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ON TWO ANALOGIES IN THE ALMOST HERMITIAN GEOMETRY

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The paper is an extension of ideas and results given by G. Stanilov, F. Tricerri and L. Vanhecke

In the first part we give some analogical theorems of the classical theorem in the Reimannian geoemetry of Schouten and Struik for the case of the almost Hermitian geometry. In the first three theorems is used the classical notion of holomorphic sectional curvature as the first theorem in another more classical form is given in our earlier paper [1]. In the second three theorems we used our notion of Kählerian defect for 4-dimensional holomorphical tangent subspace given in our paper [5].

In the second part we discuss the problem about the global constancy of the Kählerian defect. Here we have theorem 9 which is a full analog of a theorem from A. Gray and L. Vanhecke where is treated the corresponded problem for the holomorphic sectional curvature in the geometry of QK_2-manifolds.

In the third part we give an interpretation of the well-known almost Hermitian manifolds of constant and conformal type. Namely an almost Hermitian manifolds is of constan (conformal) type iff the projections $p_i(R)=0, i=5,6,\dots, 10$ $(p_i(R)=0, =6,\dots, 10)$.

In the fourth part we consider an analogous tensor $C^*(R)$ of the classical Weyl's tensor $C(R)$ in the almost Hermitian geometry. For their first components we prove theorem 13 which says that $C_1(R)$ (resp. $C_1^*(R)$) is the projection of the curvature tensor R on the 1-dimensional $U(n)$-invariant subspace orthogonal to the tensor π_1(resp. π_2).

One can say that in the paper among the results is underlined the analogy in the almost Hermitian geometry which exists and has an important meaning between:

i) the holomorphic sectional curvature and the Kählerian defect and

ii) the Weyl's tensor $C(R)$ and its analogous tensor $C^*(R)$.

1. Some analogical theorems of the classical theorem of Schouten and Struik. In [6] we find the following theorem: A Riemannian manifold of dimension $n\geq 3$ is of constant sectional curvature iff it is an Einstein manifold and conformal flat. We give now to this theorem six analogical assertions in the geometry of the almost Hermitian manifolds.

Let (M, g, J) be a $2n$-dimensional almost Hermitian manifold, R — the curvature tensor in respect to the Levi—Civita connection. The classical Ricci tensor and the scalar curvature we denote now by $\rho(R)$ and $\tau(R)$. The proper Ricci tensor and the proper scalar curvature we denote by $\rho^*(R)$ and $\tau^*(R)$ [1, 3, 7]. The projections $p_i(R)$, $i=1, 2, \dots, 10$, of R on the $U(n)$-irreducible components of the curvature tensors space are given in the following way [6]:

$$p_1(R)=\frac{\tau(R)+3\tau^*(R)}{16n(n+1)}\,(\pi_1+\pi_2),$$

$$p_2(R)=\frac{1}{16(n+2)}(\varphi+\psi)\ \{(\rho+3\rho^*)(R+L_3R)-\frac{1}{n}(\tau(R)+3\tau^*(R))g\},$$

$$p_3(R)=\frac{1}{8}(I+L_1)(I+L_2)(I+L_3)R-p_1(R)-p_2(R);$$

$$p_4(R)=\frac{\tau(R)-\tau^*(R)}{16n(n-1)}(3\pi_1-\pi_2),$$

$$p_5(R)=\frac{1}{16(n-2)}(3\varphi-\psi)\ \{(\rho-\rho^*)(R+L_3R)-\frac{1}{n}(\tau(R)-\tau^*(R))g\},$$

$$p_6(R)=\frac{1}{8}(I-L_1)(I+L_2)(I+L_3)R-p_4(R)-p_5(R);$$

$$p_7(R)=\frac{1}{4}(I-L_2)(I+L_3)R;$$

$$p_8(R)=\frac{1}{4(n-1)}\varphi(\rho(R-L_3R)),$$

$$p_9(R)=\frac{1}{4(n+1)}\psi(\rho^*(R-L_3R)),$$

$$p_{10}(R)=\frac{1}{2}(I-L_3)R-p_8(R)-p_9(R).$$

Here:

a) if S is a symmetric tensor of type (0,2) $\varphi(S)$ denotes the following curvature tensor

$$\varphi(S)(x,y,z,u)=g(x,z)S(y,u)+g(y,u)S(x,z)-g(x,u)S(y,z)-g(y,z)S(x,u);$$

b) if S is a tensor of type (0,2) and

$$S(Jx,Jy)=S(y,x),$$

$\psi(S)$ denotes the following curvature tensor

$$\psi(S)(x,y,z,u)=2g(x,Jy)S(z,Ju)+2g(z,Ju)S(x,Jy)+g(x,Jz)S(y,Ju)$$
$$+g(y,Ju)S(x,Jz)-g(x,Ju)S(y,Jz)-g(y,Jz)S(x,Ju);$$

c) $\varphi(g)=2\pi_1$, $\psi(g)=2\pi_2$ and π_1, π_2 are the basic curvature tensors invariant under the action of $U(n)$.

d) the operators L_i, $i=1, 2, 3$, are defined in the following way:

$$(L_1R)(x,y,z,u)=\frac{1}{2}\{R(Jx,Jy,z,u)+R(y,Jz,Jx,u)+R(Jz,x,Jy,u)\}$$

for all tensors R with the property

$$R(x,y,z,u)=R(Jx,Jy,z,u)+R(Jx,y,Jz,u)+R(Jx,y,z,Ju); \tag{1}$$

$$(L_2R)(x,y,z,u)=\frac{1}{2}\{R(x,y,z,u)+R(Jx,Jy,z,u)$$
$$+R(Jx,y,Jz,u)+R(Jx,y,z,Ju)\}$$

for all tensors with the property

(2) $$R(Jx, Jy, Jz, Ju) = R(x, y, z, u);$$
$$(L_3R)(x, y, z, u) = R(Jx, Jy, Jz, Ju)$$

for any curvature tensor R.

The holomorphic sectional curvature of direction x at a point $p \in M$ in respect to R is denoted by $H_R(p; x)$. Let x, y, Jx, Jy be an orthonormal basis for a holomorphical 4-dimensional tangent subspace E^4 of the tangent space M_p. We have the curvature $K_R(p; E^4)$, defined in [2] using the linear mapping

$$y \in M_p \to R(x, y, z) \in M_p$$

and is given by

$$K_R(p; E^4) = R(x, Jx, x, Jx) + R(x, Jy, x, Jy) + R(x, y, x, y)$$
$$+ R(Jx, Jy, Jx, Jy) + R(Jx, y, Jx, y) + R(y, Jy, y, Jy).$$

in the same way using the linear mapping

$$z \in M_p \to R(Jz, x, Jy) \in M_p,$$

we get the proper curvature [1, 3]

$$K_R^*(p; E^4) = R(x, Jx, x, Jx) + 2R(x, Jy, y, Jx)$$
$$+ R(y, Jy, y, Jy) + 2R(x, y, Jx, Jy).$$

In our paper [5] we have defined the notion "Kahlerian defect of a 4-dimensional holomorphical tangent subspace E^4"

$$\Delta_R(p; E^4) = K_R(p; E^4) - K_R^*(p; E^4).$$

In the Kahlerian geometry because of the identity

(3) $$R(x, y, z, u) = R(x, y, Jz, Ju)$$

(Kähler identity) the Kahlerian defect is identical zero

$$\Delta_R(p; E^4) = 0.$$

Let $R_1 = \Sigma_{i=1}^3 p_i(R)$, $R_1^\perp = \Sigma_{i=4}^6 p_i(R)$.

At first our main goal is to show that the function H_R is related to the projection R_1 and the function Δ_R — to the projection $R_1^\perp$.

We have $H_R = H_{R_1}$ [7, 5] and $\Delta_R = \Delta_{R_1^\perp}$ [5].

Using the explicit formulas for the projections $p_i(R)$, we can compute

$$H_{p_1(R)}(x) = \frac{\tau(R) + 3\tau^*(R)}{4n(n+1)},$$

$$H_{p_2(R)}(x) = \frac{1}{2(n+2)}\{(\rho + 3\rho^*)(R + L_3R)(x, x) - \frac{\tau(R) + 3\tau^*(R)}{n} g\},$$

$$H_{p_3(R)}(x) = H_R - H_{p_1(R)} - H_{p_2(R)};$$

$$\Delta_{p_4(R)}(x, y, Jx, Jy) = \frac{\tau(R) - \tau^*(R)}{n(n-1)},$$

$$\Delta_{p_5(R)}(x, y, Jx, Jy)=\frac{1}{n-2}\{(\rho-\rho^*)(R+L_3R)(x, x)$$
$$+(\rho-\rho^*)(R+L_3R)(y, y)-2\,\frac{\tau(R)-\tau^*(R)}{n}\},$$
$$\Delta_{p_6(R)}(x, y, Jx, Jy)=\Delta_R-\Delta_{p_4(R)}-\Delta_{p_5(R)}.$$

Now we have the following assertions:

Theorem 1. $H_R=H_{p_1(R)}\Leftrightarrow R_1=p_1(R)$.

The condition $H_R=H_{p_1(R)}$ means that $H_R=\frac{\tau(R)+3\tau^*(R)}{4n(n+1)}$ is pointwise constant. Then by a standard way it follows $R_1=p_1(R)$. The assertion follows also by some results in [7].

Theorem 2. $H_R=H_{p_2(R)}\Leftrightarrow R_1=p_2(R)$.

The condition $H_R=H_{p_2(R)}$ means

$$H_R(p; x)-\frac{1}{2(n+2)}(\rho+3\rho^*)(R+L_3R)(x, x)=-\frac{\tau(R)+3\tau^*(R)}{2n(n+2)}.$$

In [1] and [3] we have proved the following classification theorem:

Let (M, g, J) be $2n$-dimensional almost Hermitian manifold and for every unit direction $x\in M_p$ and at every point $p\in M$ the linear relation holds

$$\delta H_R(p; x)+\lambda S^*(p; x)=c, \quad (S^*=(\rho+3\rho^*)(R+L_3R)/8,$$

where the functions $\delta, \lambda, c\in FM$ don't depend of x. Then

(i) $\delta=0\Leftrightarrow S^*(p:x)=c^*(p)$;

(ii) $\delta\neq0,\ \lambda(n+2)+4\delta=0\Leftrightarrow p_3(R)=0$;

(iii) $\delta\neq0,\ \lambda(n+2)+4\delta\neq0\Rightarrow H_R(p; x)=c'(p)$,

In our case we have $\delta=1$, $\lambda=-4/(n+2)$ and hence we are in the case (ii): $p_3(R)=0$ From $H_R=H_{p_2(R)}$ we have also $H_{p_1(R)}=0$ which is equivalent to $p_1(R)=0$.

Theorem 3. $H_R=H_{p_3(R)}\Leftrightarrow R_1=p_3(R)$.

The proof follows by using the expression for $H_{p_3(R)}$.

Remark 1. The Theorem 3 gives an interesting information about the K_3-surfaces. It is well known that every K_3-surface is a Ricci flat Kähler manifold. In this case $R=R_1$ and $p_2(R)=0$, $p_1(R)=0$, which implies $R=p_3(R)$.

Theorem 4. $\Delta_R=\Delta_{p_4(R)}\Leftrightarrow R_1^\perp=p_4(R)$.

The condition $\Delta_R=\Delta_{p_4(R)}$ means that $\Delta_R=(\tau(R)-\tau^*(R))/n(n-1)$ is pointwise constant. In [5] we have proved the following theorem:

For an almost Hermitian manifold of dimension $2n>6$ the following assertion are equivalent:

(i) The Kählerian defect is pointwise constant

(ii) $p_6(R)=0$ and

$$(\rho-\rho^*)(R+L_3R)=\frac{\tau(R)-\tau^*(R)}{n}\,g.$$

Now it is easy to see that the last relation is equivalent to $p_5(R)=0$. Then theorem 4 follows immediately.

Theorem 5. $\Delta_R=\Delta_{p_5(R)}\Leftrightarrow R_1^\perp=p_5(R)$.

Proof. We denote $\sigma=(\rho-\rho^*)(R+L_3R)$. The condition $\Delta_R=\Delta_{p_5(R)}$ by using the expression for Δ_R and $\Delta_{p_5(R)}$ gives

$$R(x, y, x, y)+R(Jx, Jy, Jx, Jy)+R(x, Jy, x, Jy)$$
$$+R(Jx, y, Jx, y)-2R(x, y, Jx, Jy)-2R(x, Jy, y, Jx)$$
$$=\frac{1}{n-2}\{\sigma(x, x)g(y, y)+\sigma(y, y)g(x, x)-2\frac{\tau(R)-\tau^*(R)}{n(n-2)}g(x, x)g(y, y)\},$$

where (x, y, Jx, Jy) is an orthonormal basis for E^4. If (x, z, Jx, Jz) is an arbitrary basis for E^4 the last relation implies

$$R(x, z, x, z)+R(Jx, Jz, Jx, Jz)+R(x, Jz, x, Jz)$$
$$+R(Jx, z, Jx, z)-2R(x, z, Jx, Jz)-2R(x, Jz, z, Jx)$$
$$=\frac{1}{n-2}\{\sigma(x, x)g(z, z)+\sigma(z, z)g(x, x)-2g(x, z)\sigma(x, z)$$
$$-2g(Jx, z)\sigma(Jx, z)-2\frac{\tau(R)-\tau^*(R)}{n(n-2)}(g(x, x)g(z, z)-g^2(x, z)-g^2(Jx, z))\}.$$

Then by the standard way in [5] we get $R_1^{\perp}=p_5(R)$.

Theorem 6. $\Delta_R=\Delta_{p_6(R)}\Leftrightarrow R_1^{\perp}=p_6(R)$.

The proof follows by using the expression for $\Delta_{p_5(R)}$.

Remark 2. There are examples of almost Hermitian manifolds with $R=p_1(R)$. They are Kähler manifolds of constant holomorphic sectional curvature. But there do not exist almost Hermitian manifolds with $R=p_4(R)\neq 0$ [7]. With respect to the above remark 1 we state here the following problem: Do there exist almost Hermitian manifolds with curvature tensor $R=p_6(R)$?

2. The problem about the global constancy of the Kählerian defect. An almost Hermitian manifold belongs to the class of QK_2 manifolds iff the curvature tensor R satisfies the condition (1) and for the complex structure J holds

$$(\nabla_{Jx}J)y=-J(\nabla_x J)y.$$

The following theorem is well known [8]:

Let (M, g, J) be $2n$-dimensional connected manifold. If $n\geqq 2$ and the holomorphic sectional curvature $H_R(p; x)$ is the pointwise constant $H_R(p)$, then $H_R(p)$ is a global constant on the manifold.

For the same class of manifolds the same theorem is also true if we put Δ_R on the place of H_R. Namely we have the following

Theorem 7. *Let (M, g, J) be $2n$-dimensional connected QK_2 manifold. If $n\geqq 3$ and the Kählerian defect $\Delta_R(p; E^4)$ is the pointwise constant $\Delta_R(p)$, then $\Delta_R(p)$ is a global constant on the manifold.*

Proof. From the condition and using Theorem 4 it follows that $p_5(R)=0$ is equivalent to the relation

$$\rho(R)-\rho^*(R)=\frac{\tau(R)-\tau^*(R)}{2n}g.$$

Now from [11] it is well known that the tensor fields $\rho(R)$ and $\rho^*(R)$ for a QK_2 manifold satisfy the well-known identity in the Relativity theory;

$$\sum_{i=1}^{2n}(\nabla_{e_i}\rho)(x, e_i)=\frac{1}{2}x\tau.$$

Then by direct differentiation the above relation it follows that

$$\Delta_R(p)=\frac{\tau(R)-\tau^*(R)}{n(n-1)}$$

is a global constant on the manifold.

The proof follows also from our Theorem 3 in [4] if we put $\lambda=1$, $\mu=1$.

In [8] is shown that the Schur's lemma fails for the class of the Hermitian manifolds. Namely there is proved the following theorem:

Let ds^2 be the usual metric on C^n and $f: C^n \to C$ any nonlinear holomorphic function. Then $(C^n, (1+\mathrm{Re} f(z))^{-2} dS^2)$ is a Hermitian manifold with pointwise constant holomorphic sectional curvature which is not globally constant.

We state now the same problem for the Kählerian defect. At first we have the following

Theorem 8. *Let (M, g, J) be 2n-dimensional almost Hermitian manifold. If $n>3$ and the manifold is conformal flat then:*

(i) $\Delta(p; E^4)=2(H(p; x)+H(p; y))$;
(ii) $\Delta(p; E^4)=\Delta(p) \Leftrightarrow H(p; x)=H(p)$.

Proof. From [7] is known that if the manifold is conformal flat then

$$R=\frac{1}{2n-2}\varphi(\rho(R))-\frac{\tau(R)}{(2n-1)(2n-2)}\pi_1.$$

By directly computation we get

$$H(p; x)=\frac{1}{2n-2}\{\rho(R)(x, x)+\rho(R)(Jx, Jx)\}-\frac{\tau(R)}{(2n-1)(2n-2)},$$

$$\Delta(p; E^4)=\frac{2}{2n-2}\{\rho(R)(x, x)+\rho(R)(Jx, Jx)+\rho(R)(y, y)$$

$$+\rho(R)(Jy, Jy)\}-\frac{4\tau(R)}{(2n-1)(2n-2)}.$$

Then Theorem 8 follows immediatelly.

Using this theorem (assertion (ii)) and the result from [8] we can formulate the following

Theorem 9. *Let ds^2 be the usual metric on C^n and $f: C^n \to C$ any nonlinear holomorphic function. Then $(C^n, (1+\mathrm{Re} f(z))^{-2} ds^2)$ is a Hermitian manifold with pointwise constant Kählerian defect which is not globally constant.*

Hence Schur's lemma fails in respect to the Kählerian defect for the class of Hermitian manifolds.

3. Almost Hermitian manifolds of conformal type and their generalizations

An almost Hermitian manifold (M, g, J) is said to be of constant type λ [13] if

$$R(x, y, x, y)-R(x, y, Jx, Jy)=\lambda(p)$$

holds good for any antiholomorphic plane spaned by x, y (e. g. $(x, y)\perp(Jx, Jy)$) at every point $p\in M$. The constant λ is equal to $(\tau(R)-\tau^*(R))/4n(n-1)$. This condition is equivalent to requirement that the curvature tensor

$$R-\frac{\tau(R)-\tau^*(R)}{4n(n-1)}\pi_1$$

satisfies the Kähler identity (3).

These manifolds are generalized in [10]. Namely an almost Hermitian manifold is of conformal type if the tensor

$$R-\frac{1}{2(n-2)}\varphi(\rho-\rho^*)(R)+\frac{\tau(R)-\tau^*(R)}{4(n-1)(n-2)}\pi_1$$

satisfies the Kähler identity (3).

Now we will discuss both manifolds, e. g. to find their projections and to open a way for their generalizations.

We shall do this in an elegant way. From the Riemannian geometry we have the construction [1]

$$\overline{R}(x,y,z,u)=\frac{1}{6}\{2R(x,y,z,u)+2R(z,u,x,y)$$

$$+R(x,z,y,u)+R(y,u,x,z)+R(z,y,x,u)+R(x,u,z,y).$$

If R has only the properties

$$R(x,y,z,u)=-R(y,x,z,u)=-R(x,y,u,z),$$

as in the case when R is the curvature in respect to a nonsymmetric Riemannian connection, then $\overline{R}$ has the same properties and satisfies also the first Bianchi's identity Besides $\overline{R}$ is the unique tensor with the property

$$\overline{R}(x,y,x,y)=R(x,y,x,y),$$

e. g. R and $\overline{R}$ have the equal sectional curvatures.

Using the above construction we consider the mapping

$$\gamma: R\to\gamma(R)$$

defined by

$$6\gamma(R)(x,y,z,u)=2R(x,y,Jz,Ju)+2R(z,u,Jx,Jy)$$

$$+R(x,z,Jy,Ju)+R(y,u,Jx,Jz)+R(z,y,Jx,Ju)+R(x,u,Jz,Jy).$$

If R is a curvature tensor the same is also true for tensor $\gamma(R)$ and

$$\gamma(R)=L_3\gamma(R).$$

We omit now the long calculations and give some results:

$$\rho\circ\gamma(R)=\frac{1}{2}\rho^*(R+L_3R),$$

$$\rho^*\circ\gamma(R)=\frac{1}{6}(\rho+2\rho^*)(R+L_3R).$$

The projections of tensor $\gamma(R)$ are:

$$p_i(\gamma(R))=p_i(R),\quad i=1,2,3;$$

$$p_i(\gamma(R))=-\frac{1}{3}p_i(R),\quad i=4,5,6;$$

$$p_7(\gamma(R))=-p_7(R);$$

$$p_i(\gamma(R))=0,\quad i=8,9,10.$$

Thus we have the following

T h e o r e m 10. *The decomposition of tensor* $\gamma(R)$ *into irreducible orthogonal components for the action of the unitary group* $U(n)$ *is given by*

$$\gamma(R)=\sum_{i=1}^{3} p_i(R)-\frac{1}{3}\sum_{i=4}^{6} p_i(R)-p_7(R),$$

or equivalently

$$R-\gamma(R)=\frac{4}{3}\sum_{i=4}^{6} p_i(R)+2p_7(R)+\sum_{i=8}^{10} p_i(R).$$

As a consequence we have the following

C o r o l l a r y. *In the case* $R=L_3R$ *the following two assertions are equivalent*

(i) $R-\gamma(R)=\frac{4}{3}p_4(R)$;

(ii) $p_5(R)=p_6(R)=p_7(R)=0$.

This corollary is identical with the following assertion in [9]:

Let (M, g, J) be an almost Hermitian manifold of dimension $2n\geq 8$ which satisfies (2). The manifold is of constant type iff it is of conformal type and

$$\rho(R)-\rho^*(R)=\frac{\tau(R)-\tau^*(R)}{2n}g. \tag{4}$$

To show the identity first we observe that $R=\gamma(R)$ iff R satisfies the Kähle identity. Because $3\gamma(\pi_1)=\pi_2$ and (2) the condition (i) can be written as

$$\left(R-\frac{\tau(R)-\tau^*(R)}{4n(n-1)}\pi_1\right)(x, y, z, u)=\left(R-\frac{\tau(R)-\tau^*(R)}{4n(n-1)}\pi_1\right)(x, y, Jz, Ju),$$

which says that (M, g, J) is a manifold of constant type.

Now we consider the condition (ii). Because of

$$\rho\circ p_5(R)=\frac{3}{8}\left\{(\rho-\rho^*)(R+L_3R)-\frac{\tau(R)-\tau^*(R)}{n}g\right\}$$

and $R=L_3R$ the equality $p_5(R)=0$ is equivalent to the relation (4).

To finish the proof we observe that the following assertions are true:

a) if S is symmetric and J-invariant tensor of type (0,2) then

$$3\gamma(\varphi(S))=\psi(S);$$

b) if R satisfies (1) then

$$3\gamma(R)-2L_1(R)=R.$$

Using these facts and $p_7(R)=0$, $R=L_3R$ the equality $p_6(R)=0$ or equivalently $R_1^{\perp}=p_4(R)+p_6(R)$ implies

$$3R-3\gamma(R)=\frac{\tau(R)-\tau^*(R)}{4n(n-1)}(3\pi_1-\pi_2)$$

$$+\frac{1}{2(n-2)}(3\varphi-\psi)\left\{(\rho-\rho^*)(R)-\frac{\tau(R)-\tau^*(R)}{n}g\right\}$$

which says that the tensor

$$R-\frac{1}{2(n-2)}\varphi\circ(\rho-\rho^*)(R)+\frac{(\tau-\tau^*)(R)}{4(n-1)(n-2)}\pi_1$$

satisfies the Kähler identity. But this is the condition (M, g, J) to be a manifold of conformal type.

The above considerations give us the possibility to formulate the following two theorems:

Theorem 11. *(M, g, J) with (2) is an almost Hermitian manifold of constant type iff $p_i(R)=0$, $i=5, 6, 7, 8, 9, 10$.*

Theorem 12. *(M, g, J) with (2) is an almost Hermitian manifold of conformal type iff $p_i(R)=0$, $i=6, 7, 8, 9, 10$.*

4. An analogous tensor of the Weyl's tensor in the almost Hermitian geometry. In the paper [7] the classical Weyl's tensor for an almost Hermitian manifold is given in the following way:

$$C(R)=R-\frac{1}{2n-2}\varphi\circ\rho(R)+\frac{\tau(R)}{(2n-1)(2n-2)}\pi_1.$$

Besides this tensor is decomposed into 7 orthogonal components:

$$C(R)=C_1(R)+C_2(R)+\Sigma p_i(R),\quad i=3, 6, 7, 9, 10,$$

where

$$C_1(R)=\frac{\tau(R)-(2n-1)\tau^*(R)}{8n(n^2-1)(2n-1)}\{3\pi_1-(2n-1)\pi_2\},$$

$$C_2(R)=\frac{1}{n-1}\{3\varphi-(n-1)\psi\}(S_2-S_1),$$

and

$$16(n+2)S_1=(\rho+3\rho^*)(R+L_3R)-\frac{1}{n}(\tau+3\tau^*)(R)g.$$

$$16(n-2)S_2=(\rho-\rho^*)(R+L_3R)-\frac{1}{n}(\tau-\tau^*)(R)g.$$

All these components are conformal invariants and are characterized by the requirement they have Ricci tensor $\rho=0$.

At the recommendation of mine in [14] is founded an analog of the tensor $C(R)$. Namely:

$$C^*(R)=R-\frac{1}{2n+2}\psi\circ\rho^*(R)+\frac{\tau^*(R)}{(2n+1)(2n+2)}\pi_2.$$

This tensor is decomposed into 7 orthogonal components

$$C^*(R)=C_1^*(R)+C_2^*(R)+\Sigma p_i(R),\quad i=3, 6, 7, 8, 10,$$

where

$$C_1^*(R)=\frac{(2n+1)\tau(R)-3\tau^*(R)}{8n(n^2-1)(2n+1)}\{(2n+1)\pi_1-\pi_2\},$$

$$C_2^*(R)=\frac{1}{n+1}\{(n+1)\varphi-\psi\}(S_1+3S_2).$$

All these components are characterized by the requirement they have * — Ricci tensor $\rho^*=0$.

The tensor $C^*(R)$ is a new one even in the Kählerian geometry. It follows by the next theorem 13 which gives an interesting geometric meaning of the tensor $C_1(R)$ and $C_1^*(R)$.

Theorem 13. (i) *In the $U(n)$-invaiiant 2-dimensional subspace spanned by the "vectors" π_1 and π_2, the tensor $C_1(R)$ is orthogonal to the tensor π_1 and $C_1^*(R)$ is orthogonal to the tensor π_2*; (ii) *$C_1(R)$ is the projection of the tensor R on the 1-dimensional $U(n)$-invariant subspace orthogonal to π_1 and $C_1^*(R)$ is the projection of the tensor R on the 1-dimensional $U(n)$-invariant subspace orthogonal to π_2.*

Proof. The first part follows by the formulas [7]

$$\langle R_1\pi_1\rangle=2\tau(R), \quad \langle R, \pi_2\rangle=6\tau^*(R)$$

and

$$\tau(\pi_1)=2n(2n-1), \quad \tau(\pi_2)=6n,$$

$$\tau^*(\pi_1)=2n, \quad \tau^*(\pi_2)=2n(2n+1).$$

To prove the second part we take the "unit" tensor

$$e=\frac{3\pi_1-(2n-1)\pi_2}{\sqrt{48n(n^2-1)(2n-1)}},$$

orthogonal to π_1 and compute that

$$\langle R, e\rangle e=C_1(R).$$

Also

$$e^*=\frac{(2n+1)\pi_1-\pi_2}{\sqrt{16n(n^2-1)(2n+1)}}$$

is a "unit" tensor orthogonal to π_2 and

$$\langle R, e^*\rangle e^*=C_1^*(R).$$

We finish our considerations with the following

Theorem 14. Let *(M, g, J) be $2n$-dimensional almost Hermitian manifold. If $n>3$ and $C^*(R)=0$ then*

i) $\Delta(p; E^4)=-\frac{2}{3}\{H(p; x)+H(p; y)\}$;

ii) $\Delta(p; E^4)=\Delta(p) \Leftrightarrow H(p; x)=H(p)$.

The proof is going in the following way: $C^*(R)=0$ implies

$$H(p; x)=\frac{3}{n+1}\rho^*(R)(x, x)+\frac{(3\tau^*(R)}{(2n+1)(2n+2)};$$

$$\Delta(p; E^4)=-\frac{2}{n+1}\{\rho^*(R)(x, x)+\rho^*(R)(y, y)\}-\frac{4\tau^*(R)}{(2n+1)(2n+2)}$$

from which it follows (i) and then (ii).

We remark that Theorem 14 is an analogous theorem of Theorem 8.

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