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VARIETIES OF METABELIAN LIE ALGEBRAS OVER FINITE FIELDS

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Metabelian varieties of Lie algebras over a finite field are studied in this paper. It is proved that any such variety is a union of two subvarieties. One of them is nilpotent and the other is generated by algebras which are abelian-by-abelian split extensions. Any proper subvariety of the metabelian variety is embedded in the variety generated by a wreath product of two finite dimensional abelian algebras. The proofs are based on the technique of varieties of representations of Lie algebras. Some other results concerning bivarieties of Lie algebras are obtained in the paper, too.

Metabelian varieties are the simplest objects in the theory of Lie algebras with polynomial identities. In the case of an infinite base field their complete description is given by Bahturin [2]. Over a finite field the picture is rather complicated and partial results are known only (e. g. [4, 10]). For a background of the theory of varieties of Lie algebras cf. [6]. The purpose of this paper is to study varieties of metabelian Lie algebras over a finite field. The main results are that any such variety can be almost determined by its algebras which are abelian-by-abelian split extensions and it is contained in a variety generated by the (abelian) wreath product of two finite dimensional abelian algebras. Having a split extension, there exists a natural action of an abelian algebra over the commutant. Therefore, the technique of varieties of representations can be applied [3]. More precisely, we follow the exposition of Bryce [7] and some ideals of his paper have induced the present work.

1. Bialgebras and bivarieties. The definitions and notations in the paper are in the spirit of [7]. We fix a finite field K with q elements and consider split extensions $G=M\lambda B$, where G is a Lie algebra, M is an abelian ideal and B subalgebra of G.

Definition 1.1. *A bialgebra is a triple (G, M, B), where G is a Lie algebra and $G=M\lambda B$.*

In a natural way we can determine subobjects, homomorphisms and cartesian products of bialgebras.

Definition 1.2. *A subbialgebra of (G, M, B) is a bialgebra (G_1, M_1, B_1), where G_1 is a Lie subalgebra of G and $M_1=M\cap G_1$, $B_1=B\cap G_1$. A homomorphism $\varphi: (G, M, G)\to(G_2, M_2, B_2)$ is a Lie algebra homomorphism $\varphi: G\to G_2$ such that $\varphi(M)\subset M_2$, $\varphi(B)\subset B_2$. A cartesian product of a collection of bialgebras (G_j, M_j, B_j), $j\in J$, is the bialgebra (G, M, B), where $M=\Pi M_j$, $B=\Pi B_j$ and $G=M\lambda B$ with canonical action of B on M.*

In order to introduce bivarieties we need identical relations for bialgebras. Let $L(X)$ be the free Lie algebra with free generators $X=\{x_1, x_2, \ldots\}$, let $U(L(X))$ be the universal enveloping algebra of $L(X)$ (i. e. the free associative algebra on X) and let $M(Y)$ be the free right $U(L(X))$-module freely generated on $Y=\{y_1, y_2, \ldots\}$. We define a trivial multiplication on $M(Y)$ and consider the free bialgebra $(M(Y) \lambda L(X), M(Y), L(X))$. For a bialgebra (G, M, B) there exist two types of identities. One of them forces conditions on the subalgebra and the other on the action of B on M.

Defintion 1.3. *Let* $f(y_1, \ldots, y_k; x_1, \ldots, x_n) = \Sigma_{i=1}^{k} y_i f_i (x_1, \ldots, x_n) \in M(Y)$ (*where* $f_i \in U(L(X))$) *and* $g(x_1, \ldots, x_n) \in L(X)$. *We say that* $f=0$ *and* $g=0$ *are identities for the bialgebra* (G, M, B) *if for any* $m_1, \ldots, m_k \in M$, $b_1, \ldots b_n \in B$ $f(m_1, \ldots, m_k; b_1, \ldots, b_n) = g(b_1, \ldots, b_n) = 0$.

Definition 1.4. *A class* $\mathfrak{M}$ *of bialgebras is a bivariety if* $\mathfrak{M}$ *is the class of all bialgebras satisfying a given system of identities.*

In a standard way we are able to prove the Birkhoff theorem that the class $\mathfrak{M}$ is a bivariety if and only if $\mathfrak{M}$ is closed with respect to subalgebras, homomorphic images and cartesian products. Hence, it is clear that a bivariety bivar $\{(G_j, M_j, B_j) \mid j \in J\}$ generated by a class of bialgebras $\{(G_j, M_j, B_j) \mid j \in J\}$ means.

Recall that a Lie algebra G is metabelian if $(G^2)^2=0$. By analogy, a bialgebra (G, M, B) is metabelian if B is an abelian subalgebra of G. In the sequel we shall consider metabelian (bi) algebras and (bi) varieties and abelian-by-abelian split extensions only. The classes of all metabelian algebras and bialgebras are denoted by $\mathfrak{A}^2$ and $\mathfrak{A} \circ \mathfrak{A}$, respectively. We reserve the letter A for an arbitrary abelian Lie algebra and the notations $A(X)$ and A_n for the algebra with linear basis X and of dimension n, respectively.

The free metabelian bialgebra $(F(Y, X), M_a(Y), A(X))$ is obtained in the following way: $M_a(Y)$ is a free right $K[X]$-module with free generators Y, $K[X]$ being the ordinary polynomial algebra. Hence, $F(Y, X) = A(Y) \operatorname{wr} A(X)$, the abelian wreath product of two abelian Lie algebras [5]. Of course, consider metabelian bivarieties we shall take the identities from the free metabelian bialgebra.

Let $(G, M, A) \in \mathfrak{A} \circ \mathfrak{A}$ and $A \neq 0$. Because of commutativity of A, there are no non-trivial identical relations on A. Therefore, all identities of (G, M, A) are of the form $f(y_1, \ldots, y_k; x_1, \ldots, x_n) = \Sigma_{i=1}^{k} y_i f_i (x_1, \ldots, x_n)$, where $f_i \in K[X]$. Clearly, if $f=0$ is an identity for (G, M, A), then $y_i f_i (x_1, \ldots, x_n) = 0$, $i=1, \ldots, k$, are identities as well and we can examine relations of the type $yf(x_1, \ldots, x_n) = 0$ only. Hence, without loss of generality we assume $Y=\{y\}$. The following proposition is an easy exercise on the subject.

Proposition 1.5. *Consider all subbivarieties of* $\mathfrak{A} \circ \mathfrak{A}$ *different from the trivial* (G, M, O). *The mapping*

$$\mathfrak{M} \rightarrow I(\mathfrak{M}) = \{f \in K[X] \mid yf=0 \text{ is an identity for } \mathfrak{M}\}$$

is a bijective correspondence between such bivarieties and ideals $I=I(\mathfrak{M})$ *of* $K[X]$ *with the following property: for any* $f(x_1, \ldots, x_n) \in I$ *and arbitrary* $\alpha_{ij} \in K$, $j=1, \ldots, n$, $i=1, 2, \ldots$, $f(\Sigma \alpha_{i1} x_i, \ldots, \Sigma \alpha_{in} x_i) \in I$ *again.*

Remark 1.6. Let GL be the general linear group acting on the linear space with a basis X. This action is expanded on $K[X]$ canonically. Ideals of $K[X]$ which we study in Proposition 1.5 are GL-invariant and we call them GL-ideals.

Proposition 1.7. *The bivariety* $\mathfrak{A} \circ \mathfrak{A}$ *is spechtian, i. e. any subbivariety of* $\mathfrak{A} \circ \mathfrak{A}$ *has a finite basis for its identities.*

Proof. By Proposition 2 [8] $K[X]$ has a maximum condition on Φ-ideals. Here a Φ-ideal means that the ideal is invariant under special type of linear transformations. But GL-ideals are invariant under all transformations, hence $K[X]$ has a maximum condition on GL-ideals, too. In the virtue of the bijection between GL-ideals and subbivarieties of $\mathfrak{A}\circ\mathfrak{A}$, the latests satisfy a minimum condition. The bivariety $\mathfrak{A}\circ\mathfrak{A}$ itself is defined by the trivial identity $[x_1, x_2]=0$ and is finitely based. Therefore $\mathfrak{A}\circ\mathfrak{A}$ is spechtian.

Definition 1.8. *A bialgebra (G, M, B) is called residually finite if for any element $g=m+b\in G$ (where $m\in M$, $b\in B$) there exists a homomorphism θ_g of (G, M, B) on a finite dimensional bialgebra (G_g, M_g, B_g) such that $\theta_g(g)\neq 0$.*

Proposition 1.9. *Any finitely generated bialgebra from $\mathfrak{A}\circ\mathfrak{A}$ is residually finite.*

Proof. By analogy with the ordinary algebra case, any bialgebra is a subdirect product of monolithic (or subdirectly irreducible) bialgebras. Therefore it suffices to prove that any finitely generated monolithic bialgebra (G, M, A) from $\mathfrak{A}\circ\mathfrak{A}$ is residually finite. Clearly, in this case A is a finite dimensional vector space, $A=A(x_1, \dots, x_n)$ and M is a finitely generated monolithic $K[x_1, \dots, x_n]$-module. By [1] the module M is a subdirect product of finite dimensional modules M_i, $i\in I$, or equivalently, M is residually finite $K[x_1, \dots, x_n]$-module. For any $g=m+a\in G$, $m\in M$, $a\in A$, we are able to construct a homomorphism θ_g of (G, M, A) on a finite dimensional bialgebra (G_0, M_0, A) such that $\theta_g(g)\neq 0$.

Finally, we make the following conventions: If $\mathfrak{B}$ is a subvariety of $\mathfrak{A}^2$, the corresponding verbal (or T-) ideal of the free metabelian algebra $F(\mathfrak{A}^2)$ will be denoted by $T(\mathfrak{B})$. For any set of elements $\{f_j \mid j\in J\}\subset F(\mathfrak{A}^2)$, the verbal ideal which they generate is $\{f_j \mid j\in J\}^T$. Similarly, we attach a GL-ideal $I(\mathfrak{M})$ of $K[X]$ to any subbivariety $\mathfrak{M}$ of $\mathfrak{A}\circ\mathfrak{A}$ and generate a GL-ideal $\{g_j \mid j\in J\}^I$ for any collection of polynomials $\{g_j \mid j\in J\}\subset K[X]$.

All Lie products will be left normed: $[x_1, x_2]=x_1(\mathrm{ad}\, x_2)$, $[x_1, x_2, \dots, x_n]=x_1(\mathrm{ad}\, x_2)\dots(\mathrm{ad}\, x_n)$ and, by definition, if $g(x_1, \dots, x_n)\in K[X]$, then $[y, g(x_1, \dots, x_n)]=yg(\mathrm{ad}\, x_1, \dots, \mathrm{ad}\, x_n)$. Recall that the algebra $F(\mathfrak{A}^2)$ has the following basis as a linear space $[x_{i_1}, x_{i_2}, \dots, x_{i_n}]$, $i_1>i_2\leq \dots \leq i_n$, and any element of $F^2(\mathfrak{A}^2)$ has the form $\Sigma_{i=2}^n [x_i, g_i(x_1, \dots, x_n)]$. Of course, when we examine generators $\{f_j \mid j\in J\}$ of a T-ideal we may assume that f_j is a sum of Lie monomials in $x_1, \dots, x_n$ and every monomial really depends on $x_1, \dots, x_n$. In this case $f_j=\Sigma_{i=2}^n [x_i, x_1, h_i(x_1, \dots, x_n)]$.

2. Metabelian varieties and bivarieties. Let $\Lambda(\mathfrak{A}^2)$ (resp. $\Lambda(\mathfrak{A}\circ\mathfrak{A})$) be the lattice of all subvarieties of $\mathfrak{A}^2$ (resp. all subbivarieties of $\mathfrak{A}\circ\mathfrak{A}$). Any split extension $G=M\lambda A\in\mathfrak{A}^2$ can be regarded as a bialgebra (G, M, A) and we define two mappings $\sigma:\Lambda(\mathfrak{A}^2)\to\Lambda(\mathfrak{A}\circ\mathfrak{A})$ and $\tau:\Lambda(\mathfrak{A}\circ\mathfrak{A})\to\Lambda(\mathfrak{A}^2)$: If $\mathfrak{B}\subset\mathfrak{A}^2$, $\mathfrak{M}\subset\mathfrak{A}\circ\mathfrak{A}$, then $\mathfrak{B}\sigma=\{(G, M, A) \mid G=M\lambda A\in\mathfrak{B}$, A being abelian$\}$ and $\mathfrak{M}\tau=\mathrm{var}\{G=M\lambda A \mid (G, M, A)\in\mathfrak{M}\}$, the variety generated by G, when (G, M, A) runs on $\mathfrak{M}$.

The following properties of σ and τ are analogous to (1.7.2), (1.7.3) and (1.7.6) of [7].

Proposition 2.1. (i) $\mathfrak{A}^2\sigma=\mathfrak{A}\circ\mathfrak{A}$, $(\mathfrak{A}\circ\mathfrak{A})\tau=\mathfrak{A}^2$.

(ii) *If $\mathfrak{B}\subset\mathfrak{A}^2$ and $\mathfrak{M}\subset\mathfrak{A}\circ\mathfrak{A}$, then $\mathfrak{B}\sigma\tau\subset\mathfrak{B}$, $\mathfrak{M}\tau\sigma\supset\mathfrak{M}$.*

(iii) $\sigma\tau\sigma=\sigma$, $\tau\sigma\tau=\tau$, $(\sigma\tau)^2=\sigma\tau$, $(\tau\sigma)^2=\tau\sigma$.

(iv) *For $\mathfrak{B}_1, \mathfrak{B}_2\subset\mathfrak{A}^2$, $\mathfrak{M}_1, \mathfrak{M}_2\subset\mathfrak{A}\circ\mathfrak{A}$ the equalities hold* $(\mathfrak{B}_1\cap\mathfrak{B}_2)\sigma=\mathfrak{B}_1\sigma\cap\mathfrak{B}_2\sigma$, $(\mathfrak{M}_1\cup\mathfrak{M}_2)\tau=\mathfrak{M}_1\tau\cup\mathfrak{M}_2\tau$.

Proof. Property (i) is obvious because $\mathfrak{A}^2$ and $\mathfrak{A}\circ\mathfrak{A}$ are generated by the wreath product $A(y)\,\mathrm{wr}\,A(X)$ and by the relatively free bialgebra $(M_a(y)\lambda$

$A(X)$, $M_a(y)$, $A(X)$) resp. where $X=\{x_1, x_2, \dots\}$ and $A(y) \operatorname{wr} A(X)=M_a(y) \lambda A(X)$. The other proofs repeat verbatim the corresponding proofs from [7].

We extend the action of the maps σ and τ to the sets of T-ideals of $F(\mathfrak{A}^2)$ and GL-ideals of $K[X]$ by the rule $T(\mathfrak{B})\sigma=I(\mathfrak{B}\sigma)$, $I(\mathfrak{M})\tau=T(\mathfrak{M}\tau)$ for $\mathfrak{B}\subset\mathfrak{A}^2$, $\mathfrak{M}\subset\mathfrak{A}\circ\mathfrak{A}$.

Proposition 2.2. *Let* $T=\{f=\Sigma_{i=2}^n [x_i, x_1, h_i(x_1, \dots, x_n)] \mid f\in J\}^T$ *and* $I=\{g(x_1, \dots, x_n) \mid g\in J\}^I$. *Then*
(i) $T\sigma=\{x_1 h_i(x_1, \dots, x_n), \Sigma_{i=2}^n x_i h_i(x_1, \dots, x_n) \mid f\in J\}^I$.
(ii) $I\tau=\{f=\Sigma_{i=2}^n [x_i, x_1, g_i(x_1, \dots, x_n)] \mid x_1 g_i(x_1, \dots, x_n)\in I, \Sigma_{i=2}^n x_i g_i(x_1, \dots, x_n)\in I\}^T$.

Proof. (i) Let $\mathfrak{B}$ be the variety determined by the identities of T and let $G=M\lambda A$ be a split extension. Then $G\in\mathfrak{B}$ if and only if $f(c_1, \dots, c_n)=0$ for arbitrary $c_i=m_i+a_i$, $m_i\in M$, $a_i\in A$, $i=1, \dots, n$. But $f(c_1, \dots, c_n)=\Sigma_{i=2}^n [[m_i, a_1]-[m_1, a_i], h_i(a_1, \dots, a_n)]=\Sigma_{i=2}^n [m_i, a_1 h_i(a_1, \dots, a_n)]-[m_1, \Sigma_{i=2}^n a_i h_i(a_1, \dots, a_n)]$. Therefore $(G, M, A)\in\mathfrak{B}\sigma$ iff the element of the free metabelian bialgebra $g(y_1, \dots, y_n; x_1, \dots, x_n)=\Sigma_{i=2}^n y_i x_1 h_i(x_1, \dots, x_n)-y_1 \Sigma_{i=2}^n x_i h_i(x_1, \dots, x_n)$ vanishes upon substitution of arbitrary elements from (G, M, A). In other words the bivariety $\mathfrak{B}\sigma$ is determined by the identities $g(y_1, \dots, y_n; x_1, \dots, x_n)=0$. We finish the proof with the trivial observation that these identities are equivalent to a collection of identities $yx_1 h_i(x_1, \dots, x_n)=0$, $i=2, \dots, n$ $y\Sigma_{i=2}^n x_i h_i(x_1, \dots, x_n)=0$.

(ii) Denote $\mathfrak{M}$ the bivariety defined by the identities $yg(x_1, \dots, x_n)=0$ The polynomial $f=\Sigma_{i=2}^n [x_i, x_1, g_i(x_1, \dots, x_n)]$ is an identity for $\mathfrak{M}\tau$ iff $f(x_1, \dots, x_n)=0$ for arbitrary $G=M\lambda A$, where $(G, M, A)\in\mathfrak{M}$. Let $c_i=m_i+a_i\in G$ $m_i\in M$, $a_i\in A$, $i=1, \dots, n$. Then $f(c_1, \dots, c_n)=\Sigma_{i=2}^n [m_i, a_1, g_i(a_1, \dots, a_n)]-[m_1 \Sigma_{i=2}^n a_i g_i(a_1, \dots, a_n)]=0$. This is true for all $c_i\in G$, $(G, M, A)\in\mathfrak{M}$ iff $yx_1 g_i(x_1 \dots, x_n)=0$, $i=2, \dots, n$ and $y\Sigma_{i=2}^n x_i g_i(x_1, \dots, x_n)=0$ are identities for $\mathfrak{M}$, i. e $x_1 g_i(x_1, \dots, x_n)\in I(\mathfrak{M})$, $i=1, \dots, n$, $\Sigma_{i=2}^n x_i g_i(x_1, \dots, x_n)\in I(\mathfrak{M})$. The final remark is that $I(\mathfrak{M})=\{g(x_1, \dots, x_n)\}^I$.

Corollary 2.3. *Let* $\mathfrak{B}\subset\mathfrak{A}^2$ *be a variety generated by the split extensions of* $\mathfrak{B}$. *Then* $\mathfrak{B}\sigma\tau=\mathfrak{B}$.

Proof. Without loss of generality we may assume that $\mathfrak{B}$ is generated by one split extension $G=M\lambda A$. Let $\mathfrak{B}$ be determined by the identities $f(x_1, \dots, x_n)=\Sigma_{i=2}^n [x_i, x_1 g_i(x_1, \dots, x_n)]=0$, $f\in J$. By Proposition 2.2, $I(\mathfrak{B}\sigma)$ is generated by $x_1 g_i(x_1, \dots, x_n)$, $i=2, \dots, n$, $\Sigma_{i=2}^n x_i g_i(x_1, \dots, x_n)$. Assume $h(x_1, \dots, x_n)=\Sigma_{i=2}^n [x_i, x_1 h_i(x_1, \dots, x_n)]=0$ be an identity for $\mathfrak{B}\sigma\tau$. Using the relation $\sigma\tau\sigma=\sigma$ (Proposition 2.1 (iii)) we obtain that $x_1 h_i(x_1, \dots, x_n)$, $i=2, \dots, n$, $\Sigma_{i=2}^n x_i h_i(x_1, \dots, x_n)$ are in $I(\mathfrak{B}\sigma\tau\sigma)=I(\mathfrak{B}\sigma)$. For arbitrary $c_i=m_i+a_i\in G$, $m_i\in M$, $a\in A$, $i=1, \dots, n$, $h(c_1, \dots, c_n)=\Sigma_{i=2}^n [m_i, a_1 h_i(a_1, \dots, a_n)]-[m_1, \Sigma_{i=2}^n a_i h_i(a_1, \dots, a_n)]$. Having in mind that $x_1 h_i(x_1, \dots, x_n)$, $\Sigma_{i=2}^n x_i h_i(x_1, \dots, x_n)\in I(\mathfrak{B}\sigma)$ we establish that $G\in\mathfrak{B}\sigma\tau$ and therefore $\operatorname{var} G=\mathfrak{B}\subset\mathfrak{B}\sigma\tau$. By Proposition 2.1 (ii) the opposite inclusion $\mathfrak{B}\supset\mathfrak{B}\sigma\tau$ is valid. Hence $\mathfrak{B}=\mathfrak{B}\sigma\tau$.

Example 2.4. There exists a $\mathfrak{B}\subset\mathfrak{A}^2$ such that $\mathfrak{B}\sigma\tau\neq\mathfrak{B}$.

Proof. Let the base field has characteristic p and p divide n, $n>2$. Denote $\mathfrak{B}$ the subvariety of $\mathfrak{A}^2$ defined by the identity $f(x_1, \dots, x_n)=\Sigma_{i=2}^n [x_i, x_1, \dots, \widehat{x_i}, \dots, x_n]=0$ ($\widehat{\ }$ means that the corresponding variable is missing). Easy calculations show that $f(x_1, \ldots, x_n)=f(x_{i_1}, \ldots, x_{i_n})$ for arbitrary permutation $i_1, \ldots, i_n$ of $1, \ldots, n$ and the identity $[x_1, \ldots, x_{n+1}]=0$ is a consequence of $f(x_1, \ldots, x_n)=0$ but $[x_1, \ldots, x_n]=0$ is not. Hence $\mathfrak{N}_{n-1}\subset\mathfrak{B}$ and $\mathfrak{N}_{n-1}\neq\mathfrak{B}$. On the other hand, by Proposition 2.2 $I(\mathfrak{B}\sigma)$ is generated by $x_1 \ldots x_{n-1}$ and $\mathfrak{B}\sigma\tau=\mathfrak{N}_{n-1}$. Therefore $\mathfrak{B}\sigma\tau\neq\mathfrak{B}$.

Theorem 2.5. *Let $\mathfrak{M}\subset\mathfrak{A}\circ\mathfrak{A}$. Then there exists only a finite number of varieties $\mathfrak{B}\subset\mathfrak{A}^2$ such that $\mathfrak{B}\sigma=\mathfrak{M}$.*

Proof. Let $I(\mathfrak{M})$ be the GL-ideal of $K[X]$ related to $\mathfrak{M}$ and $I(\mathfrak{M})=\{g_1, \ldots, g_k\}^I$. Denote $\{\mathfrak{B}_j \mid j\in J\}$ the set of all $\mathfrak{B}_j$ such that $\mathfrak{B}_j\sigma=\mathfrak{M}$ and $V_j=T(\mathfrak{B}_j)$. We shall find two varieties $\mathfrak{M}_0$ and $\mathfrak{M}_1$ such that $\mathfrak{M}_0\supset\mathfrak{B}_j\supset\mathfrak{M}_1$ for all $j\in J$ and shall prove that there is a finite number varieties between $\mathfrak{M}_0$ and' $\mathfrak{M}_1$ only. Let $\mathfrak{M}_0$ be defined by all identities of the form $[x_{n+1}, x_{n+2}, f(x_1, \ldots, x_n)]$ when $f(x_1, \ldots, x_n)$ runs over $I(\mathfrak{M})$, $W_0=T(\mathfrak{M}_0)$. Clearly, the T-ideal W_0 is generated by $[x_{n+1}, x_{n+2}, g_i(x_1, \ldots, x_n)]$, $i=1, \ldots, k$. Let $V=V_j$ for a given $j\in J$, $V\sigma=I(\mathfrak{M})$. We shall prove that $V\supset W_0$. By Proposition 2.2, there exist $h_1, \ldots, h_r\in V$, $h_l=\Sigma_{i=2}^n [x_i, x_1, p_{i_l}(x_1, \ldots, x_n)]$ and

$$g_j\in\{x_1 p_{i_l}(x_1, \ldots, x_n), \Sigma_{i=2}^n x_i p_{i_l}(x_1, \ldots, x_n)\}^I.$$

We substitute $y_{i_0}=[z_1, z_2]+x_{i_0}$, $y_i=x_i$, $i\neq i_0$ in h_l and obtain

$$h_l(y_1, \ldots, y_n)-h_l(x_1, \ldots, x_n)=\begin{cases}[z_1, z_2, x_1 p_{i_l}(x_1, \ldots, x_n)], & i_0>1,\\ -\Sigma_{i=2}^n [z_1, z_2, x_i p_{i_l}(x_1, \ldots, x_n)], & i_0=1.\end{cases}$$

Therefore all generators of W_0 lie in V, $W_0\subset V$, and $\mathfrak{M}_0\supset\mathfrak{B}=\mathfrak{B}_j$.

Now, let $\mathfrak{M}_1=\mathfrak{M}\tau$, $W_1=T(\mathfrak{M}_1)=I(\mathfrak{M})\tau$. By Proposition 2.1 (ii) $\mathfrak{M}_1\subset\mathfrak{B}$, i. e. we found $\mathfrak{M}_0$ and $\mathfrak{M}_1$ such that $\mathfrak{M}_0\supset\mathfrak{B}\supset\mathfrak{M}_1$. The T-ideal W_1 is generated by a finite number of elements $f_i(x_1, \ldots, x_n)=\Sigma_j [x_j, x_1, q_{ij}(x_1, \ldots, x_n)]$ Using the metabelian law $[[x_1, x_2], [x_3, x_4]]=0$ we obtain that

$$[f_i(v_1, \ldots, v_n), x]=\Sigma_j [v_j, x, v_1 q_{ij}(v_1, \ldots, v_n)]-[v_1, x, \Sigma_j v_j q_{ij}(v_1, \ldots, v_n)], \quad v_k\in F(\mathfrak{A}^2).$$

Hence all consequences of f_i are linear combinations of $f_i(u_1+y_1, \ldots, u_n+y_n)$, where $u_k\in F^2(\mathfrak{A}^2)$ and y_k are polynomials of first degree. Any such consequence can be written in the form $w=\Sigma[u, h]+\Sigma\alpha_i f_i(y_1, \ldots, y_n)$. Here $u\in F^2(\mathfrak{A}^2)$, $h\in I(\mathfrak{M})$ and $\deg y_i=1$, $\alpha_i\in K$. Therefore $w\equiv\Sigma\alpha_i f_i(y_1, \ldots, y_n) \pmod{W_0}$. We have a finite number of polynomials $f_1, \ldots, f_k$ and substitute elements y_j of first degree instead of the indeterminates x_j. Let $\max(\deg f_i)=d$. Then the polynomial identity $f_i(y_1, \ldots, y_n)=0$ is equivalent to a finite set of identities $f_{ij}(x_1, \ldots, x_r)=0$, $r\leq d$, such that all indeterminates $x_1, \ldots, x_r$ enter any monomial of f_{ij}. The base field is finite and the number of all such polynomials $f_{ij}(x_1, \ldots, x_r)$ is bounded with the number of all polynomials from $F(\mathfrak{A}^2)$ in d variables and of degree $\leq d$. Hence, modulo W_0, any V, $W_1\supset V\supset W_0$, can be determined by a subset of the set $\{f_{ij}(x_1, \ldots, x_r)\}$. So, we have a finite number of possibilities for V. This completes the proof of the theorem.

Proposition 2.6. *Let G be a finite metabelian algebra. There exists a split extension G^* and a nilpotent algebra N such that* $\operatorname{var} G=\operatorname{var} G^* \cup \operatorname{var} N$.

Proof. By Corollary 8.3 [9] there exists a nilpotent subalgebra N of G, such that $G=G^2+N$. Let $c_1, \ldots, c_r$ be a linear basis of G modulo G^2 and $d_1, \ldots, d_s$ be a basis for G^2. Clearly, we may assume $c_1, \ldots, c_r\in N$. Then $[d_i, d_j]$

$=0$, $[c_i, c_j]$, $[d_i, c_j] \in G^2$ and $[d_i, c_j, d_k]=0$. We define a new algebra G^* with a basis $c_1^*, \ldots, c_r^*, d_1^*, \ldots, d_s^*$ and multiplication $[c_i^*, c_j^*]=[d_i^*, d_j^*]=0$ and $[d_i^*, c_j^*]$ $=\Sigma_k \gamma_{ijk} d_k^*$ if $[d_i, c_j]=\Sigma_k \gamma_{ijk} d_k$. We shall prove that $\operatorname{var} G = \operatorname{var} G^* \cup \operatorname{var} N$.

Let $\{f = \Sigma [x_i, x_1, f_i(x_1, \ldots, x_n)]\} \subset F(\mathfrak{A}^2)$ be a system of polynomial identities defining $\operatorname{var} G$. For arbitrary $e_i = a_i + b_i$, $a_i = \Sigma \alpha_{ij} c_j$, $b_i = \Sigma \beta_{ij} d_j$, $f(e_1, \ldots, e_n)=\Sigma [[b_i, a_1]+[a_i, b_1], f_i (a_1, \ldots, a_n)]+\Sigma [a_i, a_1, f_i (a_1, \ldots, a_n)]=0$.

But $\Sigma [a_i, a_1, f_i (a_1, \ldots, a_n)]=f(a_1, \ldots, a_n)=0$. Hence

$$\Sigma [[b_i, a_1]+[a_i, b_1], f_i (a_1, \ldots, a_n)]=0,$$

$$0=\Sigma [[b_i^*, a_1^*]+[a_i^*, b_1^*], f_i (a_1^*, \ldots, a_n^*)]=f(a_1^*+b_1^*, \ldots, a_n^*+b_n^*),$$

where $a_i^*=\Sigma\alpha_{ij} c_j^*$, $b_i^*=\Sigma\beta_{ij} d_j^*$. Consequently, $G^* \in \operatorname{var} G$. Together with trivial $N \in \operatorname{var} G$ we have $\operatorname{var} G \supset \operatorname{var} G^* \cup \operatorname{var} N$. Assume $\operatorname{var} G \neq \operatorname{var} G^* \cup \operatorname{var} N$. There exists a polynomial $g(x_1, \ldots, x_n) \in F(\mathfrak{A}^2)$ such that $g=0$ is a polynomial identity for G^* and N and $g(e_1, \ldots, e_n) \neq 0$ for suitable $e_1, \ldots, e_n \in G$. Again, let $e_i=a_i$ $+b_i$, $a_i \in N$, $b_i \in G^2$, $g(x_1, \ldots, x_n)=\Sigma [x_i, x_1, g_i (x_1, \ldots, x_n)]$. Then $0 \neq g(e_1, \ldots,$ $e_n)=\Sigma [[a_i, b_1]+[b_i, a_1], g_i (a_1, \ldots, a_n)]+g(a_1, \ldots, a_n)$. But $g(a_1, \ldots, a_n)=0$, because $a_1, \ldots, a_n \in N$. Hence $\Sigma [[a_i, b_1]+[b_i, a_1], g_i (a_1, \ldots, a_n)] \neq 0$ and as a consequence, $g(a_1^*+b_1^*, \ldots, a_n^*+b_n^*) \neq 0$. This contradicts to the assumption $g(x_1, \ldots, x_n)=0$ for G^*. Hence $\operatorname{var} G=\operatorname{var} G^* \cup \operatorname{var} N$.

Theorem 2.7. *Let $\mathfrak{B}$ be a metabelian variety. Then there exists a nilpotent variety $\mathfrak{N}$ such that $\mathfrak{B}=\mathfrak{B}\sigma\tau \cup \mathfrak{N}$.*

Proof. By Theorem 2.5 only a finite number of varieties lie between $\mathfrak{B}$ and $\mathfrak{B}\sigma\tau$. Let

$$\mathfrak{B}=\mathfrak{B}_0 \supset \mathfrak{B}_1 \supset \ldots \supset \mathfrak{B}_k=\mathfrak{B}\sigma\tau$$

be a chain of maximal length. There exist finitely generated algebras $G_1, \ldots, G_k$ such that $G_i \in \mathfrak{B}_{i-1} \setminus \mathfrak{B}_i$. Therefore $\mathfrak{B}_{i-1}=\mathfrak{B}_i \cup \operatorname{var} G_i$. The algebras G_i are residually finite [1]. Having in mind the maximality of the chain we may assume without loss of generality that G_i are finite. Then Proposition 2.6 gives that for suitable split extensions G_i^* and nilpotent algebras N_i

$$\mathfrak{B}=\mathfrak{B}\sigma\tau \cup \operatorname{var}(G_1^*, \ldots, G_k^*) \cup \operatorname{var}(N_1, \ldots, N_k).$$

By Corollary 2.3 $(\operatorname{var} G_i^*)\sigma\tau=\operatorname{var} G_i^*$ and $G_i^* \in \mathfrak{B}\sigma\tau$. Consequently we obtain $\mathfrak{B}=\mathfrak{B}\sigma\tau \cup \operatorname{var}(N_1, \ldots, N_k)$. Obviously $\mathfrak{N}=\operatorname{var}(N_1, \ldots, N_k)$ is nilpotent variety and $\mathfrak{B}=\mathfrak{B}\sigma\tau \cup \mathfrak{N}$.

The following theorem is analogous to Theorem 2.5.

Theorem 2.8. *Let $\mathfrak{B} \subset \mathfrak{A}^2$. There exists only a finite number subbivarieties $\mathfrak{M}_1, \ldots, \mathfrak{M}_k$ of $\mathfrak{A} \circ \mathfrak{A}$, such that $\mathfrak{M}_i\tau=\mathfrak{B}$.*

Proof. As in Theorem 2.5 we shall find bivarieties $\mathfrak{M}'$ and $\mathfrak{M}''$ such that there is a finite number of bivarieties between $\mathfrak{M}'$ and $\mathfrak{M}''$ and for any $\mathfrak{M}$ with $\mathfrak{M}\tau=\mathfrak{B}$, it holds $\mathfrak{M}' \subset \mathfrak{M} \subset \mathfrak{M}''$. Let $V=T(\mathfrak{B})$ and $V\sigma=S$ be the corresponding GL-ideal of $K[X]$. Fix $\mathfrak{M} \subset \mathfrak{A} \circ \mathfrak{A}$, $\mathfrak{M}\tau=\mathfrak{B}$. Clearly $\mathfrak{M} \subset \mathfrak{M}\tau\sigma=\mathfrak{B}\sigma$ and $I(\mathfrak{M}) \supset I(\mathfrak{B}\sigma)=S$. Denote $S'=\{f(x_1, \ldots, x_n) \mid x_{n+1} f(x_1, \ldots, x_n) \in S\}'$ and $\mathfrak{M}'$ the bivariety determined by the identities from S'. If $h(x_1, \ldots, x_n) \in I(\mathfrak{M})$, then $x_{n+1} h(x_1, \ldots, x_n) \in I(\mathfrak{M})$ too, $[y_1, y_2, h(x_1, \ldots, x_n)] \in V$, $x_{n+1} h(x_1, \ldots, x_n) \in S$ and $h(x_1, \ldots, x_n) \in S'$. Hence $I(\mathfrak{M}) \subset S'$ and $\mathfrak{M}' \subset \mathfrak{M}$. Now we take $\mathfrak{M}''=\mathfrak{B}\sigma$ and have $\mathfrak{M} \subset \mathfrak{M}''$.

Let $f_1, \dots, f_m$ be generators of S' as a GL-ideal and d be the maximal degree of these polynomials. Clearly, modulo S, the GL-ideal $I(\mathfrak{M})$ is generated by some polynomials of degree $\leq d$. But any identity of degree $\leq d$ is equivalent to a collection of identities in d variables $x_1, \dots, x_d$. Therefore, $I(\mathfrak{M})$ is defined modulo S by a subset of the finite set of all ordinary polynomials in d variables of degree $\leq d$. Hence there is a finite number of possibilities for $I(\mathfrak{M})$ only.

The proof of the following corollary is similar to the proof of Theorem 2.7 and makes use of Proposition 1.9.

Corollary 2.9. *For any bivariety* $\mathfrak{M}\subset\mathfrak{A}\circ\mathfrak{A}$ *there exists a finite bialgebra* (G, M, A) *such that* $\mathfrak{M}\tau\sigma=\mathfrak{M}\cup \text{bivar}\,(G, M, A)$.

3. **Varieties generated by wreath products.** The aim of this section is to describe the identities of the bialgebra related to the wreath product $A_1 \operatorname{wr} A_k$ and to show that any metabelian variety is contained in $\operatorname{var}(A_1 \operatorname{wr} A_k)$ for a suitable integer k.

Let the base field has q elements. We shall use the equalities $\alpha^{q^r}=\alpha$ and $(\alpha x+\beta y)^{q^r}=\alpha x^{q^r}+\beta y^{q^r}$, $\alpha, \beta \in K$. For any positive integer k we fix the polynomial

$$\varphi_k(x_0, \dots, x_k)=\Sigma\,(\text{sign}\,\sigma)\, x_{\sigma(0)}^{q^k} \cdots x_{\sigma(k-1)}^{q}\, x_{\sigma(k)},$$

where the summation is on all permutations σ of $0, 1, \dots, k$. As a function, φ_k is linear in any variable and skew symmetric, i. e. $\varphi_k(x_0, \dots, \alpha x_i+\beta y_i, \dots, x_k)=\alpha\varphi_k(x_0, \dots, x_i, \dots, x_k)+\beta\varphi_k(x_0, \dots, y_i, \dots, x_k)$, $\alpha, \beta\in K$, $i=0, 1, \dots, k$, and $\varphi_k(x_0, \dots, x_k)=0$ when $i\neq j$ and $x_i=x_j$.

Lemma 3.1. *The polynomial* φ_k *has the property*

$$\varphi_1(x_0, x_1)=x_1\,\Pi\,(x_0-\alpha x_1),\ \alpha\in K,$$

$$\varphi_k(x_0, \dots, x_k)=\varphi_{k-1}(x_1, \dots, x_k)\,\Pi\,(x_0-\alpha_1 x_1-\dots-\alpha_k x_k),$$

where the multiplication runs over all k-triples $(\alpha_1, \dots, \alpha_k)$ *of elements of* K.

Therefore, up to a multiplicative constant, φ_k is the only polynomial of minimal degree, being divisible by all non-zero sums $\alpha_0 x_0+\dots+\alpha_k x_k$, $\alpha_i\in K$.

Proof. All non-proportional linear combinations are $(q^{k+1}-1)/(q-1)$ and this number equals the degree of φ_k. Clearly, if $x_i=\alpha_{i+1}x_{i+1}+\dots+\alpha_k x_k$ (or $x_k=0$), then $\varphi_k(x_0, \dots, x_i, \dots, x_k)=0$ and the polynomials $x_i-\alpha_{i+1}x_{i+1}-\dots-\alpha_k x_k$ divide φ_k. Hence

$$\varphi_k(x_0, \dots, x_k)=\alpha\varphi_{k-1}(x_1, \dots, x_k)\,\Pi\,(x_0-\alpha_1 x_1-\dots-\alpha_k x_k).$$

Comparing the coefficients of $x_0^{q^k}$ and by induction on k we obtain that $\alpha=1$.

Proposition 3.2. *Let* $f(x_0, \dots, x_n)\in K[X]$ *and* $(f)^V\cap K[x_1, \dots, x_k]=0$. *Then* $f\in(\varphi_k(x_0, \dots, x_k))^V$.

Proof. The condition $(f)^V\cap K[x_1, \dots, x_k]=0$ means that any substitution of $x_0, \dots, x_n$ with linear combinations of $x_1, \dots, x_k$ turns f into 0. On the other hand, φ_k is a multilinear and skew symmetric function and

$$(\varphi_k)^V=\{\Sigma\varphi_k(x_{i_0}, \dots, x_{i_k})\,a_i\,(x_{j_1}, \dots, x_{j_m})\mid i_0<\dots<i_k,\ a_i\in K[X]\}.$$

We shall use an induction on k and n $(k\leq n)$. Let $n=k$. Then $f(x_0, \dots, x_i, \dots, x_k)=0$ for arbitrary substitutions $x_i=\Sigma_{j\neq i}\alpha_j x_j$, i. e. $(x_i-\Sigma\alpha_j x_j)\mid f$. By Lemma 3.1 $\varphi_k(x_0, \dots, x_k)\mid f(x_0, \dots, x_n)$ and $f\in(\varphi_k)^V$.

Now, let $n>k$. Rewrite f in the form $f=\Sigma\, x_0^s f_s(x_1, \dots, x_n)$, $s\geqq 0$. Substituting $x_1, \dots, x_n$ with linear combinations of the variables $y_1, \dots, y_{k-1}$ we obtain 0. By induction on k, $f_s(x_1, \dots, x_n) \in (\varphi_{k-1})^I$ and

$$f_s=\Sigma\varphi_{k-1}(x_{i_1}, \dots, x_{i_k}), g_i(x_1, \dots, x_n)$$

for suitable $g_i \in K[x_1, \dots, x_n]$. Hence

$$f=\Sigma\varphi_{k-1}(x_{i_1}, \dots, x_{i_k})\, h_i(x_0, \dots, x_n),\ 0<i_i<\dots<i_k.$$

Consider the polynomials h_i as polynomials in x_0 and divide them by $\Pi(x_0 - \alpha_1 x_{i_1} - \dots - \alpha_k x_{i_k})$, $\alpha_i \in K$. Therefore

$$f=\Sigma\varphi_{k-1}(x_{i_1}, \dots, x_{i_k})\,(\Pi(x_0-\alpha_1 x_{i_1}-\dots-\alpha_k x_{i_k})\, b_i(x_0, \dots, x_n)$$
$$+c_i(x_0, \dots, x_n)),$$

where $\deg_{x_0} c_i<q^k$,

$$f=\Sigma\varphi_k(x_0, x_{i_1}, \dots, x_{i_k})\, b_i(x_0, \dots, x_n)+g(x_0, \dots, x_n),$$
$$g(x_0, \dots, x_n)=\Sigma x_0^r\, d_r(x_1, \dots, x_n),\ r<q^k.$$

By assumption $(f)^I \cap K[x_1, \dots, x_k]=0$. Obviously the same holds for g In order to complete the proof we have to establish that $g \in (\varphi_k)^I$. Let $d_r(x_1 \dots, x_n) \notin (\varphi_k)^I$ for some r. By inductive arguments on n, there exists a substitution $\bar{x}_i=\Sigma_{j=1}^k \beta_{ij} y_j$, such that $d_r(\bar{x}_1, \dots, \bar{x}_n)=e_r(y_1, \dots, y_k)\neq 0$. Hence $g(x_0, \bar{x}_1, \dots, \bar{x}_n)=h(x_0, y_1, \dots, y_k)=\Sigma x_0^r e_r(y_1, \dots, y_k)\neq 0$. Because of $\deg_{x_0} g<q^k$, there is a linear combination $x_0-\alpha_1 y_1-\dots-\alpha_k y_k$ which does not divide $h(x_0, y_1, \dots, y_k)$. So, for $\bar{x}_0=\alpha_1 y_1+\dots+\alpha_k y_k$, $g(\bar{x}_0, \bar{x}_1, \dots, \bar{x}_n)\neq 0$ and $g(\bar{x}_0, \dots, \bar{x}_n) \in K[y_1, \dots, y_k]$. This contradicts to the assumption $(f)^I \cap K[x_1, \dots, x_k]=0$. Therefore $d_r(x_1, \dots, x_n) \in (\varphi_k)^I$ and $g \in (\varphi_k)^I$, too.

Let $G_k=A_1 \operatorname{wr} A_k$ be the wreath product of the abelian algebras $A_1=A(z)$ and $A_k=A(t_1, \dots, t_k)$. Then $G_k=M_a(z)\lambda A_k$.

T h e o r e m 3.3. *Let* $I_k=I(\operatorname{bivar}(G_k, M_a(z), A_k))$. *Then* $I_k=(\varphi_k(x_0, \dots, x_k))^I$, *i. e. any identity for the bialgebra* $(G_k, M_a(z), A_k)$ *is a consequence of*

$$y\varphi_k(x_0, \dots, x_k)=y\Sigma\,(\operatorname{sign}\sigma)\, x_{\sigma(0)}^{q^k}, \dots x_{\sigma(k-1)}^{q}\, x_{\sigma(k)}=0.$$

P r o o f. For any $yf(x_1, \dots, x_n)$ and for all substitutions

$$\bar{x}_i=\Sigma_{j=1}^k \beta_{ij} t_j, \bar{y}=z, \bar{y}\, f(\bar{x}_1, \dots, \bar{x}_n)=zg(t_1, \dots, t_k) \text{ for}$$

suitable $g \in K[t_1, \dots, t_k]$ and $yf=0$ if and only if $g(x_1, \dots, x_k)=0$. Therefore $(f)^I \cap K[k_1, \dots, x_k]=0$ and by Proposition 3.2 $f \in (\varphi_k)^I$, i. e. $I_k \subset (\varphi_k)^I$. On the other hand, the polynomials $\bar{x}_i=\Sigma_{j=1}\beta_{ij} t_j$, $i=0, \dots, k$, are linearly dependent and $z\,\varphi_k(\bar{x}_0, \dots, \bar{x}_k)=0$, i. e. $\varphi_k \in I_k$. So, $I_k=(\varphi_k)^I$.

C o r o l l a r y 3. 4. *The variety* $\mathfrak{W}_k=\operatorname{var}(A_1 \operatorname{wr} A_k)$ *is determined by a system of polynomial identities of the form* $\Sigma_{i=2}^n [x_i, x_1, f_i(x_1, \dots, x_n)]=0$ *such that* $x_1 f_i(x_1, \dots, x_n)$, $\Sigma_{i=2}^n x_i f_i(x_1, \dots, x_n) \in (\varphi_k)^I$.

P r o o f. The algebra $A_1 \operatorname{wr} A_k$ is a split extension and by Corollary 2.3 $\mathfrak{W}_k\sigma\tau=\mathfrak{W}$. The proof follows immediately from Theorem 3.3 and Proposition 2.2.

Remark 3.5. Vaughan-Lee [10] has shown that the algebra $A_1 \,\mathrm{wr}\, A_1$ has a basis for its polynomial identities

$$[x_2, x_3, x_1^q]+[x_3, x_1, x_2^q]+[x_1, x_2, x_3^q]=0,$$

$$[x_2, x_3, x_1, x_2^{q-1} x_3^{q-1}]+[x_3, x_1, x_2, x_3^{q-1} x_1^{q-1}]+[x_1, x_2, x_3, x_1^{q-1} x_2^{q-1}]=0.$$

Theorem 3.6. *Let $\mathfrak{V}$ be a variety, $\mathfrak{V}\subset\mathfrak{A}^2$ and $\mathfrak{V}\neq\mathfrak{A}^2$. Then there exists a positive integer k such that $\mathfrak{V}\subset\mathfrak{W}_k=\mathrm{var}\,(A_1 \,\mathrm{wr}\, A_k)$.*

Proof. Let the base field be of characteristic p, $\mathfrak{V}\subset\mathfrak{A}^2$, $\mathfrak{V}\neq\mathfrak{A}^2$ and let $I=I(\mathfrak{V}\sigma)$ be the GL-ideal of $K[X]$ related to $\mathfrak{V}\sigma$. Clearly, $I\neq 0$. Recall that a polynomial f is said to be a p-polynomial if any monomial of f is of degree p^k in any variable.

Step 1. There exists a non-zero p-polynomial in I.

Proof. Let $0\neq f(x_0)=f(x_0, x_1, \dots, x_n)\in I$, $f(x_0)=\Sigma x_0^r f_r(x_1, \dots, x_n)$. We begin a process of linearization in x_0, i. e. consider $g(y_1, y_2)=f(y_1+y_2)-f(y_1)-f(y_2)=\Sigma_r \Sigma_{s=1}^{r-1}\binom{r}{s} y_1^s y_2^{r-s} f_r(x_1, \dots, x_n)$. Obviously $g\neq 0$ if $\binom{r}{s}\neq 0$ for some s, and in this case $\deg_{y_1} g<\deg_{x_0} f$. Therefore the linearization process gives the zero polynomial and stops only when $\binom{r}{s}=0$, $s=1, \dots, r-1$, i. e. when $r=p^k$.

Step 2. The polynomial $\varphi_k(x_0, \dots, x_k)$ belongs to I for a suitable integer k.

Proof. Let $\Phi_k(x_0, \dots, x_k)=\Sigma\,(\mathrm{sign}\,\sigma)\, x_{\sigma(0)}^{p^k} \dots x_{\sigma(k-1)}^{p} x_{\sigma(k)}$, i. e. Φ_k is an analog of φ_k for the prime field Z_p. Consider an arbitrary p-polynomial from I

$$f(x, y, \dots, z)=x^{p^k} f_k(y, \dots, z)+ \dots +x^p f_1(y, \dots, z)+x f_0(y, \dots, z).$$

Because of the skew symmetry of Φ_k we obtain that

$$\Sigma\,(\mathrm{sign}\,\sigma)\, f(x_{\sigma(0)}, y, \dots, z)\, x_{\sigma(1)}^{p^{k-1}} \dots x_{\sigma(k-1)}^{p}\, x_{\sigma(k)}=\Phi_k(x_0, \dots, x_k)\, f_k(y, \dots, z)\in I$$

Following this way, we establish that

$$\Phi_k(x_0, \dots, x_k)\, \Phi_l(y_0, \dots, y_l) \dots \Phi_m(z_0, \dots, z_m)\in I.$$

By Lemma 3.1 the polynomials Φ_i are products of different linear factors and divide $\varphi_n(x_0, \dots, x_k, y_0, \dots, y_l, \dots, z_0, \dots, z_m)$, where $n+1=(k+1)+(l+1)+\dots+(m+1)$. Hence $\varphi_n\in I$.

Step 3 (proof of the theorem). By Theorem 2.7 $\mathfrak{V}=\mathfrak{V}\sigma\tau\cup\mathfrak{N}$ for a nilpotent variety $\mathfrak{N}$. There exists an integer n such that $\varphi_n(x_0, \dots, x_n)\in I(\mathfrak{V}\sigma)$. The varieties $\mathfrak{V}\sigma\tau$ and $\mathfrak{W}_n$ are generated by split extensions and $I(\mathfrak{W}_n\sigma)=(\varphi_n)^T$. Hence $I(\mathfrak{V}\sigma)\supset I(\mathfrak{W}_n\sigma)$ and $\mathfrak{V}\sigma\tau\subset\mathfrak{W}_n\sigma\tau=\mathfrak{W}_n$. On the other hand, $\mathfrak{N}$ satisfies the identity $[x_1, \dots, x_m]=0$ and, consequently, all identities of higher degree. By Corollary 3.4, the identities of $\mathfrak{W}_k$ are of degree $\geq(q^{k+1}-1)(q-1)^{-1}-1$. Therefore, for k large enough we have both $\mathfrak{N}\subset\mathfrak{W}_k$ and $\mathfrak{V}\sigma\tau\subset\mathfrak{W}_k$, i. e. $\mathfrak{V}\subset\mathfrak{W}_k$. This completes the proof of the theorem.

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