PLISKA

STUDIA MATHEMATICA BULGARICA

ПЛИСКА

БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

S-ASYMPTOTIC OF A DISTRIBUTION*

STEVAN PILIPOVIĆ, BOGOLJUB STANKOVIĆ

We give a definition of asymptotic behaviour at infinity of Schwartz distribution — the so-called S-asymptotic. Some basic properties of this S-asymptotic are proved and possibilities of its applications are given.

1. Introduction. In the last fifteen years several notions connected with the asymptotic behaviour of a distribution have been considered (see, for example, [2] and [9]).

The most important of them the so-called quasiasymptotic of tempered distribution ([9]) which is deeply studied and applied in the quantum field theory by the Soviet mathematicians Vladimirov, Drožinov and Zavialov (see [8] and reference there). The so-called "asymptotic by translation" is compared with the quasiasymptotic of tempered distributions in [4]. For the asymptotic by translation in $\mathcal{S}$ see also [2], where this notion is defined.

We study in this paper the asymptotic by translation of Schwartz distributions. We call this asymptotic the $\mathcal{S}$-asymptotic. The notion of the $\mathcal{S}$-asymptotic is inspired by the notion given in the book Schwartz [7, T. II, p. 97]; this is the reason for the name $\mathcal{S}$-asymptotic. In the special case this notion becomes the value of a distribution at infinity ([1, p. 44]).

2. Notations. The set of real, complex and natural numbers are denoted by **R**, **C** and **N**; $\mathbf{N}_0=\mathbf{N}\cup\{0\}$. $\mathbf{R}^n_+=\{x\in\mathbf{R}^n,\ x_i\geqq 0,\ i=1,\dots,n\}$. For $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$, $\langle x, y\rangle=x_1y_1+\dots+x_ny_n$; $\langle x, x\rangle=\|x\|^2$, $|x|=|x_1|+\dots+|x_n|$. If $p\in\mathbf{N}^n_0$, then $x^p=x_1^{p_1}\dots x_n^{p_n}$.

An element of the unit sphere $S^{n-1}\subset\mathbf{R}^n$ is denoted by $w=(w_1,\dots,w_n)$. We denote by $B(0, r)$ the closed ball in $\mathbf{R}^n$ centered at zero and with radius $r>0$. Γ is a cone in $\mathbf{R}^n$ with the vertex at zero. e_i is the n-tuple with all components equal to zero except the i-th one which is equal to 1.

We denote by Γ_a an acute cone in $\mathbf{R}^n$ with the vertex at zero. This means that $\overline{\operatorname{ch}\Gamma_a}$ does not contain straight lines.

Let $h_1, h_2\in\Gamma_a$. We say that $h_1\geqq h_2$ if $h_1\in h_2+\Gamma_a$. The set Γ_a is partially ordered and directed with respect to this relation.

Let $G(h)$, $h\in\Gamma_a$, be a complex valued functions. We write

$$\lim_{h\to\infty,\, h\in\Gamma_a} G(h)=A\in C$$

if for any $\varepsilon>0$ there is $h(\varepsilon)\in\Gamma_a$ such that $G(h)\in(A-\varepsilon,\ A+\varepsilon)$ if $h\geqq h(\varepsilon)$ in Γ_a.

---

*This material is based on work supported by the US-Yugoslav Joint Fund. for Scientific and Technological Cooperation, in cooperation with the NSF under Grant (JFP) 544.

We denote by $\Sigma(\Gamma)$ the set of all real valued functions $c(h)$, $h\in\Gamma$, which are different from zero when $h\in\Gamma$.

A function $L(\tau)$, $\tau\in[\alpha,\infty)$, $\alpha>0$, is called slowly varying at infinity if it is positive, continuous and if for every $u>0$ $\lim_{\tau\to\infty}L(u\tau)/L(\tau)=1$. A function $a(\tau)$, $\tau\in[\alpha,\infty)$, $\alpha>0$, is regularly varying if it is of the form $a(\tau)=\tau^{\nu}L(\tau)$, $\nu\in\mathbf{R}$ (see [6]).

$\mathscr{D}'$ is the space of Schwartz distributions (in n-dimensions) and $\mathscr{E}'$ the space of distributions with compact supports. The space of tempered distributions is denoted by $\mathscr{S}'$ If U is a locally integrable function, then $\tilde{U}$ is the regular distribution determined by U. For $k=(k_1,\ldots,k_n)\in\mathbf{N}_0^n$ and $f\in\mathscr{D}'$, $D^kf=\partial^{|k|}f/\partial t_1^{k_1}\ldots\partial t_n^{k_n}$.

If d is real valued function defined on some domain $\Omega\subset\mathbf{R}^n$ and $w\in S^{n-1}$, then $(D_wd)(x)$ denotes the derivative at $x\in\Omega$ of the function d in the direction w.

3. S-asymptotic. Definition and Properties.

Definition 1. *A distribution $T\in\mathscr{D}'$ has a* S*-asymptotic in the cone Γ, related to some $c(h)\in\Sigma(\Gamma)$ and with a limit $U\in\mathscr{D}'$* if there exists

$$\lim_{h\in\Gamma,\|h\|\to\infty}\langle T(t+h)/c(h),\ \varphi(t)\rangle=\langle U,\ \varphi\rangle,\quad \varphi\in\mathscr{D}. \tag{1}$$

Then we will write $T(t+h)\overset{s}{\sim}c(h)U(t)$, $\|h\|\to\infty$, $h\in\Gamma$.

In the special case when Γ is a ray $\{\beta w;\ \beta>0,\ w\in S^{n-1}\}$ this definition has the following form:

Definition 1'. *A distribution $T\in\mathscr{D}'$ has a* S*-asymptotic on the ray w related to the function $c(\beta)\in\Sigma(\mathbf{R}_+)$ and with the limit $U\in\mathscr{D}'$ if there exists*

$$\lim_{\beta\to\infty}\langle T(t+\beta w)/c(\beta),\ \varphi(t)\rangle=\langle U,\ \varphi\rangle,\quad \varphi\in\mathscr{D}. \tag{2}$$

In this case we write $T(t+\beta w)\overset{s}{\sim}c(\beta)U(t)$, $\beta\in\mathbf{R}_+$.

The following theorem gives a characteristic properties of the S-asymptotic.

Theorem 1. a) *If for every $r>0$ there exists a β_r such that the sets $\{t\in\mathbf{R}^n;\ t\in(\operatorname{supp}T-h)\cap B(0,r)\}$, $h\in\Gamma$, $\|h\|\geqq\beta_r$ are empty, then $T(t+h)\overset{s}{\sim}c(h)\cdot 0$, $\|h\|\to\infty$, $h\in\Gamma$, for every $c(h)\in\Sigma(\Gamma)$.*

b) *Let $c(h)\in\Sigma(\Gamma)$ and $\tilde{T}$ be a regular distribution defined by a locally integrable function T. Suppose that there exist locally integrable functions $U(t)$ and $V(t)$, $t\in\mathbf{R}^n$, such that for every compact set $K\subset\mathbf{R}^n$*

$$|T(t+h)/c(h)|\leqq V(t),\quad t\in K,\quad \|h\|>r_K,$$

$$\lim_{h\in\Gamma,\|h\|\to\infty}T(t+h)/c(h)=U(t),\quad t\in K.$$

Then, $\tilde{T}(t+h)\overset{s}{\sim}c(h)U(t)$, $\|h\|\to\infty$, $h\in\Gamma$.

c) *If $T(t+h)\overset{s}{\sim}c(h)U(t)$, $\|h\|\to\infty$, $h\in\Gamma$, then for every $k\in\mathbf{N}_0^n$, $T^{(k)}(t+h)\overset{s}{\sim}c(h)U^{(k)}(t)$, $\|h\|\to\infty$, $h\in\Gamma$.*

Proof. a) For each $\varphi\in\mathscr{D}$ there exists $r_\varphi>0$ such that $\operatorname{supp}\varphi\subset B(0,r_\varphi)$. The support of the distribution $T(t+h)$ is $(\operatorname{supp}T-h)$. Thus, by our supposition there exists β_{r_φ} such that for all $h\in\Gamma$, $\|h\|>\beta_{r_\varphi}$ the set $(\operatorname{supp} T-h)\cap B(0, r_\varphi)$ is empty and consequently $\langle T(t+h), \varphi(t)\rangle=0$, $h\in\Gamma$, $\|h\|\geqq\beta_{r_\varphi}$.

b) It follows from the Lebesgue's theorem.

c) It is a consequence of the definition of the derivative of a distribution.

Remark. The property given in a) shows that the S-asymptotic preserves the natural property of the asymptotic for numerical functions. The quasiasymptotic from [9] has not the same property. For example, the support of the δ-distribution is bounded and δ has the S-asymptotic 0 related to every $c(h)\in\Sigma(\mathbf{R}^n)$ but δ has the quasiasymptotic $-n$.

It is quoted in [4] that if $f\in\mathscr{S}'(\mathbf{R})$ and if it has S-asymptotic (on $\mathscr{S}(\mathbf{R})$) related to $x^\nu L(x)$, where $\nu>-1$, then it has a quasiasymptotic related to this function $(U\neq 0)$.

The statement b) shows that the S-asymptotic generalizes the asymptotic of a numerical function.

The following example (for the one-dimensional case see [1]) points out that a continuous function can have S-asymptotic related to a $c(\beta)$ without having the asymptotic: Let $T(\tau)=\int_\alpha^\tau g(x)dx$, $g\in L^1(-\infty, \infty)\cap C(-\infty, \infty)$, $\alpha>0$. Then $T(\tau+\beta) \overset{s}{\sim} 1.\int_\alpha^\infty g(x)dx$, $\beta\in\mathbf{R}_+$. By Theorem 1 c) we have $\tilde{g}(\tau+\beta) \overset{s}{\sim} 1.0$, $\beta\in\mathbf{R}_+$. But g must not have asymptotic behaviour when $\tau\to\infty$.

The S-asymptotic in a cone Γ, $\|h\|\to\infty$ is a local property. This determines the next

Theorem 2. *Let us suppose that the distributions T_1 and T_2 are equal on the open set $\Omega\subset\mathbf{R}^n$, where Ω has the following property: for every $r>0$ there exists a β_0 such that the ball $B(0, r)=\{x\in\mathbf{R}^n;\ \|x\|\leqq r\}$ is in $\{\Omega-h,\ h\in\Gamma,\ \|h\|\geqq\beta_0\}$. If we have $T_1(t+h) \overset{s}{\sim} c(h)U(t)$, $\|h\|\to\infty$, $h\in\Gamma$, then $T_2(t+h) \overset{s}{\sim} c(h)U(t)$, $h\in\Gamma$, as well.*

Proof. For a $\varphi\in\mathscr{D}$, $\operatorname{supp}\varphi\subset B(0, r)$,

$$\lim_{h\in\Gamma,\ \|h\|\to\infty} \left\langle \frac{T_1(t+h)-T_2(t+h)}{c(h)}, \varphi\right\rangle=0$$

because the complement of the set $\operatorname{supp}[T_1(t+h)-T_2(t+h)]$ contains the set $\{\Omega-h,\ h\in\Gamma,\ \|h\|\geqq\beta_0\}$. But by our supposition the number β_0 is fixed in such a way that the sets $\{\Omega-h,\ h\in\Gamma,\ \|h\|\geqq\beta_0\}$ contain $B(0, r)$ and consequently $\operatorname{supp}\varphi$.

The quasiasymptotic at infinity [9] has not the same property. The supports of δ and δ' are the same, i.e. $\{0\}$, but δ has the quasiasymptotic at infinity (in one-dimensional case) -1 and δ' has -2.

Now we give a theorem which characterizes the numerical function $c(h)$ and the limit distribution U.

Theorem 3. *Let $T\in\mathscr{D}'$, Γ be a convex cone in $\mathbf{R}^n$ with the vertex at zero and $T(t+h) \overset{s}{\sim} c(h)U(t)$, $\|h\|\to\infty$, $h\in\Gamma$, where $c(h)\in\Sigma(\Gamma)$ and $U\neq 0$.* Then:

a) *There exists $\lim_{h\in\Gamma,\|h\|\to\infty} c(h+h_0)/c(h)=d(h_0)$ for every $h_0\in\Gamma$.*

b) *The limit U satisfies the equation $U(t+x)=d(x)U(t)$, $x\in\Gamma$.*

c) *There exists the derivative $(D_w d)$ in every point $h_0\in\Gamma$ and in the direction $w\in\Gamma\cap S^{n-1}$; $d(x)$ satisfies the following equation:*

(3) $$(D_w d)(h_0)=(D_w d)(0)d(h_0);\ h_0\in\Gamma,\ w\in\Gamma\cap S^{n-1}.$$

d) $d(pw)=e^{\alpha p}$, $w\in\Gamma\cap S^{n-1}$, $p>0$, *and* $\alpha\in\mathbf{R}$ depends on w.

e) *Let* $w\in\Gamma\cap S^{n-1}$; *if* $w_i\neq 0$ *for* $i=k_1,\dots,k_m$, *then* $U(t)=V(t)\exp((\alpha/m)\times\Sigma_{i=k_1}^{k_m} t_i/w_i)$, *where* $\alpha=(D_w d)(0)$ *(it depends on* w*) and* V *is a solution of the equation*

(4) $$\sum_{i=k_1}^{k_m} w_i\frac{\partial V}{\partial t_i}=0.$$

Proof. There exists $\varphi\in\mathcal{D}$ such that $\langle U,\varphi\rangle\neq 0$. For this φ we have

$$\lim_{h\in\Gamma,\|h\|\to\infty}\frac{c(h+h_0)}{c(h)}\left\langle\frac{T(t+(h+h_0))}{c(h+h_0)},\ \varphi(t)\right\rangle$$

$$=\lim_{h\in\Gamma,\|h\|\to\infty}\left\langle\frac{T((t+h_0)+h)}{c(h)},\ \varphi(t)\right\rangle,\quad h_0\in\Gamma.$$

Hence,

$$\lim_{h\in\Gamma,\|h\|\to\infty}\frac{c(h+h_0)}{c(h)}\langle U,\varphi\rangle=\langle U(t+h_0),\varphi\rangle,\ h_0\in\Gamma.$$

It follows that there exists

$$\lim_{h\in\Gamma,\|h\|\to\infty}\frac{c(h+h_0)}{c(h)}=d(h_0)<\infty$$

and the relation under b) holds.

Using the relation from b) we have

(5) $$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\langle[U(t+h_0+\varepsilon w)-U(t+h_0)],\ \varphi(t)\rangle=\left\langle\sum_{i=1}^{n} w_i\frac{\partial U(t+h_0)}{\partial t_i},\ \varphi(t)\right\rangle$$

$$=\lim_{\varepsilon\to 0}\frac{d(h_0+\varepsilon w)-d(h_0)}{\varepsilon}\langle U,\varphi\rangle$$

which gives the existence of $(D_w d)(h_0)$, $w\in\Gamma\cap S^{n-1}$.

To prove that $d(x)$, $x\in\Gamma$, satisfies the differential equation (3) we start from

$$U(t+h_0+\varepsilon w)-U(t+h_0)=[d(\varepsilon w)-d(0)]U(t+h_0)=[d(\varepsilon w)-d(0)]d(h_0)U(t).$$

Using (5) we have $(D_w d)(h_0)=(D_w d)(0)d(h_0)$.

If we put in the last relation $h_0=pw$, the differential equation (3) becomes $((d/dp)d)(pw)=\alpha d(pw)$, $d(0)=1$. Hence, $d(pw)=e^{\alpha p}$.

From (5) it follows

(6) $$\sum_{i=1}^{n} w_i \frac{\partial U}{\partial t_i} = \alpha u, \quad \alpha = (D_w d)(0).$$

Let $V(t)$ be given by

$$U(t) = V(t) \exp\left(\frac{\alpha}{m} \sum_{i=k_1}^{k_m} t_i / w_i\right).$$

From relation (6) it follows that $V(t)$ satisfies equation (4).

Now we can give the analytical form of U which satisfies the functional equation b). The functional equation b) and

$$F(t+\beta w) = F(t), \; F(t) = U(t) \exp\left(-\frac{\alpha}{m} \sum_{i=k_1}^{k_m} t_i / w_i\right) \quad (\beta w = x)$$

are equivalent. Let A be the linear transformation $y = At$, defined by $y_i = w_i t_{k_1} - w_{k_1} t_i$, $i \neq k_1$; $y_{k_1} = t_{k_1}$. The distribution $V(y) = F(A^{-1} y)$ satisfies the relation

$$V(y_1, \ldots, y_{k_1} + \beta w_{k_1}, \ldots, y_n) = V(y), \quad y \in \mathbf{R}^n, \; \beta \in \mathbf{R}.$$

Thus we obtain that

$$U(t) = V(y) \exp\left(\frac{\alpha}{m} \sum_{i=k_1}^{k_m} t_i / w_i\right), \quad y = At,$$

where V does not depend on y_{k_i}, $i = 1, \ldots, m$.

If $\mathring{\Gamma} \neq \emptyset$ ($\mathring{\Gamma}$ is the interior of Γ), then for any $h_0 \in \mathbf{R}^n$, the sets

$$\{h + h_0;\; h \in \Gamma\} \cap \Gamma \cap \{x;\; \|x\| > R\}, \quad R > 0,$$

are non-empty. In this case if $T(x+h) \sim c(h) U(x)$, $\|h\| \to \infty$, $h \in \Gamma$, where $U \neq 0$, we obtain that for $h_0 \in \mathbf{R}^n$

(7) $$\lim_{\|h\| \to \infty, h+h_0 \in \Gamma} \frac{T(h+h_0+x)}{c(h+h_0)} = U(x), \qquad \lim_{\|h\| \to \infty, h \in \Gamma} \frac{T(x+h+h_0)}{c(h)} = U(x+h_0).$$

Thus, in the same way as in Theorem 3 we can prove

Proposition 4. *If $\mathring{\Gamma} \neq \emptyset$ and $T(x+h) \sim c(h) U(x)$, $c \in \Sigma(\Gamma)$, $\|h\| \to \infty$, $h \in \Gamma$, and $U \neq 0$, then for every $h_0 \in \mathbf{R}^n$ there exists the limit*

$$\lim_{h \in \Gamma, h+h_0 \in \Gamma} \frac{c(h+h_0)}{c(h)} = d(h_0).$$

Moreover, all the assertions from Theorem 3 *hold with* h_0, $x\in\mathbf{R}^n$, $w\in S^{n-1}$ *instead of* h_0, $x\in\Gamma$, $w\in S^{n-1}\cap\Gamma$. *As well,*

$$d(x)=\exp(\langle\alpha,\ x\rangle),\quad \alpha_i=\frac{\partial}{\partial x_i}d(0),\quad i=1,\dots,n, \tag{8}$$

and

$$U(x)=C\exp(\langle\alpha,\ x\rangle)\quad \textit{for some } C\in\mathbf{R}. \tag{9}$$

Proof. From (7) it follows that for every $x\in\mathbf{R}^n$

$$U(t+x)=d(x)U(t)$$

holds. Similarly as in Theorem 3 one can prove that

$$\left(\frac{\partial}{\partial x_i}d\right)(h_0)=\left(\frac{\partial}{\partial x_i}d\right)(0)d(h_0),\quad h_0\in\mathbf{R}^n. \tag{10}$$

If we put $d(x)=f(x)\exp\langle\alpha,\ x\rangle$, where $\alpha_i=((\partial/\partial x_i)d)(0)$, $i=1,\dots,n$, we obtain

$$\frac{\partial}{\partial x_i}f(x)=0,\quad i=1,\dots,n,\quad \text{i.e.}\quad f(x)=C.$$

Since $f(0)=d(0)=1$ we obtain that (8) holds. In order to get the analytic expression of U we have to use the fact that U satisfies the equations

$$\frac{\partial}{\partial x_i}U=\alpha_i U,\quad \alpha_i=\left(\frac{\partial}{\partial x_i}d\right)(0),\quad i=1,\dots,n.$$

An interesting conclusion for numerical functions follows from Proposition 4.

Corollary 1. *Let* f, V *and* U *be locally integrable functions,* $U\neq 0$, *such that for every compact set* $K\subset\mathbf{R}^n$ *and* $c(h)\in\Sigma(\Gamma)$, *where* Γ *is a convex cone with* $\mathring{\Gamma}\neq 0$,

$$\left|\frac{f(t+h)}{c(h)}\right|\leqq V(t),\quad t\in K,\quad \|h\|>r_K,\quad h\in\Gamma,$$

$$\lim_{h\in\Gamma,\ \|h\|\to\infty} f(t+h)/c(h)=U(t)\quad t\in K,$$

then $U(t)=c\exp(\langle\alpha,\ t\rangle)$, $\alpha\in\mathbf{R}^n$, $c\in\mathbf{R}$.

Proof. From Theorem 1 b) it follows that for the regular distribution $\tilde{f}$ is $\tilde{f}(t+h)\overset{s}{\sim} c(h)U(t)$, $\|h\|\to\infty$, $h\in\Gamma$. Using Proposition 4 we have that $U(t)$ has the form $U(t)=c\exp(\langle\alpha,\ t\rangle)$, $\alpha\in\mathbf{R}^n$ and that c is a constant.

Let Γ_a be an acute cone with the vertex at zero. For a such cone we give a more general definition of the S-asymptotic.

Definition 2. *We say that* $T \in \mathscr{D}'(\mathbf{R}^n)$ *has a* S-asymptotic *in the cone* Γ_a *related to some* $c(h) \in \Sigma_a(\Gamma_a)$ *if there exists the limit in* $\mathscr{D}'(\mathbf{R}^n)$

$$\lim_{h \to \infty, h \in \Gamma_a} T(x+h)/c(h) = U(x). \tag{11}$$

In this case we write $T(x+h) \overset{s}{\sim} c(h)U(x)$, $h \to \infty$, $h \in \Gamma_a$.

In the same way as in Theorems 1, 2, 3, Corollary 1 and Proposition 4, but with $h \to \infty$ instead of $\|h\| \to \infty$, $h \in \Gamma_a$ one can prove

Theorem 5. *Let* $T \in \mathscr{D}'(\mathbf{R}^n)$.

a) *If the condition of Theorem* 1, a) *holds* (*with* $h > h_r$, $h_r \in \Gamma_a$, *instead of* $\|h\| > \beta_r$, $h \in \Gamma$), *then* $T(t+h) \overset{s}{\sim} c(h)0$, $h \to \infty$, $h \in \Gamma$, *for every* $c(h) \in \Sigma(\Gamma_a)$.

b) *Let* $c(h) \in \Sigma(\Gamma_a)$ *and* $\tilde{T}$ *be a regular distribution such that all the conditions of Theorem* 1, b) *hold with* ($h > h_K$ *and* $h \to \infty$ *instead of* $\|h\| \geqq r_K$ *and* $\|h\| \to \infty$), *then* $\tilde{T}(t+h) \sim c(h)U(t)$, $h \to \infty$, $h \in \Gamma_a$.

c) *The same assertion as in Theorem* 1, c) *holds* ($h \to \infty$ *instead of* $\|h\| \to \infty$).

d) *The* S-*asymptotic in the cone* Γ_a, $h \to \infty$, $h \in \Gamma_a$, is *a local property* (*see Theorem* 2).

e) *Let* Γ_a *be a convex cone, as well, and let* (11) *hold with* $U \neq 0$. *All the assertions in Theorem* 3 *hold with* $h \to \infty$ *instead of* $\|h\| \to \infty$.

f) *Let* Γ_a *be convex*, $\mathring{\Gamma}_a \neq \emptyset$ *and* (11) *hold with* $U \neq 0$. *Then, all the assertions of Proposition* 4 *hold* (*again with* $h \to \infty$ *instead of* $\|h\| \to \infty$).

We can give the analytical expression for $c(\beta) \in \Sigma(\mathbf{R}_+)$ if we assume that $c(\beta)$ satisfies some additional conditions.

Let $\Sigma_0(\mathbf{R}_+) \subset \Sigma(\mathbf{R}_+)$ be the set of those functions $c(\beta)$ which have the following properties:

(i) $c(\beta)$ is positive and continuous in $[\alpha, \infty)$ for some $\alpha > 0$.

(ii) There exist $T_c \in \mathscr{D}'$, $w_c \in S^{n-1}$ and $U_c \in \mathscr{D}'$, $U_c \neq 0$, such that

$$T_c(t + \beta w_c) \overset{s}{\sim} c(\beta) U_c(t), \ \beta \in \mathbf{R}_+.$$

Theorem 6. *The necessary and sufficient condition that* $c(\beta) \in \Sigma_0(\mathbf{R}_+)$ *is that* $c(\beta) = \exp(\nu\beta) L(\exp\beta)$, $\beta \in [\alpha, \infty)$, *where* $\nu \in \mathbf{R}$ *and* L *is a slowly varying function.*

Proof. Let $c(\beta) \in \Sigma_0(\mathbf{R}_+)$. Theorem 3 implies that for some $\nu \in \mathbf{R}$

$$\lim_{\beta \to \infty} c(\beta_0 + \beta)/c(\beta) = \exp(\nu\beta_0), \ \beta_0 \in \mathbf{R}_+.$$

By setting $\beta_0 = \ln p_0$, $p_0 > 0$, and $\beta = \ln p$, $p > 0$, the last limit becomes

$$\lim_{p \to \infty} c(\ln p_0 p)/c(\ln p) = p_0^\nu, \quad p_0 > 0.$$

Hence (see [6]) $a(p) = c(\ln p)$, $p > p' > 0$, is a regularly varying function of degree ν. It follows that $c(\ln p) = p^\nu L(p)$, $p > p' > 0$, and consequently $c(\beta) = L(\exp\beta)\exp(\nu\beta)$, $\beta \geqq \alpha > 0$, for a slowly varying function L.

On the other hand, let $\tilde{T} \in \mathcal{D}'$ be defined by the function $T(t) = \exp(\nu\langle t, w\rangle) \times L(\exp(\langle t, w\rangle))$. For $c(\beta) = L(\exp\beta)\exp\nu\beta$

$$\lim_{\beta\to\infty} \tilde{T}(t+\beta w)/c(\beta) = \exp(\nu\langle t, w\rangle) \neq 0 \text{ in } \mathcal{D}'.$$

Corollary 2. *Let $c(\beta)$ be a positive and differentiable function for $\beta \geqq \alpha$. If $\lim_{\beta\to\infty} c'(\beta)/c(\beta) = \nu < \infty$, then $c(\beta) = L(\exp\beta)\exp\nu\beta$.*

Proof. Let $a(p) = c(\ln p)$. A sufficient condition that $a(p)$ is regularly varying function is the existence of $\lim_{p\to\infty} pa'(p)/a(p) = \nu < \infty$ [8]. In this case $a(p) = p^\nu L(p)$, L is a slowly varying function. We obtain the assertion by putting $a(p) = c(\ln p)$ and $\beta = \ln p$ in the last limit.

4. Multiplication by a smooth function and the S-asymptotic.

Theorem 7. *Let $g \in \mathcal{E}$, $c(h)$, $c_1(h) \in \Sigma(\Gamma)$ and $g(t+h)/c_1(h)$ converges to $G(t)$ in $\mathcal{E}$ as $h \in \Gamma$, $\|h\| \to \infty$. If $T(t+h) \overset{s}{\sim} c(h)U(t)$, $\|h\| \to \infty$, $h \in \Gamma$, then $g(t+h)T(t+h) \overset{s}{\sim} c_1(h)c(h)G(t)U(t)$, $\|h\| \to \infty$, $h \in \Gamma$.*

Proof. Since $T(t+h)c(h)$ converges weakly in $\mathcal{D}'$, the set $\{T(t+h)/c(h), h \in \Gamma, \|h\| \geqq \beta_0\}$ is a weakly bounded set and thus it is a bounded set ([7, T. I, p. 72]). From ([7, T. I, Théorème X) it follows that if B is a bounded set in $\mathcal{D}'$, then for any $\varphi \in \mathcal{D}$ and $S \in B$, $\langle S(t), [g(t+h)/c_1(h) - G(t)]\varphi(t)\rangle$ converges uniformly to zero in B as $h \in \Gamma$, $\|h\| \to \infty$. Therefore

$$\lim_{h\in\Gamma, \|h\|\to\infty} \langle g(t+h)T(t+h)/c_1(h)c(h)), \varphi(t)\rangle$$

$$= \lim_{h\in\Gamma, \|h\|\to\infty} \langle T(t+h)/c(h), [g(t+h)/c_1(h) - G(t)]\varphi(t)\rangle$$

$$+ \lim_{h\in\Gamma, \|h\|\to\infty} \langle T(t+h)/c(h), G(t)\varphi(t)\rangle = \langle U(t)G(t), \varphi(t)\rangle, \quad \varphi \in \mathcal{D}.$$

Corollary 3. *Let $g \in \mathcal{E}$, $c_1(h) \in \Sigma(\Gamma)$, where Γ is a convex cone with $\mathring{\Gamma} \neq \emptyset$ and $g(x+h)/c_1(h) \to G(x)$ as $h \in \Gamma$, $\|h\| \to \infty$ in the sense of convergence in $\mathcal{E}$, then $G(x) = C\exp\langle m, x\rangle$ for some $m \in \mathbf{R}^n$, $C \in \mathbf{R}$.*

Proof. We have to apply Theorem 7 to $T(t) \equiv 1$, $c(h) \equiv 1$ and to use Proposition 4.

Using Definition 2, in the same way as above one can prove for the accute cone Γ_a the following theorem:

Theorem 8. (i) *The same assertion as in Theorem 7 holds with $h \to \infty$ instead of $\|h\| \to \infty$.*

(ii) *Let Γ_a be convex and $\mathring{\Gamma}_a \neq 0$. The same assertion as in Corollary* 3 *holds* (*with $h \to \infty$ instead of $\|h\| \to \infty$*).

5. The S-asymptotic of some numerical functions.

1. $\exp(\langle a, x+h\rangle) \overset{s}{\sim} \exp(\langle a, h\rangle)\exp(\langle a, x\rangle)$, $\|h\|$ (or h)$\to\infty$, $h \in \Gamma$.

2. $\exp(\sqrt{(x+\beta)^2 + (x+\beta)}) \overset{s}{\sim} \exp\beta \exp\left(x + \frac{1}{2}\right)$, $\beta \in \mathbf{R}_+$, $x \in \mathbf{R}$.

3. Let $P(x) = \Sigma_{|p|\leqq m} A_p x^p$, $A_p \in \mathbf{C}$, $p \in \mathbf{N}_0^n$ and $w \in S^{n-1}$. We put

$$J=\{v_1,\dots,v_k,\ w_{v_i}\neq 0,\ i=1,\dots,k\},\ \mathscr{P}=\{p\in\mathbf{N}_0^n,\ |p|\leqq m,\ A_p\neq 0\},$$

$$\mathscr{P}_0=\{p^0=(p_1^0,\dots,p_n^0)\in\mathscr{P};\ \Sigma_{j\in J}p_i^0\geqq\Sigma_{i\in J}p_i,\ p\in\mathscr{P}\}\ \text{and}\ \gamma=\Sigma_{i\in J}p_i^0.$$

Then $P(x+\beta w)\overset{s}{\sim}\beta^\gamma.\ \Sigma_{p\in\mathscr{P}_0}A_p w^p,\ \beta\in\mathbf{R}_+$ and $P(x+\beta w)/\beta^\gamma$ converges in $\mathscr{E}$ when $\beta\to\infty$.

4. For a slowly varying function $L(t)$, $t\geqq\alpha>0$ we have $\tilde{L}(t+\beta)\overset{s}{\sim}L(\beta)\cdot 1$, $\beta\in\mathbf{R}_+$. Namely,

$$\lim_{\beta\to\infty}\langle\tilde{L}(t+\beta)/L(\beta),\ \varphi(t)\rangle=\lim_{\beta\to\infty}\int_{-r}^{r}\varphi(t)L(t+\beta)/L(\beta)dt$$

$$=\lim_{q\to\infty}\int_{e^{-r}}^{e^r}\varphi(\ln y)L(\ln yq)/L(\ln q)\frac{dy}{y}=\int_{\mathbf{R}}\varphi(t)dt,\quad\varphi\in\mathscr{D}.$$

We used above that $L(\ln t)$ is also a slowly varying function ([4, p. 19]) and that $L(\ln\beta)/(L(\ln\beta)$ converges to 1 as $\beta\to\infty$ uniformly if $u\in[\alpha_1,\ \alpha_2]$, $0<\alpha_1<\alpha_2<\infty$.

6. Some applications of the S-asymptotic. Let $\Gamma=\mathbf{R}^n$; first we have to restrain our set $\Sigma(\mathbf{R}^n)$. By $\Sigma_1(\mathbf{R}^n)$ we denote the subset of $\Sigma(\mathbf{R}^n)$: $\Sigma_1(\mathbf{R}^n)=\{c(h)\in\Sigma(\mathbf{R}^n)$;

$\lim_{\|h\|\to\infty}\|h\|^k/c(h)=0$, for every $k\in\mathbf{N}\}$.

Theorem 9. *Let for every* $c(h)\in\Sigma_1(\mathbf{R}_+^n)$ $T(t+h)\overset{s}{\sim}c(h)U_c(t)$, $\|h\|\to\infty$, $h\in\mathbf{R}^n$, *then* $T\in\mathscr{S}'$. (*U can be the zero distribution, as well.*)

Proof. One can prove that the set

$$\{T(t+h)/c(h),\quad\|h\|\geqq\beta\}$$

is weakly bounded in $\mathscr{D}'$. By [5, Ch. VII, Théorème VI], it follows that $T\in\mathscr{S}'$. We obtain that $T\in\mathscr{S}'$.

Theorem 10. *Let us suppose*:

a) $c_p(h)\in\Sigma(\Gamma)$, $p\in\mathbf{N}_0^n$, $|p|\leqq m$, $m\in\mathbf{N}_0$;

b) *For some* $p_0\in\mathbf{N}_0^n$, $|p_0|\leqq m$,

$$\lim_{h\in\Gamma,|h|\to\infty}c_p(h)/c_{p_0}(h)=a_p<\infty,\quad|p|\leqq m;$$

c) $g_p(t)\in\mathscr{E}$, $|p|\leqq m$ *and* $\lim_{h\in\Gamma,\|h\|\to\infty}g_p(t+h)/c_p(h)=G_p(t)$ in $\mathscr{E}$;

d) $H\in\mathscr{D}'$ *and* $H(t+h)\overset{s}{\sim}c_{p_0}(h)c(h)V(t)$, $\|h\|\to\infty$, $h\in\Gamma$;

e) $T\in\mathscr{D}'$ *is the solution of the partial differential equation*

$$\sum_{|p|\leqq m}g_p(t)D^pT(t)=H(t)\tag{12}$$

such that $T(t+h)\overset{s}{\sim}c(h)U(t)$, $\|h\|\to\infty$, $h\in\Gamma$.

Then

(13) $$\sum_{|p|\leqq m} a_p G_p(t) D^p U(t) = V(t).$$

Proof. We have only to use Theorem 7 and Theorem 1, c).

Remarks. Notice that equation (13) is simpler than (12) and just equation (13) gives the asymptotic condition for solutions of (12).

The S-asymptotic on a cone Γ can be determined if we assume that the cone Γ is convex and $\mathring{\Gamma} \neq \emptyset$. In this case (13) becomes

$$\tilde{C}_2 \sum_{|p|\leqq m} C_p a_p u^p \exp(\langle s_p + u, t\rangle) = \tilde{C}_1 \exp(\langle v, t\rangle),$$

where

$$G_p(t) = C_p \exp(\langle s_p, t\rangle), \quad V(t) = \tilde{C}_1 \exp(\langle v, t\rangle),$$

$$U(t) = \tilde{C}_2 \exp(\langle u, t\rangle).$$

REFERENCES

1. P. Antosik, J. Mikusiński, R. Sikorski. Theory of Distributions. The Sequential approach. Polish Scientific Publishers, Warszawa, 1973.
2. Ju. A. Bričkov, Ju. M. Širokov. On the asymptotic behaviour or Fourier transformations. *TMF*, **4**, 1970, 301-109.
3. Ju. N. Drožinov, B. I. Zavialov. Tauberian theorems for generalized functions with supports in a cone. *Math. Sb.*, **108**, 1979, 78-90 (in Russian).
4. Ju. Drožinov, B. I. Zavialov. Asymptotic properties of some classes of generalized functions. *Izv. Akad. Nauk. SSSR, Ser. Mat.*, **49**, 1985, 81-140 (in Russian).
5. J. Lavoine, O. P. Misra. Théorèmes abélians pour la transformation de Stieltjes des distributions. *C. R. Acad. Sci. Paris, Série A*, **279**, 1974, 99-102.
6. E. Seneta. Regularly varying functions. *Lecture Notes Math.*, **508**, 1976.
7. L. Schwartz. Théorie des distributions. Paris, T. I (1950). T. II (1951).
8. V. S. Vladimirov, B. I. Zavialov. Tauberian theorems in the quantum field theory (in Russian). *Itogi Nauki i Tehniki*, **15**, 1980.
9. B. I. Zavialov. Automodel asymptotic of electromagnetic form-factors and the behaviour of their Furier transformations in the neighbourhood of the right cone. *TMF*, **17**, 1973, 178-178 (in Russian).

University of Novi Sad.
Institute of Mathematics
Yugoslavia

Received 30. 5. 1986