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CONVOLUTION OPERATORS PRESERVING UNIVALENT FUNCTIONS

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For a fixed function f_0 the Hadamard product $f_0 * f$ is a linear operator in the set containing the variable function f. For a special choice of f_0 and f ranging in certain subclasses of the well-known class S of normalized univalent functions such an operator is investigated.

1. Introduction. Univalent conformal mappings of the unit disc $U=\{|z|<1\}$ form a complicated non-linear compact set S which lies in the linear topological space A of all holomorphic functions. The impossibility of equipping S with a linear structure induced from A (a linear combination of univalent functions can be multivalent) makes it difficult to construct variations of univalent functions, to solve extremal problems, to obtain integral representations and description of extreme points and convex hulls. Therefore choosing objects and structures in S which have linear nature it remains an important and non-trivial problem.

Many papers in the theory of univalent functions are devoted to questions of construction of linear integral (or integro-differential) operators which map the class S and its subclasses into themselves (see a survey in [2, § 14] and [8]).

One of the first papers in this direction is due to Biernacki [5]. Using the observaion of Alexander [1] that the operator

$$f \to Bf := \int_0^z \frac{f(t)}{t}\,dt \tag{1}$$

transfers starlike functions into convex functions, he showed that the operator B maps the class S into itself. However, Krzyz and Lewandowski [10] gave an example of univalent spirallike function, namely $f_0(z)=z\exp((i-1)\log(1-iz))$, which is transferred by the Biernacki operator into a non-univalent function.

The linear operator

$$f \to Lf := (2/z)\int_0^z f(t)\,dt \tag{2}$$

introduced by Libera [11], transfers each of the classical subclasses of convex, starlike and close-to-convex functions into itself. Later on Bernardi [4] generalized the results of Libera by introducing the operator

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$$f \to B_c f := (1+c)\, z^{-c} \int_0^z t^{c-1} f(t)\, dt, \quad c \in N, \tag{3}$$

but both authors restrict themselves with some subclasses of S. Later examples of univalent functions of the form $f_0(z) = z(1+z)^{1-i}$ were found which the operator L also transfers into non-univalent ones.

Ruscheweyh [13] and Livingston [12] investigated the differential operators

$$f \to B_n^{-1} f := z(z^{n-1} f(z))^{(n)}/n! \tag{4}$$

and

$$f \to L^{-1} f: = (zf(z))'/2 \tag{5}$$

being inverse to Biernacki ($n=1$) and Libera operators, respectively. They also do not establish the univalence of the whole class S.

By means of Hadamard's convolution we introduced in [9] the hypergeometric linear operator

$$f \to F(a, b, c) f: = zF(a, b; c; z) * f(z), \tag{6}$$

where $F(a, b; c; z) = 1 + \Sigma_{k=1}^{\infty} [(a)_k (b)_k / (c)_k (1)_k] z^k$ is the hypergeometric function of Gauss, $(a)_k = \Gamma(a+k)/\Gamma(a)$, $f \in S$.

The three-parameter family of operators given by (6) contains as special cases all operators defined above by (1)-(5). As a matter of fact, $B = F(1, 1, 2)$, $B_n^{-1} = F(1, n+1, 1)$, $B_c = F(1, c+1, c+2)$ and L, L^{-1} are equal to $F(1, 2, 3)$ and $F(1, 3, 2)$, respectively.

The investigation of the operators of Biernacki, Libera and others written down by means of $F(a, b, c)$ gives the possibility to consider (1)-(5) as special cases of (6) with integer values of the parameters (a, b, c) while it is natural to study hypergeometric operators also for arbitrary real (or complex) parameters.

In this paper we investigate convolution linear operators on the set A of holomorphic functions that map the class S into itself.

2. Linear convolution operators. In this section we give a description of general convolution operators as continuous rotation and contraction invariant operators on the space A.

Recall that a convolution operator is an operator of the form

$$L(f) : = f_0 * f,$$

where f_0 is some fixed holomorphic in U function and $*$ denotes the Hadamard product (convolution)

$$f_1 * f_2(z) = z + \sum_{n=2}^{\infty} a_n(f_1) a_n(f_2) z^n$$

for two functions $f_j(z) = z + \Sigma_{n=2}^{\infty} a_n(f_j) z^n$, $j = 1, 2$. It is not difficult to see that the convolution operators L map A into A. The set of all such operators forms a commutative semigroup with respect convolution $(*)$ with $\delta(z)=\Sigma_{n=1}^{\infty} z^n = z/(1-z)$ as a unit element.

Let us introduce the rotation and contraction operators in A defined by

$$f \to R_\alpha(f) := e^{-i\alpha} f(e^{i\alpha} z),$$

$$f \to T_r(f) := f(rz),$$

where $|\alpha| < \pi$, $0 < r < 1$.

The following conclusion holds true (cf. [15, 18]).

Theorem 1. *L is a convolution operator on A if and only if L is a continuous, rotation or contraction invariant operator.*

If we have a concrete expression of the function f_0, then we can obtain much deeper properties of convolution operators that preserve the class A together with its contraction and rotation invariant subclasses. In the following theorem we consider one of the possible choices of f_0 having a wide range of applications in the class S of univalent functions.

One of the main problems which arise in considering the hypergeometric operators (6) is: for which values of the parameters (a, b, c) does the operator $F(a, b, c)$ map the class S of univalent functions into itself? The answer to this question is given by the following

Theorem 2. *Let $F(a, b, c)$ be a hypergeometric linear operator on the space A. Suppose that $(a, b, c) \in H \subset R_+^3$ is defined by the inequalities:*

$$a > 0, \quad b > 0, \quad c > a + b + 2,$$

$$(7) \qquad \frac{\Gamma(c)}{\Gamma(c-a)} \frac{\Gamma(c-a-b-2)}{\Gamma(c-b)}((a)_2(b)_2 + 3ab(c-a-b-2) + (c-a-b-2)_2) < 2.$$

Then for each univalent $f \in A$, the function $F(a, b, c)f$ is also univalent.

In the introduction it was noted that the operator of Biernacki does not map the class of univalent functions into itself. That case corresponds to the values of the parameters $a=1$, $b=1$ and $c=2$, for which the series $F(a, b; c; 1)$ is divergent. The point $(1, 1, 2)$ does not belong to the domain H and we cannot expect that the function $F(a, b, c)f(z)$ belongs to the class S. So the assertion is especially interesting when it is received as a particular case of Theorem 2.

Corollary 1. *The generalized operator of Biernacki*

$$f \to F(1, 1, n+1)f := n!\, z^{1-n} \int_0^z \int_0^{t_n} \dots \int_0^{t_2} \frac{f(t_1)}{t_1} dt_1 \dots dt_n$$

transfers any univalent function into a univalent function for any $n > 8$.

It is easy to show that the assumptions about natural parameter c slightly increases the value of this parameter. For instance the operator $F(1, 1, c)$ maps the class S into S for any real $c > c_0 = (11 + \sqrt{33})/2$.

Now we shall deal with the generalized operators of Libera.

Corollary 2. *Let $F(1, b, c)$ be a hypergeometric operator in the class S. Suppose that $c_0(b)$ is the maximal positive root of the equation*

$$x^3-4bx^2-(1+b+5b^2)x+(b-b^2-2b^3)=0.$$

If $b>0$ and $c>c_0(b)+b+2$, then the operator $F(1, b, c)$ maps the class S into itself.

Consider an analogous problem in the class S^0 of convex univalent functions.

Theorem 3. *Let $F(a, b, c)$ be a hypergeometric linear operator in the class S^0. Suppose that $(a, b, c)\in H_0\subset R^3$ is defined by the inequalities*

$$a>0,\ b>0,\ c>a+b+1,$$

$$\frac{\Gamma(c-a-b-1)}{\Gamma(c-a)}\frac{\Gamma(c)}{\Gamma(c-b)}[ab+c-a-b-1]<2.$$

Then $F(a, b, c)$ maps any convex function to a univalent function.

3. Nonlinear convolution operators.* In 1955 Bazilevich [3] proved the univalence of all functions

$$f(z)=\left[(\alpha+i\beta)\int_0^z \Phi^\alpha(t)h(t)t^{i\beta-1}\,dt\right]^{1/(\alpha+i\beta)}, \tag{8}$$

where $\alpha\in R_+$, $\beta\in R$, Φ is a starlike function and h is a holomorphic function with positive real part. Numerous investigations in the last years are devoted to the studying of analytic and geometric properties of the class $B_{\alpha,\beta}$ of Bazilevich functions. By using the technique of convolutions and results of [16] Ruscheweyh [14] proved that the operator

$$f\to F(z):=[F(1,\ c+1,\ c+2)f^{\alpha+i\beta}(z)]^{1/(\alpha+i\beta)} \tag{9}$$

maps the class $B_{\alpha,\beta}$ into itself for any $\alpha\in R_+$, $\beta\in R$ and $c\in C$, $\operatorname{Re} c>0$. After that Prokhorov [17] proved in another way the same statement, but already for the subclass $B^\gamma_{\alpha,\beta}$ which yielded to additional restrictions on the function h in the integral representation (8).

In this section we show that the operator (9) maps also any general subclass of Bazilevich functions into itself. Meanwhile, analogous assertions will be obtained for the classes of starlike and close-to-convex functions.

Let us introduce the class B_G of holomorphic functions p in U such that $p(0)=1$ and $p(U)\subseteq G$, where G is a convex domain which does not contain the origin but contains the point $w=1$.

We prove one remarkable geometric lemma.

* The results of this part are obtained in collaboration with V. A. Tsapov.

L e m m a 1. *Let* $p \in A$, $p(0)=1$, *and let* p_1 *be a holomorphic function with nonnegative real part in* U. *Then for any* $c \in C$, $\operatorname{Re} c \geqq 0$,

$$p+\frac{zp'}{p_1+c} \in B_G \Rightarrow p \in B_G, \quad z \in U.$$

Now let us consider the classes of starlike and close-to-convex functions. By definition the class of starlike functions is

$$S_G^* = [f \in A \mid zf'/f \in B_G]$$

and the class of close-to-convex functions is

$$K_{G,G_1} = [f \in A \mid zf'/\varphi \in B_G, \quad \varphi \in S_{G_1}^*].$$

Let us first show

L e m m a 2. *Let* $\varphi \in S_G^*$ *and* $f \in K_{G,G_1}$. *Then*

$$\text{i)} \quad \psi(z) = \left[z^{-c-i\beta} \int_0^z t^{c-1+i\beta} \varphi^\alpha(t)\, dt \right]^{1/\alpha} \in S_G^* ;$$

$$\text{ii)} \quad F(z) = F(1,\ c+1,\ c+2) f(z) \in K_{G,G_1}$$

for any α, $\beta \in R$, $\alpha > 0$, *and* $c \in C$, $\operatorname{Re} c \geqq 0$.

Finally we shall deal with nonlinear convolution operators onto the class of Bazilevich functions. Let us denote by $B_{\alpha,\beta}(G)$ the class of functions (8), where $h \in G_G$. More precisely we have

T h e o r e m 4. *If* $f \in B_{\alpha,\beta}(G)$, *then*

$$F(z) = [F(1,\ c+1,\ c+2) f^{\alpha+i\beta}(z)]^{1/(\alpha+i\beta)} \in B_{\alpha,\beta}(G)$$

for any $\alpha \in R_+$, $\beta \in R$ *and* $c \in C$, $\operatorname{Re} c \geqq 0$.

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