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INFINITESIMAL DEFORMATIONS OF THE SCALAR CURVATURE OF A HYPERSURFACE

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The purpose of the present paper is to study the infinitesimal normal parallel deformations (INPD) which preserve the scalar curvature of a hypersurface M^n in a space M^{n+1} of constant curvature c. We obtain a necessary and sufficient condition in which a totally umbilical hypersurface M^n admits INPD preserving the scalar curvature and find some inequalities which the square of the length of the second fundamental tensor and the mean curvature of an arbitrary hypersurface M^n satisfy when M^n admits INPD preserving the scalar curvature as well as some conditions in which M^n is quasiumbilical. We prove Theorem 4.4 using one result of S. Chern, M. do Carmo and S. Kobayashi and H. B. Lawson (for hypersurface). We also find a sufficient condition in which a minimal hypersurface M^n does not admit INPD preserving the scalar curvature.

1. Infinitesimal deformations of the scalar curvature. Let M^{n+1} be an $n+1$-dimensional Riemannian manifold, covered by a system of coordinate neighbourhoods $\{U, x^i\}$ and denote by g_{ij}, Γ^k_{ij}, ∇_i, R^s_{ijk} and R_{ij} the metric tensor, the Christoffel symbols formed with g_{ij}, the operator of covariant differentiation with respect to Γ^s_{ij}, the curvature tensor and the Ricci tensor of M^{n+1} respectively, where, here and in the sequel, the indices $i, j, k, \dots$ run over the range $\{1, 2, \dots, n+1\}$. Let M^n be an n-dimensional Riemannian manifold, covered by a system of coordinate neighbourhoods $\{V, u^\alpha\}$, $\alpha=1, 2, \dots, n$ and immersed as a hypersurface in M^{n+1}. The analytic representation of M^n is: $x^i=x^i(u^\alpha)$. We denote by $g_{\alpha\beta}$, $\Gamma^\tau_{\alpha\beta}$, ∇_α, $R^\tau_{\alpha\beta\gamma}$, $R_{\alpha\beta}$ the metric tensor the Christoffel symbols formed with $g_{\alpha\beta}$, the operator of covariant differentiation with respect to $\Gamma^\tau_{\alpha\beta}$, the curvature tensor and the Ricci tensor of M^n.

An infinitesimal deformation of a hypersurface M^n of M^{n+1} is said to be parallel when the tangent space at a point of M^n and that at the corresponding point of the deformed hypersurface are parallel with respect to the connection Γ^k_{ij} [10], and an infinitesimal deformation is said to be normal when the deformation vector is normal to the hypersurface [10].

If we denote by $\partial x^i/\partial u^\alpha - n$ linearly independent vectors of M^{n+1} tangent to M^n and by N^i the unit normal to M^n, then the vector field z^i of an infinitesimal deformation of M^n can be represented as:

$$z^i=\xi^\alpha\frac{\partial x^i}{\partial u^\alpha}+\lambda N^i, \tag{1.1}$$

where ξ^α and $\lambda(u^\alpha)$ are respectively tangential and normal components of z^i.

When $\xi^\alpha\equiv 0$ the deformation vector z^i is normal to M^n and the deformation of M^n is normal. A normal deformation of a hypersurface is given by

$$\bar{x}^i=x^i+\lambda N^i\varepsilon. \tag{1.2}$$

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We have

Proposition 1.1 [10]. *In order for a normal deformation of a hypersurface to be parallel, it is necessary and sufficient that*

(1.3) $$\nabla_\alpha \lambda = 0$$

that is, the normal deformation displaces each point of the hypersurface by the same distance.

When we have an infinitesimal normal parallel deformation (INPD) of M^n the variation δK of the scalar curvature $K = R_{\alpha\beta} g^{\alpha\beta}$ is as follows [4]:

(1.4) $$\delta K = (-2h^{\alpha\beta} R_{\alpha\beta} - 2\nabla^\alpha \nabla_\beta h^\beta_\alpha + 2\nabla_\alpha \nabla^\alpha h)\lambda\varepsilon,$$

where $h_{\alpha\beta}$ is the second fundamental tensor of M^n and $h^\alpha_\beta = h_{\beta\tau} g^{\tau\beta}$, $h^{\alpha\beta} = h_{\tau\gamma} g^{\tau\alpha} g^{\beta\gamma}$, $h = h_{\alpha\beta} g^{\alpha\beta}$; ε is an infinitesimal.

If the ambient manifold M^{n+1} is a space of constant curvature c, then:

(1.5) $$\delta K = -2[c(n-1)h + h h_{\alpha\beta} h^{\alpha\beta} - h_{\alpha\tau} h^\tau_\beta h^{\alpha\beta}]\lambda\varepsilon.$$

Proposition 1.2. *An INPD:* $\bar{x}^i = x^i + \lambda N^i \varepsilon$, $\lambda > 0$, *of a hypersurface* M^n *of a space* M^{n+1} *of constant curvature c preserves the scalar curvature K of* M^n *if and only if*

(1.6) $$h_{\alpha\beta} h^\alpha_\tau h^{\tau\beta} = c(n-1)h + h \,.\, h_{\alpha\beta} h^{\alpha\beta}.$$

We will consider only non-totally geodesic hypersurfaces.

2. Totally umbilical hypersurfaces

Proposition 2.1. *A totally umbilical hypersurface* M^n *of a space* M^{n+1} *of constant curvature c admits INPD preserving the scalar curvature if and only if the mean curvature of* M^n *is equal to* $\pm\sqrt{-c}$.

Proof. As M^n is totally umbilical hypersurface, then the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the second fundamental tensor $h_{\alpha\beta}$ are equal, i. e.

(2.1) $$\lambda_1 = \lambda_2 = \ldots = \lambda_n.$$

For a totally umbilical hypersurface we have:

(2.2) $$\lambda_\alpha = \frac{h}{n}, \quad h_{\alpha\beta} h^{\alpha\beta} = \frac{h^2}{n}, \quad h_{\alpha\tau} h^\tau_\beta h^{\alpha\beta} = \frac{h^3}{n^2}.$$

Let M^n admits INPD preserving the scalar curvature K. Then $h_{\alpha\beta}$ and h satisfy (1.6). Substituting (2.2) into (1.6), we obtain

(2.3) $$c(n-1)h + \frac{h^3}{n} = \frac{h^3}{n^2}$$

or

(2.4) $$h = \pm n\sqrt{-c}.$$

Conversely, if (2.4) is fulfilled, we have (1.6) because of (2.2) and consequently the deformation preserves the scalar curvature.

Corollary. *A totally umbilical hypersurface* M^n *of a space* M^{n+1} *of positive curvature c does not admit INPD preserving the scalar curvature.*

3. Non-totally umbilical and non-minimal hypersurfaces. First we prove the following lemma:

Lemma 3.1. *Let* x_α, $\alpha=1,\dots,n$, *be real numbers satisfying*

$$\sum_{\alpha=1}^{n} x_\alpha=h, \quad \sum_{\alpha=1}^{n} x_\alpha^2=a^2 \quad (a\geq 0,\ na^2>h^2). \tag{3.1}$$

Then we have

$$\frac{h}{n^2}(3na^2-2h^2)-\frac{1}{n^2}(na^2-h^2)^{3/2}\frac{n-2}{\sqrt{n-1}}\leqq \sum_{\alpha=1}^{n} x_\alpha^3, \tag{3.2}$$

$$\sum_{\alpha=1}^{n} x_\alpha^3 \leqq \frac{h}{n^2}(3na^2-2h^2)+\frac{1}{n^2}(na^2-h^2)^{3/2}\frac{n-2}{\sqrt{n-1}}. \tag{3.3}$$

The equality in (3.2) *or in* (3.3) *is fulfilled only then when* $n-1$ *of* x_α *are equals.*

Proof. We will find the biggest and the smallest values of the function $y=\Sigma_{\alpha=1}^{n} x_\alpha^3$, when x_α satisfy (3.1). The set of real numbers x_α satisfying (3.1) is compact. We can use the Lagrange-multiplier method to determine the local extreme values of y. For the function

$$f(x_1, x_2,\dots, x_n)=x_1^3+\dots+x_n^3+\lambda(x_1^2+\dots+x_n^2-a^2)+\mu(x_1+\dots+x_n-h) \tag{3.4}$$

the variational equations

$$\frac{\partial f}{\partial x_\alpha}=3x_\alpha^2+2\lambda x_\alpha+\mu=0 \tag{3.5}$$

yield the result that

$$x_{\alpha_{1,2}}=(-\lambda\pm\sqrt{\lambda^2-3\mu})/3. \tag{3.6}$$

From (3.6) we see that at an extreme point $(x_1, x_2,\dots, x_n)$ everyone of x_α is equal to one of the two numbers

$$X=(-\lambda+\sqrt{\lambda^2-3\mu})/3, \quad Y=(-\lambda-\sqrt{\lambda^2-3\mu})/3. \tag{3.7}$$

Let us denote by k, $1\leq k\leq n-1$, the number of x_α from an extreme point $(x_1, x_2, \dots, x_n)$ which are equal to X. From this and (3.1) we obtain

$$X=\frac{h}{n}+\frac{\varepsilon}{n}\sqrt{\frac{n-k}{k}(na^2-h^2)}, \qquad Y=\frac{h}{n}-\frac{\varepsilon}{n}\sqrt{\frac{k}{n-k}(na^2-h^2)}, \tag{3.8}$$

where $\varepsilon=\pm 1$.

For the extreme values of the function $y=\Sigma_{\alpha=1}^{n} x_\alpha^3$ we have

$$y_{ex}=\frac{h}{n^2}(3na^2-2h^2)+\frac{\varepsilon}{n^2}(na^2-h^2)^{3/2}\frac{n-2k}{\sqrt{k(n-k)}}. \tag{3.9}$$

We will compare all extreme values of y. The function

$$\varphi=\varphi(k)=\frac{n-2k}{\sqrt{k(n-k)}} \tag{3.10}$$

is a monotonically decreasing function, because

$$\frac{d\varphi}{dk}=\frac{-n^2}{2[(n-k)k]^{3/2}}<0. \tag{3.11}$$

Hence the function $y_{ex}=y_{ex}(k)$ is also monotonically decreasing one.

1. Let $h>0$. As $3na^2-2h^2>0\,(na^2>h^2)$ then

$$y_{max}=\frac{h}{n^2}(3na^2-2h^2)+\frac{1}{n^2}(na^2-h^2)^{3/2}\frac{n-2}{\sqrt{n-1}}, \tag{3.12}$$

when $\varepsilon=1$, $k=1$.

$$y_{min}=\frac{h}{n^2}(3na^2-2h^2)-\frac{1}{n^2}(na^2-h^2)^{3/2}\frac{n-2}{\sqrt{n-1}}, \tag{3.13}$$

when $\varepsilon=-1$, $k=1$.

2. Let $h<0$. We have the same situation as $h>0$.

3. Let $h=0$. Then

$$y_{max}=a^{\frac{3}{2}}\frac{n-2}{\sqrt{n(n-1)}}, \quad y_{min}=\frac{-a^{\frac{3}{2}}(n-2)}{\sqrt{n(n-1)}}. \tag{3.14}$$

The lemma is proved.

Theorem 3.1. *Let M^n be a non-minimal, non-totally umbilical hypersurface of a space M^{n+1} of constant curvature c. If an INPD preserves the scalar curvature K of M^n, then the square of the length of the second fundamental tensor $h_{\alpha\beta}$ and the mean curvature h/n satisfy*

$$\frac{h}{n^2}(3nh_{\alpha\beta}h^{\alpha\beta}-2h^2)-\frac{1}{n^2}(nh_{\alpha\beta}h^{\alpha\beta}-h^2)^{3/2}\frac{n-2}{\sqrt{n-1}}\leq c(n-1)h+hh_{\alpha\beta}h^{\alpha\beta}, \tag{3.15}$$

$$\frac{h}{n^2}(3nh_{\alpha\beta}h^{\alpha\beta}-2h^2)+\frac{1}{n^2}(nh_{\alpha\beta}h^{\alpha\beta}-h^2)^{3/2}\frac{n-2}{\sqrt{n-1}}\geq c(n-1)h+hh_{\alpha\beta}h^{\alpha\beta}. \tag{3.16}$$

The equality in (3.15) *or in* (3.16) *exists in these points in which M^n has $n-1$ equal principal curvatures.*

Proof. Let M^n admits INPD preserving the scalar curvature. Then $h_{\alpha\beta}$ and h satisfy (1.6). On the other hand, using the eigenvalues λ_α of the second fundamental tensor which are the principal curvatures of M^n, we can rewrite (1.6) in the form

$$\sum_{\alpha=1}^{n}\lambda_\alpha^3=c(n-1)h+h\,.\,h_{\alpha\beta}h^{\alpha\beta}. \tag{3.17}$$

Then in view of (3.17) and the fact that $nh_{\alpha\beta}h^{\alpha\beta}-h^2=\Sigma_{\alpha<\beta}(\lambda_\alpha-\lambda_\beta)^2>0$ (when M^n is not totally umbilical), our theorem follows immediately from the above lemma.

Theorem 3.2. *Let M^n be a non-minimal, non-totally umbilical hypersurface of M^{n+1} of constant curvature c. If M^n admits INPD preserving the scalar curvature and if the square of the length of the second fundamental tensor $h_{\alpha\beta}$ and the mean curvature at every point p of M^n satisfy*

$$\frac{h}{n^2}(3nh_{\alpha\beta}h^{\alpha\beta}-2h^2)+\frac{1}{n^2}(nh_{\alpha\beta}h^{\alpha\beta}-h^2)^{3/2}\frac{n-2}{\sqrt{n-1}}=c(n-1)h+h\,.\,h_{\alpha\beta}h^{\alpha\beta} \tag{3.18}$$

then there exist exactly $n-1$ equal principal curvatures at every point of M^n equal to $h/n-(\sqrt{(na^2-h^2)/(n-1)}/n)$ $a^2=h_{\alpha\beta}h^{\alpha\beta}$.

The proof follows from (3.17) and the lemma.

Analogous theorem to theorem 3.2 we obtain if we take the equality

$$\frac{h}{n^2}(3nh_{\alpha\beta}h^{\alpha\beta}-2h^2)-\frac{1}{n^2}(nh_{\alpha\beta}h^{\alpha\beta}-h^2)^{3/2}\frac{n-2}{\sqrt{n-1}}=c(n-1)h+h\,.\,h_{\alpha\beta}h^{\alpha\beta} \tag{3.19}$$

instead of (3.18).

In case when $n=2$ for $y=\Sigma_{\alpha=1}^{2}\lambda_{\alpha}^{3}$ we have

$$y=\frac{3}{2}h\,.\,h_{\alpha\beta}h^{\alpha\beta}-\frac{h^3}{2}. \tag{3.20}$$

Then from (3.17), (1.6) and (3.20) we obtain

$$h_{\alpha\beta}h^{\alpha\beta}=2c+h^2. \tag{3.21}$$

Proposition 3.1. *A non-totally umbilical surface M^2 in M^3 of constant curvature c admits INPD preserving the scalar curvature if and only if the square of the length ot the second fundamental tensor $h_{\alpha\beta}$ and the mean curvature of M^n satisfy* (3.21).

Corollary. *A surface M^2 in E^3 admits INPD preserving the Gaussian curvature K if and only if $K=0$.*

4. Minimal hypersurface. Let us denote by $\lambda_{1p}, \lambda_{2p}, \dots, \lambda_{np}$ the principal curvatures at a point p of a minimal hypersurface M^n and let $\lambda_{1p}\leqq\lambda_{2p}\leqq\dots\leqq\lambda_{np}$.

Theorem 4.1. *For the square $h_{\alpha\beta}h^{\alpha\beta}$ of the length of the second fundamental tensor $h_{\alpha\beta}$ at a point p of a minimal hypersurface M^n in a Riemannian manifold M^{n+1} we have:*

$$h_{\alpha\beta}h^{\alpha\beta}\leqq n\lambda_{np}^2 \quad n - \textit{even}, \tag{4.1}$$

$$h_{\alpha\beta}h^{\alpha\beta}\leqq(n-1)\lambda_{np}^2 \quad n - \textit{odd}. \tag{4.2}$$

Inequality (4.1) is evident. Inequality (4.2) follows from the following simple algebraic lemma.

Lemma 4.1. *Let $x_1, x_2, \dots, x_n$ be $n(n\geqq 2)$ real numbers satisfying*

$$x_1+x_2+\dots+x_n=0, \tag{4.3}$$

$$\max|x_\alpha|=1, \quad \alpha=1,\dots,n, \tag{4.4}$$

then

$$\sum_{\alpha=1}^{n} x_\alpha^2\leqq n, \quad n - \textit{even}$$

$$\sum_{\alpha=1}^{n} x_\alpha^2\leqq n-1, \quad n - \textit{odd}. \tag{4.5}$$

Proof. We will prove that the function $\psi=\Sigma_{\alpha=1}^{n}x_\alpha^2$ of the variables $x_1, x_2, \dots, x_n$ satisfying (4.3) and (4.4) has maximum value only then when $x_1, x_2, \dots, x_n$ are whole numbers. Let us suppose that the maximizing point $(x_1, \dots, x_n)$ is such that there is at least one of numbers $x_1, \dots, x_n$ which is a fractional number. Then from (4.3) and (4.4), i. e. from $x_1+x_2+\dots,+x_{n-1}=-1$ and $|x_\alpha|\leqq 1$ it follows that there is another number of $x_1, x_2, \dots, x_n$ which also is a fractional number. Let us denote these numbers by x_1 and x_2. We have

$$0<|x_1|<1, \quad |x_1|=a_1, \quad x_1=\varepsilon_1 a_1,$$

$$0<|x_2|<1, \quad |x_2|=a_2, \quad x_2=\varepsilon_2 a_2,$$

$$\varepsilon_1 a_1+\varepsilon_2 a_2+x_3+\dots+x_{n-1}=-1,$$

where $\varepsilon_1=\pm 1$, $\varepsilon_2=\pm 1$.

We shall consider two cases.

1. $\varepsilon_1\varepsilon_2=-1 \quad \varepsilon_1=-\varepsilon_2$.

We choose a positive number δ such that $0<\delta<\min(1-a_1, 1-a_2)$. Then $0<\delta+a_1<1$, $0<\delta+a_2<1$ and

$$\varepsilon_1(a_1+\delta)+\varepsilon_2(a_2+\delta)+x_3+\ldots x_{n-1}=-1,$$
$$(a_1+\delta)^2+(a_2+\delta)^2+x_3^2+\ldots+x_{n-1}^2+1>a_1^2+a_2^2+\ldots+x_{n-1}^2+1.$$

But this contradicts that $(x_1, x_2, \ldots, x_{n-1}, 1)$ is the maximizing point.

2. $\varepsilon_1\varepsilon_2=1 \quad \varepsilon_1=\varepsilon_2$.

Let $a_2\geqq a_1$. We choose a positive number δ such that $0<\delta<\min(a_1, 1-a_2)$. Then $0<a_1-\delta<1$, $0<a_2+\delta<1$, and $\varepsilon_1(a_1-\delta)+\varepsilon_2(a_2+\delta)+x_3+\ldots+x_{n-1}=-1$, $(a_1-\delta)^2+(a_2+\delta)^2+x_3^2+\ldots+x_{n-1}^2+1=a_1^2+a_2^2+2\delta^2+2\delta(a_2-a_1)+\ldots+x_{n-1}^2+1>a_1^2+a_2^2+x_3^2+\ldots+x_{n-1}^2+1$.

This contradicts our assumption.

Therefore the function $\psi=\Sigma_{\alpha=1}^n x_\alpha^2$ has maximum value only then when the maximum number of the numbers x_α have modulus 1. When n is odd we have $(n-1)/2$ numbers of x_α equal to 1, $(n-3)/2$ equal to -1, one -0. The lemma is proved.

For the scalar curvature K of a minimal hypersurface M^n in a space M^{n+1} of constant curvature c we have

(4.6)
$$K=cn(n-1)-h_{\alpha\beta}h^{\alpha\beta}.$$

P r o p o s i t i o n 4.1. *Every deformation of a minimal hypersurface M^n in a space M^{n+1} of constant curvature c, which preserve the scalar curvature preserves and the square of the length of the second fundamental tensor.*

Really from (4.6) we obtain

(4.7)
$$\delta K=-\delta(h_{\alpha\beta}h^{\alpha\beta}).$$

R r o p o s i t i o n 4.2. *An INPD: $\bar{x}^i=x^i+\lambda N^i\varepsilon$, $\lambda>0$, of a minimal hypersurface M^n in a space M^{n+1} of constant curvature c preserves the scalar curvature K of* M^n if and only if

(4.8)
$$h_{\alpha\beta}h_\tau^\alpha h^{\tau\beta}=0.$$

P r o o f. In case when M^n is a minimal hypersurface $(h=0)$ from (1.6) we obtain

(4.9)
$$\delta K=-h_{\alpha\beta}h_\tau^\alpha h^{\tau\beta},$$

from where if $\delta K=0$ we obtain (4.8).

P r o p o s i t i o n 4.3. *An INPD: $\bar{x}^i=x^i+\lambda N^i\varepsilon$, $\lambda>0$, of a minimal hypersurface M^n in a space M^{n+1} of constant curvature c preserves the scalar curvature K of M^n if and only if*

(4.10)
$$R_{kji}^s N^k N_s B_\alpha^j B_\beta^i h^{\alpha\beta}=0.$$

P r o o f. Let us have an INPD of M^n. If we calculate $\delta(h_{\alpha\beta}h^{\alpha\beta})$ taking in view $\delta h_{\alpha\beta}=(R_{kji}^s N^k N_s B_\alpha^j B_\beta^i-h_{\alpha\tau}h_\beta^\tau)\lambda\varepsilon$ from [10], $h^{\alpha\beta}=h_{\tau\gamma}g^{\tau\alpha}g^{\gamma\beta}$ and $\delta g^{\alpha\beta}=h^{\alpha\beta}\lambda$ [10] we obtain

(4.11)
$$\delta(h_{\alpha\beta}h^{\alpha\beta})=2(R_{kji}^s N_s N^k B_\alpha^j B_\beta^i h^{\alpha\beta}+h_{\alpha\beta}h_\tau^\alpha h^{\tau\beta})\lambda\varepsilon.$$

From (4.7), (4.8), (4.9) and (4.11) we obtain (4.10).

Theorem 4.2. *If a minimal hypersurface M^n in a space M^{n+1} of constant curvature c admits INPD preserving the scalar curvature, then at every point $p \in M^n$ we have*

(4.12) $$2\lambda_{np}^2 \leqq h_{\alpha\beta}h^{\alpha\beta} \leqq n\lambda_{np}^2, \quad n \text{ — even},$$

(4.13) $$2\lambda_{np}^2 \leqq h_{\alpha\beta}h^{\alpha\beta} \leqq (n-1)\lambda_{np}^2, \quad n \text{ — odd}.$$

As $h_{\alpha\beta}h_\tau^\alpha h^{\tau\beta} = \Sigma_{\gamma=1}^n \lambda_\gamma^3$, $h_{\alpha\beta}h^{\alpha\beta} = \Sigma_{\gamma=1}^n \lambda_\gamma^2$, $h = \Sigma_{\gamma=1}^n \lambda_\gamma$ then the proof of this theorem follows from the following two lemmas.

Lemma 4.2. *Let $x_1, x_2, \ldots, x_n$ be n, $n \geqq 2$, real numbers satisfying* (4.3), (4.4) *and*

(4.14) $$x_1^3 + x_2^3 + \ldots + x_n^3 = 0.$$

Then we have (4.5).

The proof of the lemma is the same as that of lemma 4.1 because the whole numbers satisfying (4.3) and (4.4) also satisfy (4.14).

Lemma 4.3. *Let $x_1, x_2, \ldots, x_{n-1}$,* 1 be *n*, $n \geqq 2$, *real numbers satisfying* (4.3) *and* (4.14). *Then we have*

(4.15) $$2 \leqq \sum_{\alpha=1}^{n-1} \lambda_\alpha^2 + 1.$$

Proof. When $n=2, 3$ we determine x_α $\alpha=1, \ldots, n-1$ directly from (4.3) and (4.14) and obtain (4.15). When $n>3$ we will minimize the function $y = x_1^2 + \ldots + x_{n-1}^2 + 1$ over all real numbers $1, x_1, \ldots, x_{n-1}$ satisfying conditions (4.3) and (4.14). Since the minimum of y is not assumed at a boundary point, we can utilize the Lagrange-multiplier method to determine the local minimum.

For the function

(4.16) $$f(x_1, \ldots, x_{n-1}) = x_1^2 + \ldots + x_{n-1}^2 + 1 + \lambda(x_1 + \ldots + x_{n-1} + 1) + \mu(x_1^2 + \ldots + x_{n-1}^2 + 1)$$

the variational equations

(4.17) $$\frac{\partial t}{\partial x_\alpha} = 3\mu x_\alpha^2 + 2x_\alpha + \lambda = 0$$

yield the result that

(4.18) $$x_{\alpha_{1,2}} = \frac{-1 \pm \sqrt{1-3\mu\lambda}}{3\mu} = A \pm B,$$

where $A = -1/3\mu$, $B = \sqrt{1 - 3\lambda\mu}/3\mu$.

From this, (4.3) and (4.14) we see that in a minimizing point $(x_1, x_2, \ldots, x_{n-1}, 1)$ there are k, $1 \leqq k \leqq n-2$, numbers of x_α equal to $A+B$ and $n-1-k$ numbers equal to $A-B$. Consequently there are at most $n-2$ minimizing points at which the extreme values of the function y are different one from the other. For these values of y we obtain

(4.19) $$y_{\min} = 1 + \frac{1}{n-1} + \frac{4k}{n-1}(n-1-k)B^2, \quad 1 \leqq k \leqq \left[\frac{n-2}{2}\right],$$

where B satisfies the equation

(4.20) $$8k(n-1-k)(n-1-2k)B^3 - 12k(n-1-k)B^2 + (n-1)^2 - 1 = 0.$$

For $1 \leq k < (n-2)/2$ (4.20) has one real root, for $k=(n-2)/2$ (4.20) has three real roots, $B_1=-1/2$, $B_2=B_3=1$ and

$$(4.21) \qquad y_{\min}=\frac{n^2+2n}{4(n-1)} \geq 2.$$

The function $y_{\min}(k)$ from (4.19) is a monotonically increasing function. Really,

$$(4.22) \qquad \frac{dy(k)}{dk}=4B^3[(n-1)^3+8k^3]/3(n-1)[B(n-1-2k)-1]>0 \text{ when } B<0.$$

To prove that the root B of the equation (4.20) is negative we investigate the function

$$(4.23) \qquad z=8k(n-1-k)(n-1-2k)B^3-12k(n-1-k)B^2+(n-1)^2-1$$

of the independent variable B, when $1 \leq k < (n-2)/2$. We have $z_{\max}=(n-1)^2-1>0$ when $B=0$, $z_{\min}=(n-1)^2\{[(n-1)^2-2k]^2-1\}/(n-1-2k)>0$ when $B=1/(n-1-2k)>0$, $z \to -\infty$ when $B \to -\infty$. Then $z=0$, when $B<0$. Hence for $k=1$ from (4.19) and (4.20) we obtain $B=-1/2$, $n>4$, and

$$(4.24) \qquad y_{\min}=2; \quad x_1=-1, \; x_2= \ldots =x_{n-1}=0.$$

When $n=4$, $k=1$ we directly obtain from (4.20) that $y_{\min}=2$. The lemma is proved

Theorem 4.3. *If $h_{\alpha\beta}h^{\alpha\beta}<2\lambda^2_{np}$ at some point p of a minimal hypersurface M^n in a space M^{n+1} of constant curvature c, then M^n does not admit INPD which preserve the scalar curvature.*

Proof. If we suppose that M^n admits INPD preserving K then from theorem 4.2. it follows that at every point p of M^n $2\lambda^2_{np} \leq h_{\alpha\beta}h^{\alpha\beta}$ which contradicts our assumption

Proposition 4.4. *If an INPD: $\bar{x}^i=x^i+\lambda N^i\varepsilon$, $\lambda>0$, carries a minimal hypersurface M^n into a minimal one and preserves the scalar curvature then*

$$(4.25) \qquad 2\lambda^2_{np} \leq -R^{s}_{kji}N_sN^kB^j_\alpha B^i_\beta g^{\alpha\beta} \leq n\lambda^2_{np}, \quad n \text{ — even},$$

$$(4.26) \qquad 2\lambda^2_{np} \leq -R^{s}_{kji}N_sN^kB^j_\alpha B^i_\beta g^{\alpha\beta} \leq (n-1)\lambda^2_{np}, \quad n \text{ — odd}.$$

Proof. A deformation carries a minimal hypersurface into a minimal one when $\delta h=0$. We will calculate δh in case of INPD: $\bar{x}^i=x^i+\lambda N^i\varepsilon$, $\lambda>0$

$$(4.27) \qquad \delta h=\delta(h_{\alpha\beta}g^{\alpha\beta})=(R^{s}_{kji}N_sN^kB^j_\alpha B^i_\beta g^{\alpha\beta}+h_{\alpha\beta}h^{\alpha\beta})\lambda\varepsilon.$$

If an INPD preserves the scalar curvature K of a minimal hypersurface M^n and carries M^n into a minimal one, then from $\delta h=0$ (4.27), (4.12) and (4.13) we obtain (4.25 and (4.26).

Theorem 4.4. *An INPD preserving the scalar curvature does not carry a minimal compact orientable hypersurface $S^k(\sqrt{k/n})\times S^{n-k}(\sqrt{(n-k)/n})$ of a unit sphere S^{n+1} into the hypersurface of the same type.*

Proof. Let us denote by $M_{k,n}=S^k(\sqrt{k_\lambda/n})\times S^{n-k}(\sqrt{(n-k)/n})$. An INPD: $\bar{x}^i = x^i +\lambda N^i\varepsilon$, $\lambda=\text{const}$, $\lambda>0$, transformes $M_{k,n}$ into $\bar{M}_{k,n,\lambda}=S^k(\sqrt{k/n}+\lambda)\times S^{n-k}(\sqrt{(n-k)/n}+\lambda)$. From the main theorem of [3] it is known that $h_{\alpha\beta}h^{\alpha\beta}$ of $M_{k,n}$ is equal to n. If an INPD of $M_{k,n}$ preserves the scalar curvature, this INPD preserves and the length of the second fundamental tensor two. Thus $\bar{M}_{k,n,\lambda}$ has $\bar{h}_{\alpha\beta}\bar{h}^{\alpha\beta}=n$ and again from [3] it follows that $\bar{M}_{k,n,\lambda}$ has to be $S^k(\sqrt{k/n})\times S^{n-k}(\sqrt{(n-k)/n})$, which is impossible.

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