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ON SOME ALGEBRAS OF COMPLEX-VALUED FUNCTIONS ON DIFFERENTIABLE MANIFOLDS

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This article invesigates algebras of complex-valued functions of type C on compact C^∞ manifolds and describes a large class of such algebras by linear differential operators; it examines projective limits of such algebras; studies algebras of complex-valued functions of type C on the n-dimensional torus and on discs in the complex plane C as examples.

1. Introduction. The algebras of type C are introduced by G. E. Shilov in 1947 [1]. Interesting and significant results about homogeneous algebras of complex-valued functions of type C on the torus are received in the works of G. E. Shilov [1-6], H. Mirkil [7], K. de Leeuw and H. Mirkil [8], B. S. Mityagin [9], V. V. Grushin [10] and others. It is worthwhile to investigate not only homogeneous algebras of complex-valued functions of type C. The papers [1, 11-13] contain results in this direction. A presentation of algebras R of type C of complex-valued functions on some compacts as completion of the polynomials with norms, determined by the canonical homomorphisms $\omega_s : R \to R/J(s)$ (where $J(s)$ is the minimal closed primary ideal of R at the point s), is worked out by G. E. Shilov in some cases in [11]. Nonhomogeneous algebras of continuous complex-valued functions of type C on segments in R^1 are investigated in [1, 12, 13]. These algebras of them, which contain all complex-valued functions with continuous first derivatives, are described in [13].

The present article studies the algebras of complex-valued functions of type C on compact, C^∞ differentiable, n-dimensional manifolds and it describes a large class of such algebras by linear differential operators; this article examines projective limits of such algebras; investigates algebras of complex-valued functions of type C on n-dimensional torus and on discs in the complex plane C as examples.

2. Preliminary. Let recall the fundamental definitions about the algebras of type C:

Definition 1. *The Banach algebra R of complex-valued $(c-v)$, functions on the compact topological space G is called an algebra of complex-valued $(c-v)$, functions of type C on G if* 1) *R is an algebra without a radical* (*i. e., the intersection of all maximal ideals of the algebra R consists only of the zero element of the algebra R*); 2) *The norm $\|f\|$ in R, $f \in R$, is equivalent to the norm $\sup_{s\in G}\|f\|_s$, where $\|f\|_s = \inf\{\|g\| : g\in R,\ g=f$ in a neighbourhood of $s\}$.*

Definition 2. *An algebra R of $c-v$ functions on the topological space G is called regular if for each compact $F\subset G$ and each $s_0\in G$, with $s_0\notin F$, there exists a function $f\in R$ such that $f(s_0)\neq 0$ and $f(s)=0$ for $\forall s\in F$.*

Definition 3. *An ideal J of an algebra R is called a primary ideal of R if J is contained in a unique maximal ideal M of R. If R is an algebra of $c-v$ functions on a set G, and if M consists of all functions $f\in R$, for which $f(s_0)=0$ for some $s_0\in G$, then we shall denote M by $M(s_0)$, J by $J(s_0)$.*

It is proved (see [1, Ch. I, Theorem 5], [2, Theorem 3.3]), that if R is a regular Banach algebra of c-v functions on a compact G, without radical, then for each $s_0\in G$ there exists a minimal closed primary ideal $J(s_0)$ at the point s_0.

3. Formulation of the results. Further let G be a compact C^∞ differentiable n-dimensional manifold.

Proposition 1. *Let R be a regular algebra of c-v functions of type C on G, in which the set $\mathcal{O}$ of some continuous c-v functions on G is contained and is dense. Let $\omega_s(f)$ be the image of $f\in R$ in the canonical homomorphism $R\to R/J(s)$, let $|.|_s$ be the quotiont norm in $R/J(s)$, where $J(s)$ is the minimal closed primary ideal of the algebra R at the point $s\in G$.*

Then $|\omega_s(f)|_s$ is an upper semicontinuous function in s, and the algebra R is a completion of $\mathcal{O}$ by the norm p, $pf=\sup_{s\in G}|\omega_s(f)|_s$.

When G is a finite segment in R^1, $\mathcal{O}$ — the set of the polynomials, for some cases, Proposition 1 is contained in [12]. The announced results in [11] contain Proposition 1 for some G and some $\mathcal{O}$.

Let D_G^ν (resp., D_G^{usc}) be the algebra of all c-v functions f on G, for which $f\circ\varphi^{-1}\in C^\nu_{\varphi(U)}$ (resp. $\in C^{\mathrm{usc}}_{\varphi(U)}$) for each local chart (U,φ) of the manifold G, where $C^\nu_{\varphi(U)}$ is the algebra of all continuously differentiable c-v functions up to order ν, inclusive, on $\varphi(U)$, $\nu=0,1,\dots;\ \infty$ (resp., $C^{\mathrm{usc}}_{\varphi(U)}$ is the set of all upper semicontinuous c-v functions on $\varphi(U)$).

Definition 4. *A linear continuous map $A: D_G^\infty\to D_G^0$ is called a linear differential operator on G if for any given local chart (U,φ) of G, the transfer $A\circ\varphi^{-1}$ is a linear differential operator on $\varphi(U)$ (see [14]). If in addition the coefficients of $A\circ\varphi^{-1}$ are in $C^\nu_{\varphi(U)}$ (resp. in $C^{\mathrm{usc}}_{\varphi(U)}$), then A is called still with coefficients in D_G^ν (resp. in D_G^{usc}); The operator derivative $A^{(p)}$, $p=(p_1,\dots,p_n)$, of A is called the linear differential operator on G which transfer $A^{(p)}\circ\varphi^{-1}$, in each local chart of G, is the p-operator derivative of the transfer of A in this chart. Here the operator derivative $B^{(p)}$ of the linear differential operator $B=\Sigma b_kD^k$ in R^n is*

$$\Sigma\binom{k}{p}p!\,b_kD^{k-p},$$

with $D^r=\partial^{|r|}/\partial x^r$, $x=(x_1,\dots,x_n)$, $r=(r_1,\dots,r_n)$.

Definition 5. *The linear space α of linear differential operators is called differential-invariant if for each $A\in\alpha$ all operator derivatives $A^{(p)}$ of the operator A also belong to α, $p=(p_1,p_2,\dots,p_n)\in\mathrm{Z}^n_+$. In the case $G\subset\mathrm{R}^n$ if in addition all operators of α are with constant coefficient then α is called and homogeneous as well.*

Theorem 2. *Let R be an algebra of c-v functions of type C on the compact C^∞ differentiable n-dimensional manifold G, $n\geqq 2$, for which algebra R the algebra D_G^∞ is contained and is dense in R. Then 1. The algebra R is a completion of the algebra D_G^∞ by the norm p, $pf=\sup_{s\in G}|\omega_s(f)|_s$, where $\omega_s(f)$ is the image of f in the canonical homomorphism $R\to R/J(s)$. Here $J(s)$ is the minimal closed primary ideal of R at*

$s \in G$; $|.|_s$ is the factor norm in $R/J(s)$. Moreover $|\omega_s(f)|_s$ is an upper semicontinuous function in s. 2. The factor algebra $R/J(s)$ is isomorphous to K^N/I_s, for some N, where $K^N = \mathbb{C}[X_1, \ldots, X_n]/m^{N+1}$, ($\mathbb{C}[X_1, \ldots, X_n]$ is the formal polynomial ring over $\mathbb{C}$, m is the maximal ideal in it); I_s is an ideal in K^N, $s \in G$. 3. Let (U, φ) be a local chart of G, let F be a compact n-dimensional cube or ball in $\varphi(U)$, $\mathring{F} \neq \emptyset$. Then $R'_{F,\varphi} = \{(f \circ \varphi^{-1})|F, f \in R\}$ a) *is a regular algebra of c-v functions of type C on F relatively the norm $\|f \circ \varphi^{-1}\|_{R'_{F,\varphi}} = \sup_{s \in \varphi^{-1}(F)} \|f\|_s$;* b) *$R'_{F,\varphi}$ is a completion of $D^\infty | F$ by the norm p', $p'g = \sum_{A \in \mathfrak{B}(\alpha_F)} \sup_{x \in F} |Ag(x)|$ for some linear finite-dimensional differential-invariant space α_F of linear differential operators of order not larger than N; the absolute values of the coefficients of the operators in α_F are of D_F^{usc} — i.e. are upper semicontinuous on F; $\mathfrak{B}(\alpha_F)$ is a finite basis of α_F; D^∞ is the space of all infinitely differentiable c-v functions on $\mathbb{R}^n$ with compact supports. Moreover $I \in \alpha$.* 4.a) *R is a completion of D_G^∞ by the norm p,*

$$pf = \sum_{A \in \mathfrak{B}(\alpha)} \sup_{s \in G} |Af(s)|, \tag{1}$$

where α is a finite-dimensional linear differential-invariant space of linear differential operators, the absolute values of whose coefficients are in D_G^{usc}; $\mathfrak{B}(\alpha)$ is a finite basis of α; moreover $I \in \alpha$; b) *If α is a linear finite-dimensional differential-invariant space of linear differential operators with coefficients in D_G^∞, $I \in \alpha$, then the completion R of D_G^∞ with the norm* (1) *is an algebra of c-v functions of type C on G, with precision up to a natural isomorphism* (*where I is the identity operator*).

R e m a r k 1. As $R'_{F,\varphi}$ (resp. R) is a completion of $D^\infty | F$ (resp. D_G^∞) then for any fixed $f \in R'_{F,\varphi}$ (resp. $f \in R$) there exists a sequence (f_m), $f_m \in D^\infty | F$ (resp. $f_m \in D_G^\infty$), by which (f_m), it receives f, completing $D^\infty | F$ (resp. D_G^∞) by the norm p' (resp. p). From the kind of the norm p' in 3.b) (resp. p in 4.a)) it follows that there exists $\lim_m Af_m(v)$ for $\forall A \in \alpha_F$, $\forall v \in F$ (resp. $\forall A \in \alpha$, $\forall v \in G$). These limits are determined uniquely by the function f since $R'_{F,\varphi}$ (resp. R) is without radical. Let denote $\lim_m Af_m(v) = Af(v)$ and let call it A-generalized derivative of L. Schwartz-Sobolev type of f at the point v. It will be proved: 3.b′) (resp. 4.a′)). If Af is continuous on F (resp. on G), $A \in \alpha_F$ (resp. $A \in \alpha$), $\forall f \in R'_{F,\varphi}$ (resp. $\forall f \in R$), then the coefficients of A are continuous too.

E x a m p l e 3. Let $G = T$ be the n-dimensional torus, considered as a factor manifold $\mathbb{R}^n/\mathbb{Z}^n$ of the space $\mathbb{R}^n$ to the integer lattice $\mathbb{Z}^n$. In particular, Theorem 2 describes the algebras R of c-v functions of type C on T, in which the algebra $D_T^\infty \subset R$ is dense, by linear differential operators.

It is interesting to give such a definition of the homogeneity, which would provide the constance of the coefficients of the operators in the space α_F from Theorem 2.3. When G is an additive group, such a definition is given in [2] and the case is widely investigated in [2-10] and others. At first we shall give the definition of the homogeneity at a point $x_0 \in \mathbb{R}^n$ of an algebra R of c-v functions on $M \subset \mathbb{R}^n$, $x_0 \in \mathring{M}$. In the case $n = 1$, there is such a definition in [1]. Let $U(x, r) = \{y : |x - y| < r\}$, $r > 0$, x, $y \in \mathbb{R}^n$.

Definition 6. *An algebra R of c-v functions on $M \subset \mathrm{R}^n$ is called homogeneous at the point $x_0 \in \mathring{M}$ if there exists a neighbourhood $U(x_0, r_0) \subset M$, such that*: 1. *For each fixed $h \in \mathrm{R}^n$, $|h| < r_0$, the neighbourhood $U(x_0+h, r_0) \subset M$*; 2. *$R|U(x_0+h, r_0) = \{g(x+h)|U(x_0, r_0), g \in R\}$ (i.e., the set of the restrictions on $U(x_0+h, r_0)$ of the functions of R coincides with the set of the restrictions on $U(x_0, r_0)$ of $g(x+h)$, $g \in R$).*

Definition 7. *An algebra R of c-v functions on the connected n-dimensional manifold G is called homogeneous in the local chart (U, φ) of G if each of the algebras $R'_{F,\varphi}$, $\forall F \subset \varphi(U)$, is homogeneous at every point of $\mathring{F}$.*

Corollary 4. *In the conditions of the Theorem 2, if the algebra R is in addition homogeneous in the fixed local chart (U, φ) of the connected compact C^∞ manifold G, and $F \subset \varphi(U)$, then the corresponding space $\alpha_F = \alpha(F, \varphi)$ is still and homogeneous (see Definition 5).*

Remark. As it is clear from the results of K. de Leeuw and H. Mirkil in [8], if the algebra R of type C is homogeneous in each local chart of a complete atlas of the connected compact C^∞ manifold G, then $R = D_G^N$ for some N.

The works [1-13] convince in the significance of the algebras of c-v functions of type C. But the definition of the algebras of functions of type C claims the existence of $\sup_{s \in G} \|f\|_s$. To transfer the ideas to algebras of functions on non-compact spaces, here are examined projective limits of algebras of c-v functions of type C.

Definition 8. *An algebra R of c-v functions of type C_π on the topological space G is a projective limit of an up directed ordered family $\mathfrak{F}$ of algebras R_F of c-v functions of type C on F, $F \in \mathfrak{F}'$, such that*: 1. *Each F is a compact in G; $F = \overline{\mathring{F}}$;* 2. *$R_{F_1} \leq R_{F_2}$ if $F_1 \subseteq F_2$ and $(R_{F_2}|F_1) \subset R_{F_1}$;* 3. *$\cup_{F \in \mathfrak{F}'} F = G$.*

For the sake of simplicity of the account, we shall study only the case when the maps $i_{F_2}^{F_1} : R_{F_2} \to R_{F_1}$, $F_1 \subseteq F_2$ (which maps are included in the definition of the projective limits), are restrictions and when $\cup_{F \in \mathfrak{F}'} \mathring{F} = G$. In these requirements, the algebra R, with precision to a natural isomorphism, is equivalent to an algebra of c-v functions on G for which the canonical mappings $i^F : R \to R_F$ are $i^F(f) = f|F$.

Proposition 5. *Let R be an algebra of c-v functions of type C_π on the topological space G relatively the up directed ordered family of algebras R_F of type C on F, $F \in \mathfrak{F}'$. Let $\mathfrak{Q}'$ be another up directed ordered family of compacts $Q \subset G$, satisfying the requirements*: 1. *$Q = \overline{\mathring{Q}}$;* 2. *$Q_1 \leq Q_2$ if $Q_1 \subseteq Q_2$;* 3. *$\cup_{Q \in \mathfrak{Q}'} \mathring{Q} = G$. Then for each $Q \in \mathfrak{Q}'$ there exists some $F = F(Q) \in \mathfrak{F}'$, for which $Q \subset \mathring{F}$; the algebra $R_Q = \{f|Q, f \in R_F\}$ is an algebra of c-v functions of type C on Q in the norm ${}_Q\|f\| = \sup_{s \in Q}({}_F\|f\|_s)$, where ${}_F\|\cdot\|$ is the norm in R_F; and R, with precision up to a natural isomorphism, is a projective limit of the algebras R_Q, $Q \in \mathfrak{Q}'$.*

Proposition 6. *Let G be an open subset of R^n, $n \geq 2$. Let R be an algebra of c-v functions of type C_π on G, in which R, the algebra D_G^∞ is contained and is dense. Let $\mathfrak{F}'$ be an up directed ordered family of compacts $F \subset G$, $F = \overline{\mathring{F}}$, $F_1 \leq F_2$ if $F_1 \subseteq F_2$*

and $\cup_{F\in\mathfrak{F}'}\mathring{F}=G$. *Then* R *is a completion of* D_G^∞ *by the system of seminorms* (p_F), $F\in\mathfrak{F}'$, *where* $p_F f=\Sigma_{A\in\mathfrak{B}(\alpha)}\sup_{s\in F}|Af(s)|$ *for some linear finite dimensional differential-invariant space* α_F *of linear differential operators on* F; $\mathfrak{B}(\alpha_F)$ *is a finite basis of* α_F. (*Analogous results are true and for algebras of c-v functions of type* C_π *on n-dimensional* C^∞ *manifolds.*)

As examples, some algebras of c-v functions on closed and on open discs in $\mathbf{C}$ are investigated:

Definition. *The algebra* R *of functions on* G *is called homogeneous relatively the automorphism* $\varphi: G\to G$, *if* $f\in R$ *implies* $f\circ\varphi\in R$.

Let $Q_r=\{z: z\in\mathbf{C},\ |z|\leqq r\}$, $0<r<\infty$.

Proposition 7. *There exists an one-to-one mapping from the set of all algebras* R *of c-v functions of type* C *on* Q_r, *homogeneous relatively all analytic automorphisms of* Q_r, *in which* R, *the algebra* $D^\infty|Q_r$ *is contained and is dense, on the sets* ω *of operators of the kind*

$$(2)\quad \omega=\{\partial^N/\partial z^{m_j}\partial\bar{z}^{n_j},\ j=1,\dots,q;\ m_j+n_j=N,\ m_j-n_j=s_j\}=\omega(N, s_1,\dots,s_q),\ q\leqq N+1,$$

and such a mapping that R, (*with the corresponding* ω, $R\to\omega$), *is a completion of* $D^\infty|Q_r$ *by the norm* p,

$$p(g)=\sum_{|k|<N}\sup_{z\in Q_r}|D^k g(z)|+\sum_{j=1}^{q}\sup_{z\in Q_r}|(\partial^N/\partial z^{m_j}\partial\bar{z}^{n_j})g(z)|.$$

The algebra R, $(R\to\omega)$, *will be denoted by* $C_{Q_r}[\omega]$. *The inversity of this map signifies that if* $\omega_1\neq\omega_2$, *then* $C_{Q_r}[\omega_1]\neq C_{Q_r}[\omega_2]$.

Proposition 8. *Let* G *be either* $\mathring{Q}_r$ *or the complex plane* $\mathbf{C}$. *Then there exists a one-to-one map from the algebras* R *of c-v functions of type* C_π *on* G, *homogeneous relatively all analytic automorphisms of* G, *in which* $R\neq D_G^\infty$ *the algebra* D_G^∞ *is contained and is dense, on the sets* ω *of operators of the kind* (2); *and such a map that* R, $(R\to\omega)$, *is* (*with precision up to a natural isomorphism*), *a completion of* D_G^∞ *by the system of seminorms* p_n, $n=1,2,\dots$,

$$p_n(g)=\sum_{|k|<N}\sup_{z\in K_n}|D^k g(z)|+\sum_{j=1}^{q}\sup_{z\in K_n}|(\partial^N/\partial z^{m_j}\partial\bar{z}^{n_j})g(z)|,$$

where if $G=\mathring{Q}_r$, *then* $K_n=Q_{(rn/(n+1))}$; *if* $G=\mathbf{C}$, *then* $K_n=Q_n$. *The algebra* R, $(R\to\omega)$, *will be denoted by* $C_G[\omega]$.

K. de Leeuw and H. Mirkil receive the same in [5, 16] for very similar algebras to the algebras $C_{Q_r}[\omega]$, $C_G[\omega]$.

4. Proofs.

Proof of Proposition 1. Since R is a regular algebra of c-v functions of type C, hence for each $s\in G$ there exists a minimal closed primary ideal $J(s)$. Let in the canonical homomorphism $R\to R/J(s)$, the function $f\in R$ maps to the element $\omega_s(f)$. Obviously, the mapping $\omega_s(f)$, at a fixed point s, is a multiplicative homomorphism. We shall prove that $|\omega_s(f)|_s=F(s)$ (f arbitrary fixed), is an upper semicontinuous function in s: The upper semicontinuity of the function $F(s)$ is equivalent that all sets $\{s : F(s) \geqq A,\ s \in G\}$, $\forall A \in \mathrm{R}$, are closed. Let $(s_\nu) \to s_0$, s_ν, $s_0 \in G$, and $F(s_\nu) \geqq A$. We shall show that and $F(s_0) \geqq A$. We have the following

Theorem 9. (G. E. Shilov [1, 2]). *Let R be a regular algebra of c-v functions of type C on the compact Hausdorff space G. For each $s_0 \in G$ there exists a minimal primary ideal $\mathscr{J}(s_0)$ at the point s_0. It consists of all functions f in R, everyone of which vanishes on some neighbourhood of s_0. The closure $J(s_0)$ of $\mathscr{J}(s_0)$ is the minimal closed primary ideal at s_0. The functions in $J(s_0)$ are characterized by the property that for every $f(s) \in J(s_0)$ there exists a sequence (f_m), $f_m \in R$, such that $\|f_m\| \to 0$ and $f_m(s) = f(s)$ on some neighbourhood of s_0.*

Therefore

$$F(s) = |\omega_s(f)|_s = \inf\{\|\varphi\| : \varphi \in R,\ (\varphi - f) \in J(s)\} = \inf\{\|g\| : g \in R, \tag{3}$$

$g = f$ in some neighbourhood of $s\} = \|f\|_s$.

Since $\|f\|_{s_0} = \inf\{\|g\| : g \in R,\ g = f$ in some neighbouhood of $s_0\}$, hence for each $\varepsilon > 0$ there exists a function $g \in R$, $g(s) = f(s)$ in some neighbourhood U_{s_0} of the point s_0, such that $\|f\|_{s_0} \geqq \|g\| - \varepsilon$. The definition of the norm at a point s yields $\|f\|_s \leqq \|g\|$ for each point $s \in \mathring{U}_{s_0}$. Thus $\|f\|_{s_0} \geqq \|g\| - \varepsilon \geqq \|f\|_s - \varepsilon$, for $\forall s \in \mathring{U}_{s_0}$. For sufficiently large ν, we have $s_\nu \in \mathring{U}_{s_0}$. Hence $F(s_0) = \|f\|_{s_0} \geqq \|f\|_{s_\nu} - \varepsilon \geqq A - \varepsilon$. Since $\varepsilon > 0$ was chosen arbitrarily, $\varepsilon \to 0$ involves $F(s_0) = \|f\|_{s_0} \geqq A$. This proves that $|\omega_s(f)|_s$ is upper semicontinuous in s on G.

Furthermore the algebra R is an algebra of functions of type C on G, that is why the norm in R is equivalent to the norm p

$$pf = \sup_{s \in G} \|f\|_s,\quad f \in R. \tag{4}$$

From the definition of the point-norm $\|\,.\,\|_s$, it is evident (see Eq. (3)) that $\|f\|_s$ is equal to the norm of the image of f in $R/J(s)$ in the canonical homomorphism $R \to R/J(s)$. Since $\mathcal{O}$ is dense in R, then (3) and (4) involve that R is a completion $\mathcal{O}$ by the norm p, $pf = \sup_{s \in G} |\omega_s(f)|_s$. ■

Remark. The sets D^ν_G, $\nu = 0, 1, \ldots; \infty$, are algebras relatively the pointwise multiplication and Freche spaces, (see for instance [17, 18]). Since G is a compact C^∞ manifold, hence D^N_G, $N = 0, 1, \ldots$, are normable, for example in the following way: As G is a compact C^∞ maifold, there exists a finite number of compact cubes $F_1, \ldots, F_m$ such that 1. $F_j \subset \varphi_j(U_j)$, $j = 1, \ldots, m$, where (U_j, φ_j) are local charts of the manifold G. 2. $\cup_j \varphi_j^{-1}(\mathring{F}_j) = G$. Then the norm in D^N_G can be given by $\|f\| = \Sigma_j \Sigma_{|k| \leq N} \sup_{x \in F_j} |D^k(f \circ \varphi_j^{-1})(x)|$. This norm is evidently equivalent to the norm $\sup_{s \in G} \|f\|_s$, i.e., each D^N_G is an algebra of c-v functions of type C on the compact C^∞ manifold G, $N = 0, 1, \ldots$ (As obviously, each algebra D^N_G is without radical since: a) All maximal ideals of D^N_G are of the kind $M(s)$, $\forall s \in G$, as G is compact; $D^N_G \subset D^0_G$; $f \in D^N_G$ implies $\bar{f} \in D^N_G$ and $f \in D^N_G$ with $f(s) \neq 0$ for $\forall s \in G$ involves

$(1/f)\in D_G^N$ (see Lemma 5, 2, Ch. I of [1]). b) The former point involves that if f belongs to all maximal ideals of D_G^N then $f\equiv 0$ on G and so $\|f\|=0$.)

Let D^N be the algebra of all c-v functions with continuous derivatives till order N inclusively, with compact supports in $\mathbf{R}^n$.

Lemma 10 [19]. *The minimal closed primary ideal in D^N at the point $x=0$ is*

$$J^N(0)=\{f: f\in D^N,\ D^k f(0)=0 \quad \text{for} \quad \forall k,\ |k|\leqq N\}.$$

For $N=1$ this lemma is proved by G. E. Shilov in [4].

Proof. According to the Shilov's Theorem 9, a function $f\in J^N(0)$ if and only if f is a limit in D^N of such a sequence (f_m), $f_m\in D^N$, that each f_m is equal to zero on some neighbourhood U_m of the origin. Hence $J^N(0)\subset\{f: f\in D^N,\ D^k f(0)=0,\ \forall k,\ |k|\leqq N\}$. It is well known (for instance see [20]), that for $\forall\delta>0$ there exists an infinitely differentiable function $\varphi_{1/\delta}$, such that $0\leqq\varphi_{1/\delta}\leqq 1$; $|D^k\varphi_{1/\delta}|\leqq C_k/\delta^{|k|}$, C_k is a constant; and

$$\varphi_{1/\delta}=\begin{cases}0 & \text{on} \quad |x|\leqq\delta/4,\\ 1 & \text{on} \quad |x|>\delta, \qquad x\in\mathbf{R}^n.\end{cases}$$

For each $f\in D^N$ with $D^k f(0)=0$ for $\forall k$, $|k|\leqq N$, it is easily verified that the sequence (f_m), $f_m=f\varphi_m$, satisfies the requirements: $f_m\in D^N$, $f_m(x)=0$ on some neighbourhood of the origin; $(f_m)\to f$ in D^N. Hence $f\in J^N(0)$. ■

Let F be a compact in $\mathbf{R}^n$ with $\overline{\mathring{F}}=F\neq\emptyset$. For the further proof let investigate $(D^N|F)/J_F^N(x)$, where $J_F^N(x)$ is the minimal closed primary ideal in $D^N|F=\{f|F,\ f\in D^N\}$ at the point $x\in F$.

Lemma 11 [19]. *$(D^N|F)/J_F^N(x)$ is isomorphous to $D^N/J^N(0)=\mathscr{K}^N$ and*

$$\mathscr{K}^N\cong(\mathbf{C}[X_1,\dots,X_n]/m^{N+1})=K^N. \tag{5}$$

Proof. $D^N|F$ and D^N are locally isomorphous on $\mathring{F}$. Therefore $(D^N|F)/J_F^N(x)$ is isomorphous (and with norm i.e. preserving the norm) to $D^N/J^N(x)$, $\forall x\in\mathring{F}$, where $J^N(x)$ is the minimal closed primary ideal of D^N at the point x. Moreover D^N is homogeneous. That is why it is sufficient to investigate $D^N/J^N(0)$. According to the Weierstrass' Theorem, the polynomials in $x_1,\dots,x_n$ form an everywhere dense set in D^N. Therefore the polynomial's images in the canonical homomorphism $D^N\to D^N/J^N(0)$ are dense. Let denote $D^N/J^N(0)=\mathscr{K}^N$. Since $J^N(0)=\{f: f\in D^N,\ D^k f(0)=0$ for $\forall k,\ |k|\leqq N\}$, hence each x^k, $|k|>N$, belongs to $J^N(0)$ and the image X^k of each x^k, $|k|>N$, is equal to 0. Therefore $\mathscr{K}^N$ is finite dimensional and (5) is true. ■

Proof of Theorem 2. Proposition 1 proves the point 1 of this Theorem as R is regular since $D_G^\infty\subset R$. The point 2 of Theorem 2 follows in part by point 3 (which will be proved independently from the assertion of point 2): Let fix an arbitrary $s\in G$. Let (U,φ) be a local chart of the manifold G for which $s\in U$. Let F be a compact n-cube in $\varphi(U)$ with $\varphi(s)\in\mathring{F}$. Evidently $R/J(s)$ and $R'_{F,\varphi}/J'_{F,\varphi}(\varphi(s))$

are isomorphous and in norm (i.e. preserving the norm), where $J'_{F,\varphi}$ $(\varphi(s))$ is the minimal closed primary ideal of the algebra $R'_{F,\varphi}$ at the point $\varphi(s)$. In the point 3, independently from the point 2, will be proved that $R'_{F,\varphi}/J'_{F,\varphi}$ $(\varphi(s))$ is isomorphous to K^N/I_s for some $N=N(F, \varphi)$. Since G is a compact manifold we can choose N independently from the cube F and the chart (U, φ).

The same may be received and directly: Since R is an algebra of type C, then all functions in R are continuous. Applying to the inclusions $D_G^\infty \subset R \subset D_G^0$ the Closed Graph Theorem and the method from the point 3.a) we get again $R/J(s) \cong K^N/I_s$.

Remark 2. In this article, the Closed Graph Theorem is used to compare the topologies of some algebras $R_1 \subset R_2$. In all such cases here, R_1 and R_2 are regular algebras of c-v functions on the same compact G (or F) without radical and moreover R_2 is of type C, R_1 is Fréchet space. For the norm in R_2, $\|.\|$, is true $|f(s)| \leqq \|f\|$, $\forall f \in R_2$, $\forall s \in G$, where $|f(s)|$ is the absolute value of f at the point s: The norm in the quotient algebra $R_2/M(s)$ is not larger than the norm in R_2, where $M(s)$ is the maximal ideal of R_2 at the arbitrary fixed $s \in G$. But $R_2/M(s) \cong \mathrm{C}$, and the image of $f \in R_2$ in $R_2/M(s)$ (by the canonical homomorphism $R_2 \to R_2/M(s)$) is $f(s)$. Hence $\|f\| \geqq |f(s)|$. So $\|f\| \geqq \sup_G |f(s)|$. If $D_G^\infty \subset R_2$; D_G^∞ is dense in R_2, then $R_2 \subset D_G^0$).

Then by the convergence of a sequence (f_m), $f_m \in R_2$, in norm of R_2, it follows its pointwise convergence on G. Since in all our cases from the convergence in the topology of R_1 also it follows pointwise convergence, therefore, applying the Closed Graph Theorem, we get the continuity of the inclusion map $i: R_1 \to R_2$. At last, the continuity of the inclusion map i permits to compare the topologies in R_1 and R_2.

Proof of point 3 of Theorem 2. Since G is a C^∞ manifold and as $D_G^\infty \subset R$, it is obvious that $R'_{F,\varphi}$ is a regular algebra of c-v functions of type C on F relatively the indicated norm in 3.a).

Furthermore $D^\infty|F \subset R'_{F,\varphi}$, since $D_G^\infty \subset R$. Let $g \in D^\infty|F$, let F_1 be a compact cube or a ball in $\varphi(U)$ such that $F \subset \mathring{F}_1$. There exists a function $g_1 \in D^\infty$ such that supp $g_1 \subset F_1$ and $g_1 = g$ on F. Then the function

$$v(s) = \begin{cases} (g_1 \circ \varphi)\ (s) & \text{if } s \in U \\ 0 & \text{if } s \notin U \end{cases} \quad \text{belongs to } D_G^\infty \text{ and } g|F = (v \circ \varphi^{-1})|F.$$

The algebra $D^\infty|F$ is dense in $R'_{F,\varphi}$ since the algebra D_G^∞ is dense in R. In each algebra of c-v functions of type C, the norm is stronger than the pointwise convergence (see Remark 2), and hence we can use the Closed Graph Theorem to the inclusion $D^\infty|F \subset R'_{F,\varphi}$, which implies that the identity map $: D^\infty|F \to R'_{F,\varphi}$ is continuous, which is equivalent to, that there exists some $N = N(F, \varphi)$, such that $D^N|F \subset R'_{F,\varphi}$. Thus let $D^N|F \subset R'_{F,\varphi} \subset D^Q|F$, $0 \leqq Q \leqq N$. Further let assume N be the least integer with this property; Q – the largest, and that $Q < N$.

The following Shilov's Theorem will be used:

Theorem 12. (G. E. Shilov [1, 2]). *Let $R_1 \subset R_2$ be two regular Banach algebras without radical and with a same space G of maximal ideals. Then there* *exists a continuous algebraic homomorphism λ for which the following diagram is commutative*

$$\begin{array}{ccc} R_1 & \xrightarrow{\quad\theta\quad} & R_2 \\ \gamma_1\downarrow & & \downarrow\gamma_2 \\ R_1/J_1(t_0) & \xrightarrow{\quad\lambda\quad} & R_2/J_2(t_0) \end{array}$$

where $J_1(t_0)$ and $J_2(t_0)$ are the minimal closed primary ideals of R_1 and R_2 respectively, at the arbitrary fixed point $t_0 \in G$; γ_i, $i=1, 2$, are the canonical homomorphisms; θ is the including map.

Applying this theorem at first to $R_1 = R'_{F,\varphi}$, $R_2 = D^Q|F$, and afterwards to $R_1 = D^N|F$, $R_2 = R'_{F,\varphi}$, we receive continuous algebraic homomorphisms λ_1 and λ_2 for which:

$$K^N = (D^N|F)/J_F^N(\varphi(s)) \xrightarrow{\lambda_2} R'_{F,\varphi}/J'_{F,\varphi}(\varphi(s)) \xrightarrow{\lambda_1} K^Q = (D^Q|F)/J_F^Q(\varphi(s)),$$

(see also Lemma 11).

The composition of these homomorphisms is the natural projection, as follows from the commutativity of the diagram in Shilov's Theorem 12. Therefore $R'_{F,\varphi}/J'_{F,\varphi}(\varphi(s)) \cong K^N/I_{\varphi(s)}$, where $I_{\varphi(s)} = I_s$ is an ideal in K^N, generated by elements of the form $\Sigma_{Q<|k|\leq N}\, a_k X^k$.

From Proposition 1, since $R'_{F,\varphi}$ is a regular algebra of c-v functions of type C on F and $D^\infty|F$ is contained and is dense in $R'_{F,\varphi}$, hence $R'_{F,\varphi}$ is a completion of $D^\infty|F$ by the norm p_1,

$$p_1 g = \sup_{\varphi(s)\in F} |\omega'_{\varphi(s)}(g)|_{\varphi(s)}, \tag{6}$$

where $\omega'_{\varphi(s)}(g)$ is the image of $g \in D^\infty|F$ in the canonical homomorphism $\omega'_{\varphi(s)}: R'_{F,\varphi} \to R'_{F,\varphi}/J'_{F,\varphi}(\varphi(s)) = K_{\varphi(s)}$, and where $|\,.\,|_{\varphi(s)}$ is the norm in $K_{\varphi(s)}$. Since $K_{\varphi(s)} \cong K^N/I_{\varphi(s)}$, if $g = \Sigma_k c_k (a_k x - \varphi(s))^k$, then

$$\omega'_{\varphi(s)}(g) = \sum_{|k|\leq N} c_k (a_k X)^k, \tag{7}$$

where $X = (X_1, \ldots, X_n)$ is the image of x, and X is the generator in K^N. The norm in the finite dimensional algebra $K^N/I_{\varphi(s)}$ up to an equivalence is determined by each basis of the linear functionals on K^N, which are zero on $I_{\varphi(s)}$. In $D^N|F$ these are the continuous linear functionals concentrated at the point $\varphi(s)$, which are zero on $I^*_{\varphi(s)}$ — the closure of the proimage of $I_{\varphi(s)}$ in $D^N|F$. As it is well known, the general form of the continuous linear functionals in D^N, concentrated at the point $\varphi(s)$, is $\Sigma_{|k|\leq N}\, d_k[\varphi(s)]\, D_x^k[\delta(x-\varphi(s))]$, where $d_k(\varphi(s))$ are c-v functions in $\varphi(s)$, $\delta(x)$ is the Dirac's distribution and $D_x^k = \partial^{|k|}/\partial x_1^{k_1} \ldots \partial x_n^{k_n}$. This is evident and directly from (6) and (7) since for $g(x) = \Sigma_k (x-t)^k D^k g(t)/k!$ with $t = \varphi(s)$, we get

$\omega'_{\varphi(s)}(g)=\Sigma_{|k|\leqq N} X^k D^k g(t)/k!$ in finite dimensional space $K^N/I_{\varphi(s)}$, and since the set of all g of this kind is dense in $R'_{F,\varphi}$.

Let α^* be the C-linear space of all linear differential operators of order not larger than N on F, which are zero on $I_{\varphi(s)}$ at the point $t=\varphi(s)$ for each $t\in F$. To obtain the mapping $R'_{F,\varphi}\to\alpha\subset\alpha^*$, where α is with the properties of point 3.b) of Theorem 2, let construct it in the following way: Each ideal $I_{\varphi(s)}=I_t$ has a basis $\mathfrak{B}[I_t]$ of the form $\mathfrak{B}[I_t]=\{\Sigma_{Q<|k|\leqq N} B^j_k X^k,\ j=0,\ 1,\dots,p\}$, where the elements $\Sigma_{Q<|k|\leqq N} B^j_k X^k,\ j=0,\ 1,\dots,p$, with $B^0_k=0$, are linearly independent; B^j_k are constants; $\varphi(s)=t$ is fixed. Therefore we may determine

$$X^{k_j}=\sum_{\substack{Q<|k|\leqq N\\ k\neq k_q,\,|k|\leqq|k_j|}} A^j_k X^k \pmod{I_{\varphi(s)}},\quad q,\ j=0,\ 1,\dots,p,\quad A^0_k=0,$$

where s is fixed, $A^j_k\in\mathbf{C}$, and if $|k|>|k_j|$ then $A^j_k=0$. Thus

$$K_{\varphi(s)}=K_t=K^N/I_{\varphi(s)}=\left\{\sum_{|k|\leqq N} a_k X^k,\quad X^{k_j}=\sum_{\substack{Q<|k|\leqq N\\ k\neq k_q,\,|k|\leqq|k_j|}} A^j_k X^k,\quad j,\ q=0,\ 1,\dots,p\right\}.$$

Let

$$K_t\ni\sum_{|k|\leqq N} a_k X^k=\sum_{|k|\leqq N,\,k\neq k_j} a_k X^k+\sum_{j=0}^{p} a_{k_j}\sum_{\substack{Q<|k|\leqq N\\ k\neq k_q,\,|k|\leqq|k_j|}} A^j_k X^k=\sum_{|k|\leqq Q} a_k X^k$$

$$+\sum_{\substack{Q<|k|\leqq N\\ k\neq k_q,\,|k|\leqq|k_j|}}\left(a_k+\sum_{j=0}^{p} a_{k_j}A^j_k\right)X^k,\quad j,\ q=0,\ 1,\dots,p,$$

where if $|k|>|k_j|$ then $A^j_k=0$. The space K^N is finite dimensional, that is why the norm in $K^N/I_{\varphi(s)}$ is determined uniquely, with precision up to equivalence by:

$$\left|\sum_{|k|\leqq N} a_k X^k\right|_{\varphi(s)}=\sum_{|k|\leqq Q} b_k(t)|a_k|+\sum_{\substack{Q<|k|\leqq N\\ k\neq k_q,\,|k|\leqq|k_j|}} d_k(t)\left|a_k+\sum_{j=0}^{p} a_{k_j}A^j_k\right|,$$

where $A^j_k(t)=0$ if $|k|>|k_j|$: $t=\varphi(s)=(t_1,\dots,t_n)$; $b_k(t)>0$, $d_k(t)>0$.

This implies, for $P(x)=\Sigma_k (x-t)^k (D^k P_*)(t)/k!$ on F, that

$$(8)\qquad |\omega'_t(P)|_t=\sum_{|k|\leqq Q} b_k(t)|D^k P_*(t)/k!|$$

$$+\sum_{\substack{Q<|k|\leqq N\\ k\neq k_q,\,|k|\leqq|k_j|}} d_k(t)\left|\frac{D^k P_*(t)}{k!}+\sum_{j=0}^{p(t)} A^j_k(t) D^{k_j}P_*(t)/k_j!\right|.$$

According to the Proposition 1, $R'_{F,\varphi}$ is a completion of $D^\infty|F$ by the norm p_1:

(9)
$$p_1 P=\sup_{t\in F}|\omega_t'(P)|_t=\sup_{t\in F}\sum_{|k|\leqq Q} b_k(t)\left|\frac{D^k P_*(t)}{k!}\right|$$
$$+\sum_{\substack{Q<|k|\leqq N\\ k\neq k_q,|k|\leqq|k_j|}} d_k(t)\left|\frac{D^k P_*(t)}{k!}+\sum_{j=0}^{p(t)} A_k^j(t)\,D^{k_j}P_*(t)/k_j!\right|.$$

Since $R'_{F,\varphi}$ is an algebra of type C, then $\|f\|\geqq|f(t)|$ for $\forall f\in R'_{F,\varphi}$, $\forall t\in F$ (see Remark 2). Hence the topology in $R'_{F,\varphi}$ is stronger than the pointwise convergence and the Closed Graph Theorem is applicable to the inclusion $R'_{F,\varphi}\subset D^Q|F$. Then by continuity of the inclusion mapping, we get that the norm in $R'_{F,\varphi}$ is equivalent to a norm of the kind (9) with all $b_k(t)\equiv 1$ on F, $|k|\leqq Q$. Further we shall consider $R'_{F,\varphi}$ with the latter completing norm.

As the norm in $R'_{F,\varphi}$ is not less than $\|f\|_s=|\omega_t'(f)|_t$, (since the quotient norm is not larger than the norm of $R'_{F,\varphi}$, or since $R'_{F,\varphi}$ is an algebra of type C), and from (8), (9) we conclude that at a fixed point t, for each function f of $R'_{F,\varphi}$, there exist all derivatives $(D^k/k!+\Sigma_j A_k^j D^{k_j}/k_j!)f(t)$, $Q<|k|\leqq N$, $k\neq k_q$, $|k|\leqq|k_j|$, $q,j=0,1,\dots,p(t)$, of generalized Schwartz-Sobolev type in the following sense (see also [21]): $R'_{F,\varphi}$ is a completion of $D^\infty|F$ by the norm (9). Then for each $f\in R'_{F,\varphi}$ there exists a sequence (f_m), $f_m\in D^\infty|F$, such that $(f_m)\xrightarrow{\text{uniformly on } F} f$ and moreover (f_m) is a Cauchy sequence in the norm (9). Thus for $\forall t\in F$ with $d_k(t)\neq 0$ there exist the limits $\lim_m (D^k/k!+\Sigma_{j=0}^{p(t)} A_k^j(t)D^{k_j}/k_j!)(f_m)(t)$ for $\forall k$, $Q<|k|\leqq N$, $k\neq k_q$, $|k|\leqq|k_j|$, j, $q=0,1,\dots,p(t)$. Since $R'_{F,\varphi}$ is an algebra of c-v functions without radical, hence these limits depend only on the function $f\in R'_{F,\varphi}$ and do not depend on the choice of the sequence (f_m) with the cited properties (If we assume the contrary, i.e. that there exists a sequence (g_m) with the same properties for which (g_m), the corresponding limit for some k is another, then $(f_m-g_m)\xrightarrow{\text{uniformly on } F} 0$, moreover (f_m-g_m) is a Cauchy sequence in the norm (9). Thus the element of $R'_{F,\varphi}$, determined by (f_m-g_m), belongs to the radical of $R'_{F,\varphi}$ but it is not the zero of $R'_{F,\varphi}$.) Also since $R'_{F,\varphi}$ is an algebra of c-v functions without radical hence these generalized derivatives exist and in some neighbourhood of t. Therefore there exists an equivalent completing norm of the form (9) in which $X_{j_1},\dots,X_{j_p}$ can be chosen the same as at the fixed point t_0 and in a neighbourhood U_* of t_0. The next steps of the proof of 3.b) are:

(i) Let $k\neq k_j$, $Q<|k|\leqq N$. For each $t_0\in\mathring{F}$, there exists a ball-neighbourhood U^0 of t_0, $U^0\subset U_*$, such that:

(i.1) All $d_k(t)$ are upper semicontinuous on U^0.

(i.2) For each d_k there exists the alternative: (i.2′) either $\min_{U_0} d_k(t)\neq 0$ on some neighbourhood U_0 of t_0, $U_0\subset U^0$, or the other items in (9) dominate the corresponding operator $M_k=(D^k/k!+\Sigma_j A_j^k(t)D^{k_j}/k_j!)$ on some neighbourhood U_0

of t_0, $U_0 \subset U^0$, in $D^\infty | U_0$. Thus in the case (i.2′) the norm p_1 (by which we complete $D^\infty | F$ to receive $R'_{F,\varphi}$) is equivalent to a norm of the same form (9) with $\min_{U_0} d_k(t) \neq 0$. This is p_1 is equivalent to a norm of the kind (9) with $d_k(t) = 1$ on U_0. (i.2″) The remaining cases are: $\min_V d_k(t) = 0$ on each neighbourhood V of t_0, and the other items in (9) do not dominate the operator M_k in $D^\infty | V$ for each neighbourhood V of t_0. Let E_0^k be the set of all points in U_0 with (i.2″), let $E_0 = \cup_k E_0^k$, $Q < |k| \leqq N$, $k \neq k_j$.

(i.3) $d_k(t)$ has in addition the following properties: (i.3′) If $d_k(t^*) \neq 0$ for some $t^* \in U^0$ then $d_k(t) \neq 0$ for $\forall t \in U^0$ (U^0 is from (i.1)).

Remark 3. Let fix an arbitrary $f \in R'_{F,\varphi}$. Let (f_m), $f_m \in D^\infty | F$, be a sequence by which is received f, completing $D^\infty | F$ by the norm (9). Let $d_k(t^*) \neq 0$, $t^* \in U^0$. Since $d_k(t) \neq 0$ on U^0, hence for each $t \in U^0$ there exists $\lim_m M_k f_m(t)$. As $R'_{F,\varphi}$ has no radical, this limit is unique, i.e., it depends only on f and on t, but it does not depend on the choice of (f_m). This limit will be denoted by $M_k f(t)$ and will be called M_k generalized derivative of Schwartz-Sobolev type of the function f at the point t.

(i.3″) If $d_k(t_0) \neq 0$, then $\lim_{t \to t^*, t \neq t^*} d_k(t) \neq 0$, t, $t^* \in U^0$.

(i.3‴) Evidently if $d_k(t_0) \neq 0$ and $\min_V d_k(t) = 0$ on $\forall V \subset U^0$ (V is a neighbourhood of t_0), then there exists a sequence of points $(t_m) \to t_0$, $t_m \in U^0$, with $(|t_0 - t_m|)$ decreasing, such that $d_k(t_m) < 1/m$. As $d_k(t)$ is upper semicontinuous, then there exists an n-ball $\Delta_m, \mathring{\Delta}_m \neq \emptyset$, centered at t_m, such that $d_k(t) < 1/m$ on $\Delta_m \subset U^0$. Moreover (i.3″) and the boundness of $d_k(t)$ on every compact $\subset U^0$ yield that there exists a point-sequence $(v_m) \to t_0$ with $\lim_m d_k(v_m) = a > 0$.

(i.3$^{\text{IV}}$) α) If $s_0 \in U^0$ and if (s_q) is a sequence with $s_q \to s_0$, $s_q \in U^0$, $d_k(s_q) \geqq a > 0$, then $\lim_q M_k f(s_q) = M_k f(s_0)$ (see Remark 3), $\forall f \in R'_{F,\varphi}$; β) If for some cube $L \subset U^0$, $\mathring{L} \neq \emptyset$, there exists such $a > 0$ that the set $L_a = \{t \in L,\ d_k(t) \geqq a\}$ is dense in L, then the completing norm p_1 of $R'_{F,\varphi}$ is equivalent to the norm of the form (9) with $d_k(t) \equiv 1$ on L.

(i.3$^{\text{V}}$) Up to equivalence, the norm (9) can be chosen such that the set E_0 of all points in U^0 with (i.2$^{\text{II}}$) for some k, $Q < |k| \leqq N$, is not dense in any n-ball $B \subset U^0$, $\mathring{B} \neq \emptyset$.

(ii) I) All $A_k^j(t)$ are continuous on $U^0 - E_0$. II) All $|d_k(t) A_k^j(t)|$ are upper semicontinuous on U^0, $Q < |k| \leqq N$, $k \neq k_j$, $j = 0, \ldots, p(t)$.

(iii) Since F is a compact, contained in F_1 (where F_1 was a compact n-cube or n-ball in $\varphi(U)$, to whose algebra $R'_{F_1,\varphi}$ we can apply the obtained results), hence there exists a finite number of points $t_1, \ldots, t_r$ in F, such that $\cup_m^r U_m \supset F$, where U_m is a neighbourhood of t_m corresponding to U^0. Let $E = \cup_m^r E_m$ (E_m is the set for U_m corresponding to E_0 from (i)). From (i.3$^{\text{V}}$) follows that E is not dense in any n-ball $B \subset F$, $\mathring{B} \neq \emptyset$.

Further (6)-(9) and (i)-(iii) yield that there exists a C-linear finite dimensional space α_F of linear differential operators on F, which α_F is generated by the operators: $\{D^k,\ \forall k$ with $|k| \leqq Q$; $B_1 = \Sigma_{Q < |k| \leqq N} B_{lk} D^k$, $l = 1, \ldots, m\}$. The absolute values of the coefficients B_{lk} are upper semicontinuous. Let E be the set of all points in which some of the coefficients B_{lk} is not continuous. E is not dense in any ball $B \subset F$, $\mathring{B} \neq \emptyset$; Moreover for $\forall V$ with $V \cap E \neq \emptyset$, there exists some l' that $\min_V |B_{l'k}| = 0$, for $\forall k$, $Q < k \leqq N$.

In the case 3.b′) of Theorem 2, since (i.3.^{III}) and (i.3^{IV}) hold, then up to equivalence of the norm (9), we can assume $d_k(t)=1$ on U_m, $\forall k$, if $d_k(t_m)\neq 0$. Thus $E=\emptyset$. Further, this space α_F, corresponding to $R'_{F,\varphi}$, is such that $R'_{F,\varphi}$ is a completion of $D^\infty|F$ with the norm p', indicated in point 3.b). Thus we obtained a correspondence of $R'_{F,\varphi}\to\alpha_F$, asserted in 3.b), with an exception of the differential-invariance of α_F, which will be proved in Lemma 13. So, to finish the proof of 3.b) and 3.b′) in Theorem 2, it remains to prove (i), (ii) and Lemma 13.

The proof of (i.1) is by induction: Let $|k'|=Q+1$, $k'\neq k_j$. Let put in (8) $P(x)=(x-t_0)^{k'}$. Then

$$|\omega'_{t_0}(P(x))|_{t_0}=\sum_{|k|\leqq Q}(D^kP)(t_0)+d_{k'}(t_0).$$

Since $|\omega'_t(P(x))|_t$ is an upper semicontinuous function in t, hence $d_{k'}(t)$ is also upper semicontinuous in t at least on a neighbourhood of U'_0 of t_0, $U'_0\subset U_*$. The intersection U''_0 of all U'_0 for $\forall k$ with $|k|=Q+1$, $k\neq k_j$, is also a neighbourhood of t_0 in which all d_k, $|k|=Q+1$, $k\neq k_j$, are upper semicontinuous in t. Analogously, by inductive arguments, all $d_k(t)$, $Q<|k|\leqq N$, $k\neq k_j$, are upper semicontinuous on a ball-neighbourhood U^0 of t_0, $U^0\subset U_*$ Point (i.2) consists of an exhaustive alternative.

P r o o f o f (i.3′). Let suppose the contrary. For the sake of simplicity, let consider only the case $n=1$, $U^0=[0, 1]$, $d_k(1)=0$. $d_k(t)$ is upper semicontinuous. Hence for each $\varepsilon>0$ there is $\delta=\delta(\varepsilon)>0$ such that if $|1-t|<\delta$, then $d_k(t)<\varepsilon$. The sequence (f_m), $f_m=k!t^m/\binom{m}{k}$, is with the properties $f_m\in D^\infty|[0, 1]$; $D^lf_m\xrightarrow{\text{uniformly on }[0,1]}0$ for $\forall l$, $|l|\leqq|k|$, $l\neq k$. Moreover $|d_kD^kf_m(t)|\leqq d_k(t)<\varepsilon$ if $|1-t|<\delta$. On $[0, 1-\delta]$, the nonnegative upper semicontinuous function $d_k(t)$ is bounded, $|d_k(t)|\leqq M$. Thus on $[0, 1-\delta]$ we have $|d_k(t)D^kf_m(t)|\leqq MD^kf_m(t)$ for $\forall m>\nu^*$, where $\nu^*=\nu^*(\delta)$ is sufficiently large. Hence the convergence on [0, 1] of $(d_kD^kf_m)$ to 0 as $m\to\infty$ is uniform. But

$$D^kf_m(t)\to\begin{cases}0 \text{ at } t\in(0, 1)\\ 1 \text{ at } t=1\end{cases}.$$

Thus if for some $t^*\in[0, 1]$, $d_k(t^*)\neq 0$, then the corresponding algebra $R'_{F,\varphi}$ would have a radical. ■

Sketch of the proof of (i.3″). Let suppose the contrary. For the sake of simplicity we shall consider only the case $n=1$, $U^0=[0, 1]$, $t^*=1$. (From (i.3′), $d_k(1)\neq 0$). Since $\lim_{t\to 1, t\neq 1}d_k(t)=0$, hence for $\forall\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ such that $d_k(t)<\varepsilon$ if $1-t<\delta$, $t\neq 1$. Then, in the same way as in (i.3′), is proved that the corresponding algebra $R'_{F,\varphi}$ would have a radical. ■

P r o o f o f (i.3^{IV}). Let (f_m), $f_m\in D^\infty|F$, be a sequence by which it is received f, completing $D^\infty|F$ by the norm (9). On U^0 there exists a limit of $H_m=(D^k/k!+\Sigma_jA^j_k(t)D^{k_j}/k_j!)f_m$ (see Remark 3). Let $\lim H_m(t)=H(t)$. Point α): Let fix an arbitrary $\varepsilon>0$. From (9) the convergence of the sequence $\{d_k H_m\}_m$ is uniform on U^0, i.e. for each $\varepsilon>0$ $\exists \nu=\nu(\varepsilon)$ such that if $m>\nu$ then $|d_k(t)H_m(t)-d_k(t)H(t)|<\varepsilon$, $\forall t\in U^0$. Thus $|H(s_q)-H_m(s_q)|\leqq a^{-1}|d_k(s_q)H(s_q)-d_k(s_q)H_m(s_q)|<\varepsilon a^{-1}$ if $m>\nu$ for $\forall q$. As $H_m\to H$ on U^0, then for s_0 there exists such ν_0 that $|H_m(s_0)-H(s_0)|<\varepsilon$ if $m>\nu_0$. Let fix an arbitrary $m_*>\nu_0$, ν. The function H_m is continuous on U^0. Then there is μ such that if $q>\mu$ then $|H_{m_*}(s_q)-H_{m_*}(s_0)|<\varepsilon$. Hence

$$|H(s_q)-H(s_0)|\leqq|H(s_q)-H_{m_*}(s_q)|+|H_{m_*}(s_q)-H_{m_*}(s_0)|$$

$$+|H_{m_*}(s_0)-H(s_0)|<3\varepsilon \text{ if } q>\mu. \blacksquare$$

Since (i.3$^{\text{III}}$) and (i.3$^{\text{IV}}$), α) hold, for a proof of (i.3$^{\text{IV}}$), β) it is sufficient to show that $H(t)$ is continuous on L for each $f\in R'_{F,\varphi}$. Let s_0, $s\in L$, s_1, $s_2\in L_a$. We have

$$|H(s)-H(s_0)|\leqq|H(s)-H_{m'}(s)|+|H_{m'}(s)-H_{m'}(s_1)|+|H_{m'}(s_1)-H(s_1)|$$

$$+|H(s_1)-H_{m_*}(s_1)|+|H_{m_*}(s_1)-H_{m_*}(s_2)|+|H_{m_*}(s_2)-H(s_2)|$$

$$+|H(s_2)-H_{m''}(s_2)|+|H_{m''}(s_2)-H_{m''}(s_0)|+|H_{m''}(s_0)-H(s_0)|.$$

Let fix arbitrary $\varepsilon>0$. There exists such ν^* that if m', m'', $m_*>\nu^*$ we get

$$|H(s)-H(s_0)|<4\varepsilon a^{-1}+|H(s)-H_{m'}(s)|+|H_{m'}(s)-H_{m'}(s_1)|$$

$$+|H_{m_*}(s_1)-H_{m_*}(s_2)|+|H_{m''}(s_2)-H_{m''}(s_0)|+|H_{m''}(s_0)-H(s_0)|.$$

Let fix an arbitrary $m_*>\nu^*$. For the continuous function H_* on each fixed compact neighbourhood V^0 of s_0, $V^0\subset U^0$, there is $\delta^*=\delta^*(m_*)>0$ such that if $|s_1-s_2|<\delta^*$ then $|H_{m_*}(s_1)-H_{m_*}(s_2)|<\varepsilon$. For s_0 there exists ν_0 such that if $m''>\nu_0$ then $|H_{m''}(s_0)-H(s_0)|<\varepsilon$ (see Remark 3). So, let fix $m''>\max(\nu_0, \nu^*)$. For this fixed m'', for the continuous function $H_{m''}$ there is $\delta=\delta(m'')>0$, $\delta<\delta^*/3$, such that if $|s_2-s_0|<\delta$, then $|H_{m''}(s_2)-H_{m''}(s_0)|<\varepsilon$. For each s with $|s-s_0|<\delta$ there exists ν_s such that if $m'>\nu_s$ then $|H(s)-H_{m'}(s)|<\varepsilon$. Let fix such $m'>\max(\nu_s, \nu^*)$. For the arbitrary fixed s with $|s-s_0|<\delta$, for the continuous function $H_{m'}$ there is such $\delta_s>0$, $\delta_s<\delta^*/3$, that $|H_{m'}(s)-H_{m'}(s_1)|<\varepsilon$ if $|s-s_1|<\delta_s$. Choose and fix such $s_1\in L_a$. Then $|s_1-s_2|\leqq|s_1-s|+|s-s_0|+|s_0-s_2|<\delta_s+\delta+\delta<\delta^*$. Therefore for each arbitrary fixed $s\in L$ with $|s-s_0|<\delta$ we get $|H(s)-H(s_0)|<4\varepsilon a^{-1}+5\varepsilon$, where $\varepsilon>0$ was arbitrary fixed. $\blacksquare$

Proof of (i.3$^{\text{V}}$). Suppose the contrary. Let B be such a ball. As all k with $Q<|k|\leqq N$ are a finite number and $E_0=\cup E_0^k$, hence there exists $E_0^{k'}$ which also is dense in B. Let $a\in E_0^{k'}\cap\mathring{B}$. From (i.3$^{\text{III}}$) there is a ball $\underline{B_1}$, $\mathring{B}_1\neq\emptyset$, in a neighbourhood U_a of a, $B_1\subset U_a\subset B$, such B_1 that $d_{k'}(t)<1$ on $\overline{B}_1$. Since $E_0^{k'}$ is dense in B, hence there is a point $a_1\in E_0^{k'}\cap\mathring{B}_1$. Applying again (i.3$^{\text{III}}$), we get a ball

$B_2 \subset \mathring{B}_1$, $\mathring{B}_2 \neq \emptyset$, such that $d_{k'}(t) < 1/2$ on $\overline{B_2}$. By induction there is a ball $\overline{B_m} \subset \overline{B_{m-1}}$, $\mathring{B}_m \neq \emptyset$, with $d_{k'}(t) < 1/m$ on $\overline{B_m}$. Thus we get a sequence $\overline{B_1} \supset \overline{B_2} \supset \ldots \supset \overline{B_m} \supset \ldots$, $\forall \mathring{B}_m \neq \emptyset$, with $d_{k'}(t) < 1/m$ on $\overline{B_m}$. Therefore there is a point $b \in \overline{B_m}$ for $\forall m$ and hence $d_{k'}(b) < 1/m$ for $\forall m$. Then $d_{k'}(b) = 0$, which contradicts with (i.3[I]). ■

Proof of (ii). Proposition 1, the equality (8) and (i), (together with the made already remark that all b_k, $|k| \leqq Q$, can be chosen $\equiv 1$, up to equivalence of the completing norm (9)), imply that

$$(10) \qquad \mathscr{T}_{Q}^{N}{}_{P}(t) = \sum_{\substack{Q<|k|\leqq N \\ k\neq k_q,\ |k|\leqq|k_j|}} d_k(t) \left| D^k P_*(t)/k! + \sum_{j=0}^{p(t)} A_k^j(t) D^{k_j} P_*(t)/k_j! \right|$$

for each fixed P are upper semicontinuous functions in t on U^0. W shall prove that this involves that I). All A_k^j, $k \neq k_j$, are continuous on $U^0 - E_0$. II) Respectively all $|d_k A_k^j|$ are upper semicontinuous on U^0: At first the following two Lemmas A and B will be proved:

Lemma A. *If* 1. *for* $\forall k_j$, $|k_j| < \lambda$, *let* I) *all* A_k^j *be continuous on* $U^0 - E_0$ (*resp.* II) *all* $|d_k A_k^j|$ *be upper semicontinuous on* U^0) *and* 2. *let all* $\mathscr{T}_{M-1}^{M}{}_{P}(t)$, $Q < M \leqq \lambda$, *with* M *fixed, be upper semicontinuous functions in* t *on* U^0 *for each fixed polynomial* P *of degree not larger than* λ, *then the functions* $A_k^{j_0}(t)$ *with* $|k| = M$ *and* $|k_{j_0}| = \lambda$, *are continuous in* t *on* $U^0 - E_0$ (*resp.* II) *the functions* $d_k |A_k^j(t)|$ *with* $|k| = M$ *and* $|k_{j_0}| = \lambda$, *are upper semicontinuous on* U^0).

Proof. For $P_*(t) = ct^{k_{j_0}} + \Sigma_{|k|=M, k\neq k_j} \varepsilon_k a_k t^k$, $\varepsilon_k = \pm 1$, a_k — constants, we have

$$(11) \qquad \begin{aligned} &\mathscr{T}_{M-1}^{M}{}_{P}(t) = \sum_{|k|=M;\,|k|\leqq|k_j|;\,k\neq k_q} d_k |\varepsilon_k a_k + f_k + cA_k^j|, \\ &f_k = cD^k t^{k_{j_0}}/k! + c\sum_{j=0}^{p} \sum_{k_j < k_{j_0}} A_k^j D^{k_j} t^{k_{j_0}}/k_j! \end{aligned}$$

I) Let F be an arbitrary compact $\subset (U^0 - E_0)$. Then on $F \subset (U^0 - E_0)$ we can consider $d_k(t) \equiv 1$ on F. Since $A_k^j = 0$ if $|k_j| < |k|$, then and all A_k^j with $|k_j| < \lambda = |k_{j_0}|$ are continuous on $U^0 - E_0$, according to the upper semicontinuity (requirement 2. of Lemma A), of $\mathscr{T}_{M-1}^{M}{}_{P}(t)$, hence $A_k^{j_0}$, $k \neq k_j$, are bounded on the arbitrary fixed compact $F \subset (U^\circ - E_0)$. Let choose the constants $a_k > |cA_k^{j_0} + f_k|$ on F. Successively for $P'_*(t) = P_*(t)$ with $\varepsilon_k = 1$ and $P''_*(t) = P_*(t)$ with

$$\varepsilon_k = \begin{cases} 1 & \text{if } k = k_*, \quad k \neq k_j, \quad |k_*| = M, \\ -1 & \text{if } k \neq k_*, \end{cases}$$

where k_* is fixed and $k_* \neq k_j$, we get from (11) that the functions

$$\sum_{|k|=M;\,k\neq k_j} (a_k+f_k+cA_k^{j_0}), \quad \left\{ \sum_{|k|=M;\,k\neq k_j;\,k\neq k_*} (a_k-f_k-cA_k^{j_0})+a_{k_*}+f_{k_*}+cA_{k_*}^{j_0} \right\}$$

are upper semicontinuous on F. Therefore $(f_{k_*}+cA_{k_*}^{j_0})$ is upper semicontinuous on F. Since f_{k_*} is continuous on U^0-E_0, thus $cA_{k_*}^{j_0}$ is upper semicontinuous on F. But the constant c is arbitrary. That is why $A_{k_*}^{j_0}(t)$ is continuous on F. Here the compact $F\subset(U^0-E_0)$ was arbitrary fixed, and so was k_*, with $|k_*|=M$, $k_*\neq k_j$.

Respectively 1.II). Let $d_k(t)|A_k^{j_0}(t)|$ is with $|k|=M$, $|k_{j_0}|=\lambda$. We got $\mathscr{T}_{M-1\,P}^{M}(t)$ from (11) was upper semicontinuous in t on U°. Moreover each term $g_k(t)=d_k(t)|\varepsilon_k a_k+f_k(t)+cA_k^{j_0}(t)|$ in (11) is upper semicontinuous in t on U^0: Let suppose the contrary, i.e. that some $g_{k'}(t)$ with $k'\neq k_j$, $|k'|=M$, $|k'|\leqq|k_j|$, is not upper semicontinuous for some $a_{k'}$, $\varepsilon_{k'}$, c', at $t'\in U^0$. Then there is $\varepsilon_0>0$ such that $g_{k'}(t')+\varepsilon_0\leqq g_{k'}(t_m)$ for some sequence $(t_m)\to t'$, $t_m\in U^0$. Therefore, for the appropriate choice of the constants ε_k, a_k with $k\neq k'$, $|k|=M$, $k\neq k_j$, $|k|\leqq|k_{j_0}|$, (and $a_{k'}$, c', $\varepsilon_{k'}$ — the same as in $g_{k'}$) then the corresponding function in t in (11) $\mathscr{T}_{M-1\,P}^{M}(t)$ would not be upper semicontinuous at t'. This contradiction prove that each function $g_k(t)$ is also upper semicontinuous in t on U^0. Further let fix an arbitrary $t''\in U^0$. The degrees of t in all tems of f_k are strictly positive. Without losing the generality we can suppose $t''=0$. From the condition 1.II) of Lemma A we get that $|d_k(t)f_k(t)|<\varepsilon$ in some neighbourhood V of t'', $V\subset U^0$, where $\varepsilon>0$ is arbitrary fixed. For $|d_k(t)A_k^{j_0}(t)|$ we receive

$$|d_k(t)A_k^{j_0}(t)|\leqq|d_k(t)f_k(t)|+d_k(t)|f_k(t)+A_k^{j_0}(t)|.$$

If $a_k=0$ we have that $g_k(t)=d_k(t)|f_k(t)+A_k^{j_0}(t)|$ is upper semicontinuous on U^0. Hence there is a neighbourhood V' of t'', $V'\subset V$, such that in V' we obtain

$$|d_k(t)A_k^{j}(t)|<2\varepsilon+d_k(t'')|f_k(t'')+A_k^{j_0}(t'')|=2\varepsilon+d_k(t'')|A_k^{j_0}(t'')|.$$

This implies that $d_k(t)|A_k^{j_0}(t)|$ is upper semicontinuous at t''. As t'' was an arbitrary fixed element of U^0, then $d_k(t)|A_k^j(t)|$ is upper semicontinuous on U^0. ∎

Remark 4. In the similar way as in Lemma A.II) we can prove that if $\min_V|d_{k'}(t)A_{k'}^{j_0}(t)|\neq 0$ on some $V\subset U^0$, and the requirements of Lemma A. II) hold then all $A_{k'}^{j_0}(t)$ and $d_{k'}(t)$ are continuous on V, $|k'|=M$, $k'\neq k_j$, $|k'|\leqq|k_{j_0}|$.

Lemma B. *If* 1) *the functions* $\mathscr{T}_{M'\,P}^{M''}(t)$, $Q\leqq M'<M''\leqq N$, M', M'' *— fixed, are upper semicontinuous in t on* U^0 *for each polynomial P of degree not larger than* λ, $M''\leqq\lambda\leqq N$, *and if* 2) *all functions* $d_k(t)|A_k^j(t)|$, $|k_j|\leqq\lambda$, *are bounded on each fixed compact* $F\subset U^0$, *then the functions* $\mathscr{T}_{M''-1\,P}^{M''}(t)$, *(also as* $\mathscr{T}_{K\,P}^{M}(t)$, $M'\leqq K<M\leqq M''$*), are upper semicontinuous in t on* U^0 *for each polynomial P of degree not larger than* λ.

Proof. Let

$$P_*(t)=P'_*(t)+\sum_{\substack{|k|=M'+1\\ k\neq k_j}} \varepsilon_k a_k t^k,$$

with $P'_*(t)$ a polynomial of degree not larger than λ; $\varepsilon_k=\pm 1$; a_k are constants, dependent on the choice of P'_*. Then

$$\mathscr{T}_{M'}^{M''}{}_P(t)=\sum_{\substack{|k|=M'+1\\ k\neq k_j}} d_k|\varepsilon_k a_k+D^k P'_*(t)/k!+\sum_{j=0}^{p(t)} A^j_k D^{k_j} P'_*(t)/k_j!|+\mathscr{T}_{M'}^{M''}{}_{P'}(t)$$

since: let $|k'|=M'+1$, then

$$A^j_k D^{k_j} t^{k'}=\begin{cases}0 \text{ for } |k_j|\leqq M', \text{ as } A^j_k=0 \text{ when } |k_j|<|k|,\\ 0 \text{ for } |k_j|\geqq M'+1, \text{ as } D^{k_j}t^{k'}=0.\end{cases}$$

Since $\mathscr{T}_{M'}^{M''}{}_P(t)$ are upper semicontinuous functions in t on U^0, hence such must be also the function $\mathscr{T}_{M'}^{M''}{}_{P'}(t)$ in t on U^0 for each fixed polynomial P'_* of degree not larger than λ. Let assume the contrary for some polynomial $P''_*(t)$ of degree not larger than λ and for some $t'\in U^0$. Then for an appropriate choice of the constants ε_k, a_k, $\forall k$, $|k|=M'+1$, $k\neq k_j$ (this choice does not change $P''_*(t)$), we chall get that the corresponding (to P''_*, a_k, ε_k) function $\mathscr{T}_{M'}^{M''}{}_P(t)$ also would not be upper semicontinuous at the point t'. Thus the functions $\mathscr{T}_{M'+1}^{M''}{}_P(t)$ are upper semicontinuous in t on U^0 for each fixed $P'(x,t)=\Sigma_k(x-t)^k D^k P'_*(t)/k!$ of degree not larger than λ. ■

Let now fix j_0 such that $|k_{j_0}|=\min_j|k_j|=\varkappa$. For $P_*(t)=t^{k_{j_0}}$, the equality (8) yields, that

$$\mathscr{T}_Q^N{}_P(t)=\sum_{Q<|k|\leqq\varkappa} d_k(t)|D^k t^{k_{j_0}}/k!+A^{j_0}_k|$$

are upper semicontinuous in t on U^0. Hence all functions $|d_k(t)A^{j_0}_k(t)|$, $|k|=\varkappa$, $k\neq k_j$, are bounded on each compact $F\subset U^0$. According to Lemma B the functions $\mathscr{T}_{\varkappa-1}^{\varkappa}{}_P(t)$ are upper semicontinuous in t on U^0. Then Lemma A implies that for $|k_{j_0}|=\varkappa$, $|k|\leqq\varkappa$, $k\neq k_j$, the functions $A^{j_0}_k(t)$ are continuous on U^0-E_0, moreover, the functions $d_k(t)|A^{j_0}_k(t)|$ are upper semicontinuous on U^0. (Recall that when $|k|>|k_j|$ then $A^j_k=0$). Let now suppose that I). all A^j_k, $|k_j|\leqq\nu$, are continuous on U^0-E_0 (resp. II) all $d_k|A^j_k|$, $|k_j|\leqq\nu$, are upper semicontinuous on U^0), $k\neq k_j$, $|k|<|k_j|$. Let $\mu=\min_{|k_j|>\nu}|k_j|$, and let $|k_{j_0}|=\mu$. For $P_*=t^{k_{j_0}}$, (8) and (10) yield that

$$\mathscr{T}_Q^N{}_P(t)=\sum_{\substack{Q<|k|<\mu\\ k\neq k_j}} d_k\left|D^k t^{k_{j_0}}/k!+\sum_{\substack{j=0,\, k_j<k_{j_0}\\ k\neq k_j}} A^j_k D^{k_j} t^{k_{j_0}}/k_j!+A^{j_0}_k\right|$$

$$+\sum_{\substack{|k|=\mu\\ k\neq k_j}} d_k|D^k t^{k_{j_0}}/k!+A^{j_0}_k|+0,$$

are upper semicontinuous in t on U^0. Hence all $d_k A_k^{j_0}$, $|k_{j_0}|=\mu$, are bounded on each compact $F\subset U^0$ for an arbitrary j_0 with $|k_{j_0}|=\mu$, $k\neq k_j$, $|k|<|k_{j_0}|$. Then Lemma B, Lemma A and the fact that when $|k|>|k_j|$, then $A_k^j=0$, involve that all $A_k^j(t)$, $|k_j|=\mu$, are continuous in t on U^0-E_0 (resp. II) all $d_k(t)|A_k^j(t)|$ are upper semicontinuous on U^0, $|k_j|=\mu$). By induction all $A_k^j(t)$ are continuous on U^0-E_0 (resp. II) all $d_k(t)|A_k^j(t)|$ are upper semicontinuous on U^0). ■

The space α_F is still differential-invariant, according to the following lemma:

Lemma 13 ([19]). *Let Γ be a set of linear differential operators of order not larger than N, with coefficients on $M\subset R^n$. Let fix $x_0\in M$. Let the set $\mathscr{I}=\mathscr{I}_\Gamma=\{f: f\in D^N$ and $Af(x_0)=0$ for all $A\in\Gamma\}$. $\mathscr{I}$ is a closed ideal in D^N, for $x_0\in M$, if and only if Γ is differential-invariant.*

Proof. Let $g\in D^N, f\in\mathscr{I}, A\in\Gamma$. Since $A(fg)=\Sigma_p(D^p g)A^{(p)}f/p!$, $p=(p_1,\dots,p_n)$, and as $g\in D^n$ are sufficiently much, therefore the Lemma 13 is proved. ■

Since the norm $p'f\geqq|f(s)|$, $\forall s\in F$, $\forall f\in R'_{F,\varphi}$ (see Remark 2), hence $p'f\geqq\sup_F|f(s)|$. Then we can assume that the identity operator $I\in\alpha_F$.

This finish the proof of points 3.b), 3.b') of Theorem 2.

The proof of points 4.a), 4a') is an immediate consequence of point 3.

The proof of Point 4.b) consists of the following steps: (i') R is a set of c-v functions on G, with precision up to a natural isomorphism. (ii'). R is a Banach space of c-v functions on G, with a norm $\|f\|, f\in R$, equivalent to $\sup_{s\in G}\|f\|_s$. (iii'). R is a Banach algebra of c-v functions on G with pointwise multiplication. (iv'). R is without radical.

Proof of (i'). Let A be a linear differential operator on G with coefficients in D_G^∞.

Lemma 14. *Let A be a linear differential operator on G with coefficients in D_G^∞. Let for the function $h\in D_G^0$ there exists such a sequence $\{\varphi_m\}$, $\varphi_m\in D_G^\infty$, that $\{\varphi_m\}\xrightarrow{\text{uniformly on } G} h$ and $\{A\varphi_m\}\xrightarrow{\text{uniformly on } G} H$. Then, if for another sequence $\{\psi_m\}$, $\psi_m\in D_G^\infty$, we have $\{\psi_m\}\xrightarrow{\text{uniformly on } G} h$ and $\{A\psi_m\}\xrightarrow{\text{uniformly on } G} M$, it follows that $H\equiv M$ on G.*

The unique function $H\in D_G^0$ will be denoted by Ah and will be called a generalized A derivative of the function h.

Proof of Lemma 14. Let the linear differential operator B be the conjugate of the operator A, i.e., the linear continuous differential operator $B: D_G^\infty\to D_G^0$ is such that its transfer $B\circ\varphi^{-1}$ for any local chart (U,φ) of G is the conjugate operator of the transfer $A\circ\varphi^{-1}$ of A, i.e., $\int[(A\circ\varphi^{-1})\Phi]\Psi=\int\Phi(B\circ\varphi^{-1})\Psi$, for each Φ, $\Psi\in D^\infty_{\varphi(U)}$. Such an operator exists; moreover the coefficients of B are also in D_G^∞. Since

$$\int\{A\circ\varphi^{-1})[(\varphi_m-\Psi_m)\circ\varphi^{-1}]\}\,\Phi=\int[(\varphi_m-\psi_m)\circ\varphi^{-1}](B\circ\varphi^{-1})\Phi\to 0$$

as $m\to\infty$ for $\forall\Phi\in D^\infty_{\varphi(U)}$, hence $\lim_m(A\circ\varphi^{-1})(\varphi_m\circ\varphi^{-1})(x)\equiv\lim_m(A\circ\varphi^{-1})\times(\psi_m\circ\varphi^{-1})(x)$ at $\forall x\in\varphi(U)$ and on each local chart (U,φ) of G. Therefore $H\equiv M$ on G. ■

From the construction of R as a completion, it follows that each element of R is determined by a sequence (φ_m), $\varphi_m \in D_G^\infty$, with $(\varphi_m) \xrightarrow{\text{uniformly on } G} f$, $(A\varphi_m) \xrightarrow{\text{uniformly on } G} g^A$ for $\forall A \in \alpha$, where the c-v functions f, $g^A \in D_G^0$, for $\forall A \in \alpha$. Lemma 14 yields that each g^A is determined uniquely by the function f. Therefore this arbitrary element of R can be treated as the c-v function $f \in D_G^0$. Thus the elements of R can be treated as all these functions $f \in D_G^0$ for which there exist sequences (φ_m), $\varphi_m \in D_G^\infty$, with $(\varphi_m) \xrightarrow{\text{uniformly on } G} f$, $(A\varphi_m) \xrightarrow{\text{uniformly on } G} g^A$, $\forall A \in \alpha$. The functions $g^A \in D_G^0$ are uniquely determined by the function f, and g^A will be denoted by Af. ■

Proof of (ii′). From (i′) it follows that for each $f \in R$ there exist $Af \in D_G^0$, $\forall A \in \alpha$. So we can consider R with the norm $\Sigma_{A \in \mathfrak{B}(\alpha)} \sup_{s \in G} |Af(s)|$. Evidently this is a norm of R, and R with this norm is a Banach space of c-v functions on G, since R is a completion of D_G^∞ by the same norm.

Let remind that $\|f\|_s = \inf\{\|g\| : g \in R,\ g = f$ in some neighbourhood of $s\}$. The differentiability is a local property, thus for each such g and for $\forall A \in \alpha$, we get $Ag(s) = Af(s)$. Then $\|f\|_s \geqq \Sigma_{A \in \mathfrak{B}(\alpha)} |Af(s)|$. Hence $\sup_{s \in G} \|f\|_s \geqq \sup_{s \in G} \Sigma_{A \in \mathfrak{B}(\alpha)} |Af(s)|$. By other hand, obviously, $\|f\|_s \leqq \|f\|$ and therefore $\sup_{s \in G} \|f\|_s \leqq \|f\|$. But the norm $\sup_{s \in G} \Sigma_{A \in \mathfrak{B}(\alpha)} |Af(s)|$ and the norm $\Sigma_{A \in \mathfrak{B}(\alpha)} \sup_{s \in G} |Af(s)|$ are equivalent. Then the chosen norm in R is equivalent to the norm $\sup_{s \in G} \|f\|_s$. ■

Proof of (iii′). Let N be the largest of the orders of the operators in the finite dimensional C-linear space α. Let fix an arbitrary chart (local) (U, φ) of G and a compact n-dimensional cube F in $\varphi(U)$, $\mathring{F} \neq \emptyset$. Let $\alpha' = \{A \circ \varphi^{-1},\ A \in \alpha\}$. Obviously α' is a C-linear differential-invariant finite dimensional space of linear differential operators with coefficients in $D^\infty | \varphi(U)$. Let $\varphi(s) \in F$; let denote $\varphi(s) = t = (t_1, \dots, t_n)$ and let fix an arbitrary such t. The set $\mathscr{J}_{F,\alpha'}(t) = \{g : g \in D^N | F$, $(A \circ \varphi^{-1}) g(t) = 0$ for $\forall A \in \alpha\}$ is a closed ideal in $D^N | F$ (see Lemma 13). Let investigate the factor (quotient) algebra $(D^N | F)/\mathscr{J}_{F,\alpha'}(t)$. Evidently $\mathscr{J}_{F,\alpha'}(t) \supseteq J_F^N(t)$ (see Lemma 11). Hence the algebras $(D^N | F)/\mathscr{J}_{F,\alpha'}(t)$ and K^N/I_t are isomorphous, where I_t is an ideal of K^N, determined by $\mathscr{J}_{F,\alpha'}(t)/J_F^N(t)$. Let remind that the isomorphism $D^N/J^N(t) \to K^N$ is realized by the mapping $(x - t) \to X = (X_1, \dots, X_n)$ — the generator of K^N; $x = (x_1, \dots, x_n) \in F$; and I_t is the image of $\mathscr{J}_{F,\alpha'}(t)$ in the last isomorphism. Since the identity operator $I' \in \alpha'$, hence we can choose the basis $\mathfrak{B}(\alpha')$ consisting of the operators I', $A'_1, \dots, A'_q$, $(I' = I \circ \varphi^{-1}$, $A'_1 = A_1 \circ \varphi^{-1}, \dots, A'_q = A_q \circ \varphi^{-1})$, such that if $A'_l = \Sigma a_{kl}(t) D^k$ then $a_{0l} \equiv 0$, $l = 1, \dots, q$, $A_1, \dots, A_q \in \alpha$, $A'_0 = I'$.

Let the factor algebra K^N/I_t be with the norm $|.|_t^*$ such that if $\Sigma c_k X^k \in K^N/I_t$, then $|\Sigma c_k X^k|_t^* = \Sigma_{l=0}^q |\Sigma_{|k| \leq N} (k! \, a_{kl} c_k)|$. Since $\mathfrak{B}(\alpha')$ is a basis of α' and since I_t is the image of $\mathscr{J}_{F,\alpha'}(t)$, hence $|\rho|_t^* = 0$, $\rho \in K^N$, if and only if $\rho \in I_t$. Also it follows that this norm is a factor norm in the factor algebra K^N/I_t.

Let ω'_t be the multiplicative homomorphism,

$$\omega'_t : D^N | F \to K^N/I_t \cong (D^N | F)/\mathscr{J}_{F,\alpha'}(t) \cong D^N/\mathscr{J}_{\alpha'}(t), \text{ with}$$

$$\mathcal{J}_{\alpha'}(t)=\{g\in D^N,\ A'g(t)=0 \text{ for } \forall A'\in\alpha'\}.$$

Let construct a new algebra K_F^* of c-v functions, generated by $D^\infty|F : K_F^*=\{f;\ f\in D^\infty|F$, with the norm $\|.\|^*$, where $\|f\|=\sup_{t\in F}|\omega_t'(f)|_t^*\}$.

Let verify that K_F^* fulfils the axioms of a normed algebra: 1) $\|\lambda f\|^*=|\lambda|.\|f\|^*$ is true as $\|\lambda f\|^*=\sup_F|\omega_t'(\lambda f)|_t^*=|\lambda|\sup_F|\omega_t'(f)|_t^*$. 2) $\|f_1+f_2\|^*\leqq\|f_1\|^*+\|f_2\|^*$, since $\|f_1+f_2\|^*=\sup_F|\omega_t'(f_1+f_2)|_t^*=\sup_F|\omega_t'(f_1)+\omega_t'(f_2)|_t^*\leqq\sup_F(|\omega_t'(f_1)|_t^*+|\omega_t'(f_2)|_t^*)\leqq\|f_1\|^*+\|f_2\|^*$. 3) $\|f_1f_2\|^*=\sup_F|\omega_t'(f_1f_2)|_t^*$. According to the multiplicativity of ω_t', we obtain $\|f_1f_2\|^*=\sup_F|\omega_t'(f_1).\omega_t'(f_2)|_t^*\leqq\sup_F[|\omega_t'(f_1)|_t^*.|\omega_t'(f_2)|_t^*]\leqq\|f_1\|^*.\|f_2\|^*$. 4) $\|1\|^*=\sup_F|1|_t^*=1$ in the chosen basis $\mathfrak{B}(\alpha')$ of α'.

Therefore, completing the normed algebra K_F^* by the indicated norm we receive a Banach algebra, which will be denoted by $K_\omega^*(F)$. (The algebra $K_\omega^*(F)$ is a "continuous sum". The "continuous sums" are worked out by G. E. Shilov [1, 2, 11, 12]. The idea of "continuous sums" goes back to I. M. Geljfand). Let now investigate the norm $\|f\|^*=\sup_F|\omega_t'(f)|_t^*$. We receive for any polynomial P, $P=\Sigma(x-t)^kD^kP_*(t)/k!$ of order $\leqq N$, that $\|P\|^*=\sup_{t\in F}\Sigma_{l=0}^{q}|\Sigma_{|k|\leqq N}a_{kl}(t)D^kP_*(t)|$, which norm is equivalent to the norm (1). Therefore $K_\omega^*(F)=(R\circ\varphi^{-1})|F=\{(f\circ\varphi^{-1})|F,\ f\in R\}$. Hence $R\circ\varphi^{-1}$ is an algebra of c-v functions on F with pointwise multiplication. Since the local chart $(U,\ \varphi)$ of G and $F\subset\varphi(U)$ were chosen arbitrarily, then it follows that R is also an algebra of c-v functions on G with pointwise multiplication. ■

Proof of (iv'). R is without radical: As $D_G^\infty\subset R$ then all maximal ideals of R are of the kind $\{f:f\in R,\ f(t_0)=0\}$ for some $t_0\in G$ (see [1, p. 18]). This involves, that if the function $f_*\in R$ belongs to all maximal ideals of R then $f_*\equiv0$. According to Lemma 14 all $Af_*\equiv0$, $\forall A\in\alpha$. So f_* is the zero element of R.

Proof of Corollary 4. Let F^* be a compact cube or ball in $\varphi(U)$, for which $F\subset\mathring{F}^*$. According to the definition of the homogeneity of the algebra R in the local chart $(U,\ \varphi)$, the algebra $R'_{F^*,\varphi}$ is homogeneous at each point of $\mathring{F}^*$. Then the connectedness of F and the definition of the homogeneity at a point imply that all factor algebras $R'_{F^*,\varphi}/J'_{F^*,\varphi}(x')\cong R'_{F,\varphi}/J'_{F,\varphi}(x')$ and $R'_{F^*,\varphi}/J'_{F^*,\varphi}(x'')\cong R'_{F,\varphi}/J'_{F,\varphi}(x'')$ are isomorphous for $\forall x',\ x''\in F$. Therefore, in accordance with the construction of α_F in the proof of Theorem 2, the space $\alpha_F=\alpha(F,\ \varphi)$ is moreover and homogeneous. ■

Proof of Proposition 5. After the compactness of Q and after $Q\subset G=\cup_{F\in\mathfrak{F}'}\mathring{F}$, it follows that from $\mathfrak{F}'$ it can be chosen a finite covering $F_1,\dots,F_m$ of the set Q, $\cup_1^m\mathring{F}_i\supset Q$. $\mathfrak{F}'$ is up directed, whence there exists $F\in\mathfrak{F}'$, $F>F_i$, $i=1,\dots,m$, which yields $\mathring{F}\supset Q$. Obviously: 1. Each algebra $R_Q=R_Q(F)$ is an algebra of functions of type C on Q respectively the indicated norm. 2. The projective limit of the algebras R_Q, $Q\in\mathfrak{Q}'$, coincides with R with exactness to an isomorphism. ■

Proof of Proposition 6. Let R be a projective limit of algebras $R_{F'}$ of functions of type C on F^*, $F^*\in\mathfrak{F}^*$. The application of the Proposition 5 yields that for each $F\in\mathfrak{F}'$ there exists $F^*\in\mathfrak{F}^*$, such that $F\subset\mathring{F}^*$ and: 1. The algebra $R_F=\{f|F,\ f\in R_{F^*}\}$ is an algebra of functions of type C on F relatively the norm

$p'_F = \sup_{s\in G}({}_{F^*}\|f\|_s)$. 2. R is a projective limit of the algebras R_F, $F \in \mathfrak{F}'$, with exactness to an isomorphism. Since D_G^∞ is contained and is dense in R and R is a projective limit for which the mappings $i_{F_2}^{F_1}: R_{F_2} \to R_{F_1}$ are restrictions for $F_1 \subset F_2$, hence $D^\infty | F$ is contained and is dense in R_F. So R_F is a completion of $D^\infty | F$ by the norm p'_F; and R is a completion of D_G^∞ by the system of seminorms (p'_F), $F \in \mathfrak{F}'$. Since $F \subset G \subset \mathbf{R}^n$, there exists a compact $F_\sim \subset G$, $F_\sim$ is a sum of a finite number of hypercubes, for which $\mathring{F}_\sim \supset F$. Using again Proposition 5 and the arguments of the proof of Theorem 2.3.b), we obtain that the norm p'_F is equivalent to the norm p_F, $p_F f = \sum_{A\in\mathfrak{B}(\alpha_F)} \sup_{x\in F} |Af(x)|$ for some C-linear finite dimensional differential-invariant space α_F of linear differential operators on F; $\mathfrak{B}(\alpha_F)$ is a finite basis of α_F. ■

P r o o f o f P r o p o s i t i o n 7. Let R be an algebra of c-v functions of type C on $Q_r = Q$ for which $D^\infty | Q \subset R$. Applying the Closed Graph Theorem to the inclusion map $D^\infty | Q \to R$ (see Remark 2), it follows that there exists N such that $D^N | Q \subset R$. Let N be the minimal one for which $D^N | Q \subset R$. As $D^\infty | Q$ is contained and is dense in R, then the space of all maximal ideals of the algebra R coincides with the compact Q (see [1, p. 17, 18]). Hence the algebra R is isomorphous to some algebra R' without a radical of continuous c-v functions on Q, but R is an algebra of c-v functions of type C on Q, therefore $R \subset D_Q^0$. Let M be the largest integer for which $D^M | Q \supset R$. Thus $D^N | Q \subset R \subset D^M | Q$, $0 \leqq M \leqq N$. Analogous arguments as in the proof of Theorem 2 lead to $R/J(0) \cong K^N/I$, where $J(0)$ is the minimal closed primary ideal of R at 0, I is an ideal of K^N, generated by elements of the kind $\sum_{M<|k|\leqq N} c_k X^k$. As R is homogeneous relatively all analytic automorphisms of the disk Q, and as for any two points of $\mathring{Q}$ there exists an analytic automorphism of Q, which transfers these points one to another, then all the factor algebras $R/J(z_0)$, $z_0 \in \mathring{Q}$, are isomorphous to $R/J(0)$. These isomorphisms are realized by the mappings $f \to f \circ \varphi_{z_0}$, where $\varphi_{z_0}(z)$ is the analytic automorphism of Q for which $\varphi_{z_0}(z_0) = 0$ ($J(z_0)$ is the minimal closed primary ideal of R at the point z_0). The norm in the finite dimensional factor algebra K^N/I is determined, with exactness to equivalency, by any finite basis of the linear functionals on K^N which are zero on I. Analogously as in the proof of Theorem 2.3.b) the norm in K^N/I determines a C-linear finite dimensional differential-invariant space of linear differential operators with constant coefficients, such that R is a completion of $D^\infty | Q$ by the norm

$$pf = \sum_{A\in\mathfrak{B}(\alpha)} \sup_{z\in Q} |Af(z)|, \tag{12}$$

where $\mathfrak{B}(\alpha)$ is a finite basis of α.

It is verified as in the proof of Theorem 2.4.b) that for each such linear finite dimensional differential invariant homogeneous space α the completion R of $D^\infty | Q$ by the norm (12) is an algebra of c-v functions of type C on Q, homogeneous at each point $z \in \mathring{Q}$. The algebra R to which corresponds α ($R \to \alpha$) will be denoted by $C_Q(\alpha)$. The rotations of Q are also analytic automorphisms of the disk. Evidently the algebra $C_Q(\alpha)$ is homogeneous relatively the rotations of

Q if and only if the space α is invariant relatively the rotations, i.e., $A\in\alpha$ implies $A_w\in\alpha$, if $|w|=1$, where $A_w g(z)=A(g(wz))$.

Further, results and a method of K. de Leeuw and H. Mirkil from [15] will be used.

The theory of the representations leads to

Theorem 15 (K. de Leeuw, H. Mirkil [15]). *If α is a finite dimensional $\mathbf{C}$-linear space of linear constant-coefficient differential operators on $\mathbf{C}$ and α is rotation-invariant (i.e., $A\in\alpha$ implies $A_w\in\alpha$ for $|w|=1$), then α has a basis of operators A for which $A_w=w^sA$, when $|w|=1$, s is an integer.*

Let denote such a basis by $\mathfrak{B}_r(\alpha)$. If $A=\Sigma_{m+n\leqq N}a_{mn}\partial^{m+n}/\partial z^m\partial\bar{z}^n$, then for $|w|=1$ we have $A_w=\Sigma_{m+n\leqq N}w^{m-n}a_{mn}\partial^{m+n}/\partial z^m\partial\bar{z}^n$. Let moreover $A\in\mathfrak{B}_r(\alpha)$, i.e., $A_w=w^sA$ if $|w|=1$, where s is some integer. Hence for each $A\in\mathfrak{B}_r(\alpha)$, the numbers $(m-n)$ for all additives of A are equal to some integer s. Thus any homogeneous part of order L of the operator $A\in\mathfrak{B}_r(\alpha)$ consists only of one additive $a_{mn}\partial^{m+n}/\partial z^m\partial\bar{z}^n$, where $m+n=L$, $m-n=s$. Therefore each operator $A\in\mathfrak{B}_r(\alpha)$ is elliptic. As it is well known

Theorem 16 (K. de Leeuw, H. Mirkil [22]). *If A and B are linear constant-coefficient differential operators; if the operator A is elliptic and the order of B is less than the order of A, then $\|Bg\|_{\mathrm{C}}\leqq\varkappa(\|g\|_{\mathrm{C}}+\|Ag\|_{\mathrm{C}})$, for $\forall g\in D^\infty$, for some constant $\varkappa$, where $\|f\|_{\mathrm{C}}=\sup_{x\in\mathbf{R}^l}|f(x)|$.*

Let N be the largest order of the operators in α and let A, $A\in\mathfrak{B}_r(\alpha)$, have order N. Then K. de Leeuw, H. Mirkil's Theorems 15 and 16 imply that the norm in $D^{N-1}|Q$ is either weaker than the norm in $C_Q(\alpha)$ or equivalent. Hence $C_Q(\alpha)\subset D^{N-1}|Q$. So it is clear that for a basis of α we can choose the set $\omega\cup\{\partial^{m+n}/\partial z^m\partial\bar{z}^n, \forall(m+n)<N\}$, where ω is of the kind (2). Such a space α is still invariant relatively any analytic automorphism of Q.

For finishing the proof it remains to show that $C_Q[\omega_1]=C_Q[\omega_2]$ only if $\omega_1=\omega_2$: Let assume $C_Q[\omega_1]=C_Q[\omega_2]$. Then the Closed Graph Theorem (see Remark 2) involves that for each operator $A\in\omega_1$ there exist a constant $\varkappa$ and operators $B_1,\dots,B_{q'}\in\omega_2$ (since ω_2 consists of elliptic operators), such that $\|Ag\|_{\mathrm{C}}\leqq\varkappa\{\|g\|_{\mathrm{C}}+\|B_1g\|_{\mathrm{C}}+\dots+\|B_{q'}g\|_{\mathrm{C}}\}$, for $\forall g\in D^\infty|Q$. Let apply

Theorem 17 (K. de Leeuw, H. Mirkil [22]). *Let A, $B_1,\dots,B_q$ be constant-coefficient linear differential operators in l-variables for which there exists such constant $\varkappa$ that $\|Ag\|_{\mathrm{C}}\leqq\varkappa\{\|B_1g\|_{\mathrm{C}}+\dots+\|B_{q'}g\|_{\mathrm{C}}\}$ for $\forall g\in D^\infty$. Then the order of A is not larger than the maximal order of $B_1,\dots,B_{q'}$. Let the largest order of $B_1,\dots,B_{q'}$ be N, and let A^N, $B_1^N,\dots,B_{q'}^N$ be the homogeneous parts of order N of A, $B_1,\dots,B_{q'}$ correspondingly. Then moreover $A^N=\Sigma_jc_jB_j^N$, where c_j, $j=1,\dots,q'$, are constants.*

This theorem yields that the order of $A\in\omega_1$ is not larger than the order of $B_1,\dots,B_{q'}$. Let N_1 and N_2 be the orders of the operators in ω_1 and ω_2 correspondingly. Thus $N_1\leqq N_2$. Analogously it follows that $N_2\geqq N_1$. So $N_1=N_2$. In this case, K. de Leeuw, H. Mirkil's Theorem 17 leads to $A=\Sigma_jc_jB_j$ for some constants c_j, $j=1,\dots,q'$. Since in addition ω_1 and ω_2 are of the kind (2), hence $\omega_1\subset\omega_2$. Analogously $\omega_2\subset\omega_1$. ■

Proof of Proposition 8. Let R be an algebra of c-v functions of type C_π on G, a projective limit of the up-directed ordered family of algebras R_F, $F\in\mathfrak{F}'$, of c-v functions of type C on F. According to Proposition 5 we can suppose that $\mathfrak{F}'$ is (K_l), $l=1, 2, \dots$, where if $G=\mathring{Q}_r$ then $K_l=Q_{(rl/(l+1))}$ and if $G=\mathbb{C}$ then $K_l=Q_l$. After the additional requirements for the mappings $i_{K_m}^{K_l}$, R is equivalent (up to a natural isomorphism) with an algebra of c-v functions of type C_π on G for which the cannonical mappings $i^K: R\to R_K$ $(K=K_l)$, are $i^K f=f|K$. Then the set $R_K^*=\{f|K, f\in R\}\subset R_K$. Since R is a homogeneous algebra relatively all analytic automorphisms of G, hence R_K^* is an algebra, homogeneous relatively rotations and homogeneous on K. Obviously that $D^\infty|K\subset R_K^*$ also and $D^\infty|K\neq R_K^*$. Analogously to the proof of Theorem 2.3.b), each $R_{K_l}^*$ is a completion of $D^\infty|K_l$ by the norm p_l, $p_l f=\Sigma_{A\in\mathfrak{B}(\alpha_l)} sup_{z\in K_l}|Af(z)|$, where α_l is a C-linear finite dimensional differential-invariant space of linear differential operators on K_l. Let α_l^* be the maximal C-linear space of linear constant-coefficient differential operators such that $A\in\alpha_l^*$ implies that there exist the generalized continuous derivatives Af for $\forall f\in R_{K_l}^*$ in the sense of Remark 1 and Lemma 14. Since $R_{K_l}^*\subset R_{K_l}\subset D^0|K_l$, hence α_l^* is not empty — the identity operator $I\in\alpha_l^*$. As $R\neq D_G^\infty$, then α_1^* does not contain all operators $D^k=\partial^{|k|}/x_1^{k_1}x_2^{k_2}$ for $\forall k=(k_1, k_2)$. Further as R is homogeneous relatively the rotations of G, hence α_l^* is rotation-invariant. After the K. de Leeuw, H. Mirkil's Theorem 15, each finite dimensional rotation-invariant subspace β_l of α_l^* has a basis of operators A such that $A_w=w^sA$, $\forall w$ with $|w|=1$, where $A_w(g(z))=A(g(wz))$. But in the latter proof it was obtained that such operators are elliptic. Therefore if α_l^* is not finite dimensional then (see K. de Leeuw, H. Mirkil Theorem 16) α_1^* would contain all D^k, which contradicts with $R_{K_l}^*\neq D^\infty|K_l$. Thus each α_l^* is finite dimensional and has a basis $\mathfrak{B}(\alpha_l^*)=\omega_l\cup_{|k|<N_l}D^k$, where ω_1 is of the kind (2) for some $N=N_l$. Let investigate the allgebras $C_{K_l}[\omega_l]$. Evidently $C_{K_l}[\omega_l]=R_{K_l}^*$. Then the algebra is a projective limit, up to a natural isomorphism, of the algebras $\{C_{K_l}[\omega_l]\}_l$. As R is homogeneous relatively all analytic automorphisms of G then all $\omega_l=\omega$, $\forall l$. ∎

REFERENCES

1. Г. Е. Шилов. О регулярных нормированных кольцах. *Труды Мат. ин-та им. Стеклова, АН СССР*, **21**, 1947, 1—118.
2. Г. Е. Шилов. Однородные кольца функции. *Успехи мат. наук*, **6**, 1951, №91—137.
3. Г. Е. Шилов. Векторно-гладкие функции. *Успехи мат. наук*, **6**, 1951, №5, 176—184.
4. Г. Е. Шилов. О кольце функции на n-мерном торе. *Киевский университет, Мат. сборник*, **6**, (Киев, *Науч. записки ун-та*, **7**), 1952, 17—23.
5. Г. Е. Шилов. Об однородных кольцах функций на торе. *Доклады АН СССР*, **82**, 1952, №5, 681—684.
6. Г. Е. Шилов. О двояко периодических векторно-гладких функциях. *Украинский мат. журнал*, **4**, 1952, №1, 25—35.
7. H. Mirkil. The works of Shilov on commutative semi-simple Banach algebras. University of Chicago, 1952 (mimeographed).
8. K. de Leeuw, H. Mirkil. Intrinsic algebras on the torus. *Trans. Amer. Math. Soc.*, **81**, 1956, 320—330.
9. Б. С. Митягин. О некоторых свойствах функций двух переменных. *Вестник МГУ*, **5**, 1959, 137—152.
10. В. В. Грушин. О строении замкнутых идеалов в кольце двояко периодических векторно-гладких функций. *Вестник МГУ*, **1**, 1961, 17—23.
11. Г. Е. Шилов. Кольца типа С. *Доклады АН СССР*, **66**, 1949, №5, 813—816.

12. Г. Е. Шилов. О непрерывных суммах конечномерных колец. *Матем. сборник*, **27**, 1950, №3, 471–484.
13. Г. Е. Шилов. Описание одного класса нормированных колец функций. *Мат. сборник*, **26**, 1950, №2, 291–310.
14. F. Treves. Introduction to Pseudodifferential and Fourier Integral Operators. I, II. New York, 1980.
15. K. de Leeuw, H. Mirkil. Algebras of differentiable functions. *Bull. Amer. Math. Soc.*, **68**, 1962, 411–415; Algebras of differentiable functions in the plane. *Ann. Inst. Fourier*, **13**, 1963, 75–90.
16. K. de Leeuw, H. Mirkil. Algebras of differentiable functions on Riemann surfaces. *Ann. Inst. Fourier*, **14**, 1964, 145–181.
17. M. Valdivia. Topics in locally convex spaces. Amsterdam, North-Holland, 1982.
18. А. С. Мадгерова. О некоторых классах однородных банаховых алгебр. (Кандидатская диссертация, Москва, МГУ, 1978).
19. А. С. Мадгерова. Об однородных алгебр функций на торе и их примарных идеалах. — В: Сборник научных работ болгарских аспирантов в СССР, **3**, 1978, 532–541.
20. B. Malgrange. Ideals of differentiable functions. Oxford, 1966.
21. А. С. Мадгерова. Пространства функций с сильными обобщенными производными типа Соболева. *Доклады БАН*, **34**, 1981, №7, 923–926.
22. K. de Leeuw, H. Mirkil. A priori estimates for differential operators in L_∞ norm. *Illinois J. Math.*, **8**, 1964, №1, 112–124.

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