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MAXIMAL DOMAINS OF UNIVALENCE OF THE FULL ELLIPTIC INTEGRALS OF THE FIRST AND THE SECOND KIND CONSIDERED AS ANALYTIC FUNCTIONS OF THE COMPLEX MODULUS

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We determine the maximal domains of univalence of the full elliptic integrals of the first and the second kind $K(z)=\int_0^{\pi/2}(1-z^2\sin^2\varphi)^{-1/2}\,d\varphi$ and $E(z)=\int_0^{\pi/2}(1-z^2\sin^2\varphi)^{1/2}\,d\varphi$ considered as analytic functions of the complex modulus z.

Let us consider the full elliptic integrals of the first and the second kind in the Legendre normal form

$$(1)\qquad W=K(z)=\int_0^{\pi/2}(1-z^2\sin^2\varphi)^{-1/2}\,d\varphi,\quad w=E(z)=\int_0^{\pi/2}(1-z^2\sin^2\varphi)^{1/2}\,d\varphi$$

which for the principal values of the powers represent analytic functions of the complex modulus z in the z-plane with the cut $\{z\,|\,z\leqq -1,\ z\geqq 1\}$. If we itroduce an arbitrary fixed branch of the function $z=\sqrt{\zeta}$, the integrals (1) obtain the form

$$(2)\qquad w=K(\sqrt{\zeta})=\int_0^{\pi/2}(1-\zeta\sin^2\varphi)^{-1/2}\,d\varphi,\quad w=E(\sqrt{\zeta})=\int_0^{\pi/2}(1-\zeta\sin^2\varphi)^{1/2}\,d\varphi$$

and they represent analytic functions in the ζ-plane with the cut $\{\zeta\,|\,\zeta\geqq 1\}$. In our papers [1, Section 3, p. 438-440] and [2, Section 2, p. 62-64] we have found some domains in the ζ-plane which the integrals (2) map univalently. Now we shall solve the general problem finding the images of the cut ζ-plane onto the w-plane by the integrals (2), respectively. This will yield corresponding results for the integrals (1).

Definition. *Maximal domain of univalence of a function of a complex variable is such widest connected set containing an open domain and points of its boundary in which the function is univalent.* (Compare with the same concept for a class of functions in [3, p. 180].)

Theorem 1. *The sets* $G^+=\{z\,|\,\mathrm{Im}\,z>0\}\cup\{z\,|\,0\leqq z\leqq 1\}$ *and* $G^-=\{z\,|\,\mathrm{Im}\,z<0\}\cup\{z\,|\,-1\leqq z\leqq 0\}$ *separately represent maximal domains of univalence of the full elliptic integral of the first kind* $w=K(z)$ *from* (1) *and are mapped by it onto one and the same domain* D *in the* w*-plane which is the interior of the curve* $\gamma=\gamma^+\cup\gamma^-$ *where* $\gamma^\pm$ *are the curves with equations*

$$(3) \qquad \gamma^{\pm}: w = K(\pm x + i0) = \frac{1}{x}\left[K\left(\frac{1}{x}\right) \pm iK\left(\sqrt{1-\frac{1}{x^2}}\right)\right], \quad 1 \leqq x \leqq +\infty,$$

representing the images of the upper banks of the cuts $\{z \mid 1 \leqq z \leqq +\infty)$ *and* $\{z \mid -\infty \leqq z \leqq -1\}$, *respectively, or with equations*

$$(4) \qquad \gamma^{\pm}: w = K(\pm x - i0) = \frac{1}{x}\left[K\left(\frac{1}{x}\right) \pm iK\left(\sqrt{1-\frac{1}{x^2}}\right)\right], \quad 1 \leqq x \leqq +\infty,$$

representing the images of the lower banks of the cuts $\{z \mid -\infty \leqq z \leqq -1\}$ *and* $\{z \mid 1 \leqq z + \infty\}$, *respectively.*

Remark. The domain D is a curvilinear half-band contained in the rectilinear half-band $\{w \mid \operatorname{Re} w \geqq 0\} \cap \{w \mid -(\pi/2) \leqq \operatorname{Im} w \leqq \pi/2$, the half-lines $\{w = u \pm i(\pi/2) \mid 0 \leqq u \leqq +\infty\}$ of which are asymptotes of the curves $\gamma^{\pm}$, respectively, having one and the same origin $w = 0$.

Proof. In view of the evident relation $K(\sqrt{\bar{\zeta}}) = \overline{K(\sqrt{\zeta})}$ for our aim it is sufficient to map the upper ζ-half-plane $\{\zeta \mid \operatorname{Im} \zeta > 0\}$ only. In order to find its image we shall apply the principle of the correspondence of the boundaries in the following way:

For $\zeta = \xi$, $-\infty < \xi < 1$, the derivative $dw/d\xi$ of the first integral in (2) is positive so that when ζ runs in the interval $[-\infty, 1]$ of the real axis in the ζ-plane, then w runs in the positive real semi-axis $[0, +\infty]$ of the w-plane.

Let us determine the values of the first integral in (2) on the upper bank of the cut $\{\zeta \mid 1 \leqq \zeta \leqq +\infty\}$ basing ourselves on the concept of continuity. Going around the point $\zeta = 1$ along the half-circle $\zeta = 1 + re^{i\theta}$, $0 < r < +\infty$, $\pi \leqq \theta < 0$, the real part of the expression $1 - \zeta \sin^2 \varphi = 1 - (1 + r\cos\theta)\sin^2\varphi - ir\sin\theta \sin^2\varphi$ will decrease from $1 - (1 - r)\sin^2\varphi > 0$ to $1 - (1 + r)\sin^2\varphi$ for each fixed φ of the interval $0 < \varphi \leqq \pi/2$ and will be equal to 1 for $\varphi = 0$ where the imaginary part remains nonpositive. Hence, as a consequence of the continuity, the argument of this expression in the circuit of the point $\zeta = 1$ will become nonpositive changing from 0 to -0 for each fixed φ of the interval $0 < \varphi < \arcsin(1/\sqrt{1+r})$ and from 0 to $-\pi + 0$ for each fixed φ of the interval $\arcsin(1/\sqrt{1+r}) < \varphi \leqq \pi/2$. In this way we conclude that after the circuit for $\theta \to +0$ the integrand function of the first integral in (2) does not change its own form for $0 \leqq \varphi < \arcsin(1/\sqrt{1+r})$ and will acquire the multiplier i for $\arcsin(1/\sqrt{1+r}) < \varphi \leqq \pi/2$, i.e. following the continuity if we set $x = 1 + r$, $1 < x < +\infty$, as values of the first integral in (2) on the upper bank of the cut $\{\zeta = x \mid 1 < x < +\infty\}$ we shall accept

$$(5) \quad w = K(\sqrt{x + i0}) = \int_0^{\arcsin(1/\sqrt{x})} (1 - x\sin^2\varphi)^{-1/2} d\varphi + i \int_{\arcsin(1/\sqrt{x})}^{\pi/2} (x\sin^2\varphi - 1)^{-1/2} d\varphi.$$

Performing in (5) the substitutions $\sin\varphi = (1/\sqrt{x})\sin\psi$ in the first integral and $\cos\varphi = \sqrt{1 - (1/x)}\sin\psi$ in the second one, we shall reduce them to the first integral in (2) for the modulus $1/\sqrt{x}$ and for the complementary modulus $\sqrt{1-(1/x)}$, respectively, so that (5) acquires the form

$$w=K(\sqrt{x+i0})=\frac{1}{\sqrt{x}}\left[K\left(\frac{1}{\sqrt{x}}\right)+iK\left(\sqrt{1-\frac{1}{x}}\right)\right],\quad x>1. \tag{6}$$

By letting $x\to 1$ and $x\to+\infty$ in (6), we find $w=K(\sqrt{1+i0})=+\infty+i(\pi/2)$ and $w=K(\sqrt{+\infty+i0})=0$, respectively. Also it is immediately seen that if x increases from 1 to $+\infty$, then the real part of (6) decreases from $+\infty$ to 0, and the imaginary part decreases from $\pi/2$ to 0. Hence, we can determine the integral $w=K(\sqrt{\zeta})$ in (2) on the infinite segment $\{\zeta\,|\,1\leqq\zeta\leqq+\infty\}$ by means of the formula (5) or (6) by which the integral $w=K(\sqrt{\zeta})$ becomes a continuous function in the extended sense in the upper half-plane $\{\zeta\,|\,\mathrm{Im}\,\zeta\geqq 0\}$ of the extended ζ-plane where at the boundary points $\zeta=\infty$ and $\zeta=1$ we have set $K(\sqrt{\infty})=0$ and $K(\sqrt{1})=\infty$. From the previous conclusions we see that the first integral in (2) maps in a one-to-one manner the real axis $[-\infty,+\infty]$ of the extended ζ-plane onto this closed Jordan curve of the extended w-plane which consists of the positive real semi-axis $[0,+\infty]$ and the curve γ^+ with the equation (6) for $1\leqq x\leqq+\infty$ (γ^+ belongs to the half-band $\{w\,|\,\mathrm{Re}\,w\geqq 0,\ 0\leqq\mathrm{Im}\,w\leqq\pi/2\}$ the upper half-line of which is its asymptote), where the upper ζ-half-plane $\{\zeta\,|\,\mathrm{Im}\,\zeta>0\}$ is mapped onto its interior D^+ due to the fact that the directions of the circuits of their boundaries are the same.

Now let us take some finite point w_0 in the upper w-half-plane $\{w\,|\,\mathrm{Im}\,w\geqq 0\}$ which does not lie in the domain D^+ or on its boundary $\gamma^+\cap[0,+\infty]$. From what has been already proved and from the relation $K(\sqrt{\bar{\zeta}})=\overline{K(\sqrt{\zeta})}$ if $\mathrm{Im}\,\zeta\geqq 0$ we conclude that the point w_0 does not belong to the set of the values of the first integral in (2). Then the function $w_1=f(\zeta)=1/(K(\sqrt{\zeta})-w_0)$, continuous everywhere in the half-plane $\{\zeta\,|\,\mathrm{Im}\,\zeta\geqq 0\}$ and analytic in the half-plane $\{\zeta\,|\,\mathrm{Im}\,\zeta>0\}$, maps in a one-to-one manner the real axis $[-\infty,+\infty]$ of the ζ-plane onto some closed Jordan curve in the finite w_1-plane which is the image of the boundary of the domain D^+ under the mapping $w_1=1/(w-w_0)$. From here according to the principle of the correspondence of the boundaries we conclude that the function $f(\zeta)$, and $K(\sqrt{\zeta})$ as well is univalent in the upper half-plane $\{\zeta\,|\,\mathrm{Im}\,\zeta\geqq 0\}$.

We shall obtain the mapping of the lower ζ-half-plane $\{\zeta\,|\,\mathrm{Im}\,\zeta\leqq 0\}$ by means of the first integral in (2) if we reproduce the reasonings adduced. However, this mapping can be obtained immediately thanks to the symmetry $K(\sqrt{\bar{\zeta}})=\overline{K(\sqrt{\zeta})}$ for $\mathrm{Im}\,\zeta\geqq 0$. Thus, the values of the first integral in (2) on the lower bank of the cut $\{\zeta\,|\,1\leqq\zeta\leqq+\infty\}$ will be $w=K(\sqrt{x-i0})$, where $K(\sqrt{x-i0})$ is a magnitude, complex conjugate to the expression (5) or (6). Hence, the integral $W=K(\sqrt{\zeta})$ in (2) realizes an univalent mapping of the half-plane $\{\zeta\,|\,\mathrm{Im}\,\zeta<0\}$ onto the domain D^- in the w-plane, symmetric to the domain D^+ with respect to the real axis $\{w\,|\,\mathrm{Im}\,w=0\}$, the boundary of which consists of the segment $[0,+\infty]$ and the curve γ — with equation $w=K(\sqrt{x-i0})$ for $1 \leqq x \leqq +\infty$ (γ^- belongs to the half-band $\{w \mid \operatorname{Re} w \geqq 0, \ -(\pi/2) \leqq \operatorname{Im} dv \leqq 0\}$ the lower half-line of which is its asymptote).

Joining the both just obtained results we obtain that the integral $w=K(\sqrt{\zeta})$ in (2) maps univalently the ζ-plane with the cut $[1, +\infty]$ onto the domain $D=D^+ \cup [0, +\infty] \cup D^-$ in the w-plane with the boundary $\gamma=\gamma^+ \cup \gamma^-$. When the point ζ goes around the cut $[1, +\infty]$, beginning from 1 on the upper bank, goes to $+\infty$ and returns to 1 on the lower bank, then the point w goes around the curve γ, beginning from $+\infty+i(\pi/2)$, passes through 0 and returns to $+\infty-i(\pi/2)$.

Now let us turn to the integral $w=K(z)$ in (1). If we assume that $0 \leqq \arg \zeta \leqq 2\pi$ where the equality in the left-hand side is missing if ζ is on the upper bank of the cut $[1, +\infty]$, and in the right-hand side it is missing if ζ is on the lower bank, then by means of the principal branch and the second branch of the function $z=\sqrt{\zeta}$, we obtain that the integral $w=K(z)$ maps the upper z-half-plane $\{z \mid \operatorname{Im} z>0\}$ and the lower z-half-plane $\{z \mid \operatorname{Im} z<0\}$, respectively, onto the domain D of the w-plane with the cut $\{w \mid \pi/2 \leqq w \leqq +\infty\}$ representing the double image of the segment $\{z \mid -1 \leqq z \leqq 1\}$. The values of the integral $w=K(z)$ on the upper banks of the cuts $\{z \mid 1 \leqq z \leqq +\infty\}$ and $\{z \mid -\infty \leqq z \leqq -1\}$ are given by (3), respectively, and on the lower banks of the cuts $\{z \mid -\infty \leqq z \leqq -1\}$ and $\{z \mid 1 \leqq z \leqq +\infty\}$ are given by (4), respectively, i. e. the curves $\gamma^{\pm}$ composing the boundary γ of the domain D are the corresponding images of these banks in the w-plane under the mapping $w=K(z)$.

Hence, the sets $G^+=\{z \mid \operatorname{Im} z>0\} \cup \{z \mid 0 \leqq z \leqq 1\}$ and $G^-=\{z \mid \operatorname{Im} z<0\} \cup \{z \mid -1 \leqq z \leqq 0\}$ are maximal domains of univalence of the integral $w=K(z)$ and are mapped by it onto one and the same domain in the w-plane, namely, the domain D.

This completes the proof of Theorem 1.

Corollary 1. *The full elliptic integral of the first kind $K(z)$ considered as an analytic function of the complex modulus z realized a bivalent conformal mapping of the z-plane with the cut $\{z \mid z \leqq -1, z \geqq 1\}$, where the conformality is violated at the point $z=0$ only, where the derivative $K'(0)$ is 0.*

Corollary 1 improves our previous result for bivalence of $K(z)$ obtained by an alternate method in our paper [2, p. 63, Theorem 3, item 1].

In consequence of the bivalence of the integral $K(z)$ we conclude as well that each straight line l, passing through the origin and different from the real axis, divides the z-plane with the cuts $[1, +\infty]$ and $[-\infty, -1]$ also into two maximal domains of univalence where the one half-line of l determined by $z=0$ is integrated with the one domain of univalence, and the opposite half-line of l is integrated with the other domain of univalence, where, of course, the integral $K(z)$ maps them again onto the domain D.

Hence, complete information about the integral $K(z)$ can be obtained in any of the indicated maximal domains of univalence. Evidently, the most convenient one is the set G^+ in Theorem 1.

Conversely, in the closure $\bar{D}=D \cup \gamma$ the inverse one-valued function $\zeta=\alpha(w)$ and the inverse two-valued function $z=A(w)=\sqrt{\alpha(w)}$ are determined so that the point $w=\pi/2$, the image of the point $z=0$ under the mapping $w=K(z)$, is an algebraic branch point of the first kind.

Further, for $E(z)$ we obtain the following result:

Theorem 2. *The sets* $G^+=\{z|\operatorname{Im} z>0\}\cup\{z|0\leqq z\leqq 1\}$ *and* $G^-=\{z|\operatorname{Im} z<0\}\cup\{z|-1\leqq z\leqq 0\}$ *separately represent maximal domains of univalence of the full elliptic integral of the second kind* $w=E(z)$ *from* (1) *and are mapped by it onto one and the same domain* Δ *which is the part of the w-plane containing the segment* $\{w|w>1\}$ *and lying to the right of the curve* $\delta=\delta^-\cup\delta^+$ *where* δ^+ *are the curves with equations*

$$(7)\qquad \delta^{\mp}:=wE(\pm x+i0)$$

$$=x\left\{E\left(\frac{1}{x}\right)-\left(1-\frac{1}{x^2}\right)K\left(\frac{1}{x}\right)\mp i\left[E\left(\sqrt{1-\frac{1}{x^2}}\right)-\frac{1}{x^2}K\left(\sqrt{1-\frac{1}{x^2}}\right)\right]\right\},\ 1\leqq x\leqq+\infty,$$

representing the images of the upper banks of the cuts $\{z|1\leqq z\leqq+\infty\}$ *and* $\{z|-\infty\leqq z\leqq -1\}$, *respectively, or with equations*

$$(8)\qquad \delta^{\mp}:w=E(\pm x-i0)$$

$$=x\left\{E\left(\frac{1}{x}\right)-\left(1-\frac{1}{x^2}\right)K\left(\frac{1}{x}\right)\mp i\left[E\left(\sqrt{1-\frac{1}{x^2}}\right)-\frac{1}{x^2}K\left(\sqrt{1-\frac{1}{x^2}}\right)\right]\right\},\ 1\leqq x\leqq+\infty,$$

representing the images of the lower banks of the cuts $\{z|-\infty\leqq z\leqq -1\}$ *and* $\{z|1\leqq z\leqq+\infty\}$, *respectively.*

Remark. The curves $\delta^{\mp}$ lying in the right half-plane $\{w|\operatorname{Re} w>0\}$, have for their origin the point $w=1$ and for asymptotes the negative imaginary half-axis and the positive imaginary half-axis, respectively.

Proof. It is analogous to the proof of Theorem 1. In view of the relation $\overline{E(\sqrt{\zeta})}=E(\sqrt{\bar\zeta})$, it is sufficient to find the image of the upper half-plane $\{\zeta|\operatorname{Im}\zeta>0\}$ only be the integral $w=E(\sqrt{\zeta})$. By means of the principle of the correspondence of the boundaries we obtain successively:

For $\zeta=\xi$, $-\infty<\xi<1$ the derivative $dw/d\xi$ of the second integral in (2) is negative and hence the segment $\{\zeta|-\infty\leqq\zeta\leqq 1\}$ is mapped in a one-to-one manner onto the segment $\{w|+\infty\geqq w\geqq 1\}$.

In the proof of Theorem 1 we have investigated the variation of the expression $1-\zeta\sin^2\varphi$ in the circuit of the point $\zeta=1$ in the half-plane $\{\zeta|\operatorname{Im}\zeta>0\}$. On this basis the values of the second integral in (2) on the upper bank of the cut $\{\zeta=x|1<x<+\infty\}$ will be

$$(9)\quad w=E(\sqrt{x+i0})=\int_0^{\arcsin(1/\sqrt{x})}(1-x\sin^2\varphi)^{1/2}d\varphi-i\int_{\arcsin(1/\sqrt{x})}^{\pi/2}(x\sin^2\varphi-1)^{1/2}d\varphi.$$

With the help of the substitutions $\sin\varphi=(1/\sqrt{x})\sin\psi$ and $\cos\varphi=\sqrt{1-(1/x)\sin\psi}$ in the first and the second integral of (9), respectively, we obtain for $(x>1)$

(10) $$w=E(\sqrt{x+i0})=\frac{\sqrt{x}}{x}\int_0^{\pi/2}\frac{\cos^2\psi\,d\psi}{\sqrt{1-(1/x)\sin^2\psi}}$$
$$-i\sqrt{x}\left(1-\frac{1}{x}\right)\int_0^{\pi/2}\frac{\cos^2\psi\,d\psi}{\sqrt{1-(1-1/x)\sin^2\psi}}.$$

Now by means of the identities

$$\frac{1}{x}\cos^2\psi=\left(1-\frac{1}{x}\sin^2\psi\right)-\left(1-\frac{1}{x}\right),\quad \left(1-\frac{1}{x}\right)\cos^2\psi=\left[1-\left(1-\frac{1}{x}\right)\sin^2\psi\right]-\frac{1}{x}$$

from (10) and (2) it follows that for $x>1$

(11) $$w=E(\sqrt{x+i0})$$
$$=\sqrt{x}\left\{E\left(\frac{1}{\sqrt{x}}\right)-\left(1-\frac{1}{x}\right)K\left(\frac{1}{\sqrt{x}}\right)-i\left[E\left(\sqrt{1-\frac{1}{x}}\right)-\frac{1}{x}K\left(\sqrt{1-\frac{1}{x}}\right)\right]\right\}.$$

It is clear from (10) that letting $x\to1$ and $x\to+\infty$ we have: $w=E(\sqrt{1+i0})=1$ and $w=E(\sqrt{+\infty+i0})=0-i\infty$. Hence, we can determine the integral $w=E(\sqrt{\zeta})$ in (2) on the infinite segment $\{\zeta\,|\,1\leqq\zeta\leqq+\infty\}$ by means of the formulas (9), (10) and (11) by which the integral $w=E(\sqrt{\zeta})$ becomes a continuous function in the extended sense in the upper half-plane $\{\zeta\,|\,\mathrm{Im}\,\zeta\geqq0\}$ of the extended ζ-plane where at the boundary point $\zeta=\infty$ we have set $E(\sqrt{\infty})=\infty$. Thus, we conclude that the second integral in (2) maps in a one-to-one manner the real axis $[-\infty, +\infty]$ of the extended ζ-plane onto this closed Jordan curve of the extended w-plane which consists of the segment $\{w\,|\,+\infty\geqq w\geqq1\}$ and the curve δ^- with the equation (11) for $1\leqq x\leqq+\infty$ (δ^- lies in the fourth quadrant, has the point $w=1$ as its origin and the negative imaginary half-axis as its asymptote), where the upper ζ-half-plane $\{\zeta\,|\,\mathrm{Im}\,\zeta>0\}$ is mapped onto its interior Δ^- due to the fact that the circuits of their boundaries are in the same direction.

Further, if we follow the line of the proof of Theorem 1, we shall complete the proof of Theorem 2.

Corollary 2. *The full elliptic integral of the second kind $E(z)$ considered as an analytic function of the complex modulus z realizes a bivalent conformal mapping of the z-plane with the cut $\{z\,|\,z<-1,\ z>1\}$, where the conformality is violated at the point $z=0$ only, where the derivative $E'(0)$ is zero.*

Corollary 2 is obtained by us by an alternate method in our paper [2, p. 63, Theorem 3, item 2].

Again, an arbitrary straight line l, passing through the origin and different from the real axis, divides the z-plane with the cut $\{z\,|\,z<-1,\ z>1\}$ into two maximal domains of univalence where the two opposite half-lines of l determined by $z=0$ are integrated with these domains, respectively. In each one of them we can obtain complete information about $E(z)$. Evidently, the most convenient one is the set G^+ in Theorem 2.

Also, in the closure $\bar{\Delta}=\Delta\cup\delta$ the inverse one-valued function $\zeta=\beta(w)$ and the inverse two-valued function $z=B(w)=\sqrt{\beta(w)}$ are determined so that the point $w=\pi/2$, the image of the point $z=0$ under the mapping $w=E(z)$, is an algebraic branch point of the first kind.

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