved.

5. Proof of Theorem 1. Now all conjugacy classes of the candidates for maximal subgroups of G are determined in Sections 3 and 4. As any 3-local or 13-local subgroup of G is contained in a subgroup isomorphic to Aut $(PSL_3(3))$, these reduce to the eight classes listed in Theorem 1. Obviously no member of any of these classes is contained in any other. Thus all these subgroups are maximal in G. This completes the proof of Theorem 1.

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