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ПЛИСКА

БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

AN AXIOM ON GENERALIZED BOCHNER FLAT SUBMANIFOLDS IN ALMOST HERMITIAN MANIFOLDS

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Totally quasiumbilical σ-submanifolds in almost Hermitian manifolds are studied. A characterization of generalized Bochner flat almost Hermitian manifolds is given. Kählerian manifolds which are Bochner flat are also investigated.

1. Introduction. On the axioms of submanifolds in Riemannian geometry the results of Cartan [1], Leung and Nomizu [7], Lindt and Verstraelen [8], Schouten [11], Yano and Mutô [15] are well known.

In the almost Hermitian geometry the following theorems are known.

Theorem A [13]. *Let M be a $2m$-dimensional RK-manifold. If M satisfies the axiom of antiholomorphic p-planes (p-spheres) for some p, $1 \leq p \leq m$, then M is with pointwise constant holomorphic sectional curvature.*

Theorem B [13]. *Let M be a $2m$-dimensional RK-manifold. If M satisfies the axiom of holomorphic $2p$-spheres ($2p$-planes) for some p, $1 \leq p < m$, then M is with pointwise constant holomorphic sectional curvature.*

Theorem C [13]. *Let M be a $2m$-dimensional connected NK-manifold or a parakählerian manifold. If M satisfies the axiom of antiholomorphic (holomorphic) p-spheres, for some p $(1 \leq p < m)$ and $\dim M \geq 6$, then M is a complex-space-form.*

A summary of these results has been made by Kassabov [6] in following lines:

Theorem D. *Let M be a $2m$-dimensional almost Hermitian manifold, $m \geq 2$. If M satisfies the axiom of holomorphic $2n$-planes ($2n$-spheres) for some n, $2 \leq n < m$, then M is an RK-manifold with pointwise constant holomorphic sectional curvature.*

Theorem E. *Let M be a $2m$-dimensional connected almost Hermitian manifold, $m \geq 3$. If M satisfies the axiom of antiholomorphic n-planes or the axiom of antiholomorphic n-spheres for some n, $2 \leq n \leq m$, then M is one of the following:*

1) *a real-space-form, or*

2) *a complex-space-form.*

Theorem F [9]. *If the Kählerian manifold M of real dimension $2m \geq 6$ satisfies the axiom of holomorphic (antiholomorphic) $2n$-planes and $1 \leq n \leq m-1$ (respectively $2 \leq n \leq m$), then M is a complex-space-form.*

Theorem G [3]. *A Kählerian manifold of real dimension $2m \geq 6$ with complex structure J is a complex-space-form if and only if it satisfies the axiom of $J\xi$-quasiumbilical hypersurfaces, where ξ is the hypersurface normal.*

Theorem H [8]. *A Kählerian manifold of real dimension $2m \geq 6$ with complex structure J is flat if and only if it satisfies the axiom of $J\xi$-hypercylinders, where ξ is the hypercylinder normal.*

Our aim is to study totally quasiumbilical submanifolds in almost Hermitian and Kählerian manifolds and axioms related to them.

2. Preliminaries. Let $\tilde{M}$ be a $2n$-dimensional almost Hermitian manifold with metric $\tilde{g}$, almost complex structure $\tilde{J}$, Levi — Civita connection $\tilde{\nabla}$ and curvature tensor $\widetilde{R}$. Let M be a complex submanifold of $\widetilde{M}$ of real dimension $2m$ $(1\leqq m\leqq n-1)$ with induced metric tensor g, almost complex structure J, Levi — Civita connection ∇ and curvature tensor R.

For each point $p\in M$, let $U(p)$ be a neighbourhood of p, and $\{\xi_i, \widetilde{J}\xi_i\,;\, i=1,2,\dots,n-m\}$, be mutually orthogonal unit vector fields, normal to M. Then from the equation of Gauss $\widetilde{\nabla}_x y=\nabla_x y+\sigma(x, y)$ for arbitrary vector fields $x, y\in\mathfrak{X}(M)$, we have

$$\sigma(x, y)=\sum_{i=1}^{n-m}(h_i(x, y)\xi_i+k_i(x, y)\widetilde{J}\xi_i), \tag{1}$$

where $h_i(x, y)$ and $k_i(x, y)$ for each fixed i are symmetric covariant tensor field of type (0,2) on $U(p)$.

D e f i n i t i o n [10]. *A 2m-dimensional invariant submanifold M of $\widetilde{M}$ is said to be a σ-submanifold if the second fundamental form σ is complex bilinear, i. e.*

$$\widetilde{J}\sigma(x, y)=\sigma(Jx, y)=\sigma(x, Jy), \tag{2}$$

for arbitrary vector fields $x, y\in\mathfrak{X}(M)$.

In [3] and [12] with every LC-tensor R is associated a K-tensor R^*, which has the same holomorphic sectional curvatures as R. The tensor R^* is expressed by R

$$\begin{aligned} R^*(x, y, z, u)=\frac{3}{16}(&R(x, y, z, u)+R(x, y, Jz, Ju)+R(Jx, Jy, z, u)\\ &+R(Jx, Jy, Jz, Ju))+\frac{1}{16}(R(Jx, Jz, y, u)+R(x, z, Jy, Ju)\\ &-R(Jy, Jz, x, u)-R(y, z, Jx, Ju)+R(y, Jz, Jx, u)\\ &+R(Jy, z, x, Ju)-R(x, Jz, Jy, u)-R(Jx, z, y, Ju)). \end{aligned} \tag{3}$$

From the formula (3) under condition (2) and the equation of Gauss

$$\widetilde{R}(x, y, z, u)=R(x, y, z, u)+\widetilde{g}(\sigma(x, z), \sigma(y, u))-\widetilde{g}(\sigma(x, u), \sigma(y, z)) \tag{4}$$

as a result of long calculations we find the following equation connecting the tensors $\widetilde{R}^*$ of $\widetilde{M}$ and R^* of an arbitrary σ-submanifold M of $\widetilde{M}$:

$$\widetilde{R}^*(x, y, z, u)=R^*(x, y, z, u)+\widetilde{g}(\sigma(x, z), \sigma(y, u)-\widetilde{g}(\sigma(x, u), \sigma(y, z)). \tag{5}$$

The generalized Bochner tensor ([4], [12]) associated with R^* is

$$\begin{aligned} B^*(x, y, z, u)=R^*(x, y, z, u)-\frac{1}{2(m+2)}\{&g(y, z)S^*(x, u)+g(x, u)S^*(y, z)\\ &-g(x, z)S^*(y, u)-g(y, u)S^*(x, z)+g(Jy, z)S^*(Jx, u)\\ &+g(Jx, u)S^*(Jy, z)-g(Jx, z)\,S^*(Jy, u)-g(Jy, u)S^*(Jx, z)\\ &-2g(Jx, y)S^*(Jz, u)-2g(Jz, u)S^*(Jx, y)\}\\ &+\frac{S^*_R(p)}{4(m+1)(m+2)}\{g(y, z)g(x, u)-g(x, z)g(y, u)+g(Jy, z)g(Jx, u)\\ &-g(Jx, z)g(Jy, u)-2g(Jx, y)g(Jz, u)\}, \end{aligned} \tag{6}$$

where $S^*(x, y)$ and $S^*_R(p)$ denotes the Ricci tensor and the scalar curvature with respect to the tensor R^*.

Definition [2]. *A submanifold M of $\tilde{M}$ is said to be totally quasiumbilical if there exist functions α_i, $\bar{\alpha}_i$, β_i, $\bar{\beta}_i$ and unit 1-forms w_i, $\bar{w}_i$ $(i=1, 2, \ldots, n-m)$,* such that

$$h_i=\alpha_i g+\beta_i w_i \otimes w_i, \tag{7}$$

$$k_i=\bar{\alpha}_i g+\bar{\beta}_i \bar{w}_i \otimes \bar{w}_i,$$

in $U(p)$. In particular, if $\alpha_i=\bar{\alpha}_i=0$ for each i, then $\tilde{M}$ is said to be totally cylindrical submanifold of $\tilde{M}$; if $\alpha_i=\bar{\alpha}_i=\beta_i=\bar{\beta}_i=0$ for each i, then M is said to be totally geodesic submanifold of $\tilde{M}$ and if $\beta_i=\bar{\beta}_i=0$ for each i, then M is said to be totally umbilical.

3. Generalized Bochner flat almost Hermitian manifolds. We shall call an almost Hermitian manifold generalized Bochner flat if the generalized Bochner curvature tensor is zero. The following Lemma is valid for such manifold:

Lemma. *An almost Hermitian manifold M of real dimensional $2m\geqq 8$ is generalized Bochner flat if and only if $R^*(x, y, z, u)=0$ for every orthonormal antiholomorphic quadruple in every point.*

Proof. From (6) it follows that $R^*(x, y, z, u)=0$ for every orthonormal antiholomorphic quadruple.
The opposite follows as in Theorem 12 [14].

Definition. *An almost Hermitian manifold $\tilde{M}$ of real dimension $2n>8$ is said to satisfy the axiom of generalized Bochner flat totally quasiumbilical σ-submanifolds if for every $p\in\tilde{M}$ and any $2m$-dimensional section H in the tangential space $T_p(\tilde{M})$, $4\leq m<n$ there exists a generalized Bochner flat totally quasiumbilical σ-submanifold M passing through p such that $T_p(M)=H$.*

Theorem 1. *If an almost Hermitian manifold $\tilde{M}$ of real dimension $2n>8$ satisfies the axiom of generalized Bochner flat totally quasiumbilical σ-submanifolds, then it is generalized Bochner flat.*

Proof. *From* (5) and (7) for every totally quasiumbilical σ-submanifold M of $\tilde{M}$ we have

$$\begin{aligned}\tilde{R}^*(x, y, z, u)=R^*(x, y, z, u)+\sum_{i=1}^{n-m}\{&(\alpha_i^2+\bar{\alpha}_i^2)(g(x, z)g(y, z)-g(x, u)g(y, z))\\ &+(\alpha_i\bar{\alpha}_i+\beta_i\bar{\beta}_i)[g(x, z)(w_i(y)w_i(u)+\bar{w}_i(y)\bar{w}_i(u))\\ &+g(y, u)(w_i(x)w_i(z)+\bar{w}_i(x)\bar{w}_i(z))-g(x, u)(w_i(y)w_i(z)+\bar{w}_i(y)\bar{w}_i(z))\\ &-g(y, z)(w_i(x)w_i(u)+\bar{w}_i(x)\bar{w}_i(u))]\},\end{aligned} \tag{8}$$

where x, y, z, u are arbitrary vector fields tangent to M. Now we assume that $\tilde{M}$ satisfies the axiom of generalized Bochner flat totally quasiumbilical σ-submanifolds and $2m\geqq 8$. Then from the Lemma for every orthonormal antiholomorphic quadruple x, y, z, u in $p\in M$ (8) implies

$$\tilde{R}^*(x, y, z, u)=0.$$

Now it follows from the Lemma that $\tilde{M}$ is generalized Bochner flat.

Corollary. *If a Kählerian manifold $\tilde{M}$ of real dimension $2n>8$ satisfies the axiom of Bochner flat totally quasiumbilical submanifolds, then it is Bochner flat.*

The following Theorem is valid for totally quasiumbilical σ-submanifolds M in generalized Bochner flat almost Hermitian manifold $\tilde{M}$.

Theorem 2. *Every totally quasiumbilical σ-submanifold M of real dimension 2m>6 in a generalized Bochner flat almost Hermitian manifold is generalized Bochner flat.*

The proof follows from (7), (5) and $\tilde{B}^*\equiv 0$ by a direct computation.

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