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ПЛИСКА

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THE GAUGE INVARIANT CURRENTS OF THE MAXWELLIAN FIELD

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We show that a conserved current for the Maxwellian field, which is invariant under the gauge group of that field, is the sum of two currents $\Phi+T$, where Φ corresponds to a Poincaré symmetry of the field, and T is a topological form that is conserved under every dynamics.

1. Introduction and summary. In a Lagrangean field theory, the extremals are certain 4-dimensional submanifolds of a bundle B over space time $R^4(=M)$. A dynamic differential form is of degree 3 and is closed on each extremal and thus constitutes a conserved quantity. A 3-form ε is a current if

$$\varepsilon=J^1d^{234}-J^2d^{134}+J^3d^{124}-J^4d^{123} \tag{1.1}$$

in the notation of [2, 529].

The dynamic currents of the Maxwellian system fall into five classes [2, loc. cit.]. Although each represents a conserved quantity, some of them do not correspond to any dynamic symmetry of the system, that is they are not Noetherian.

We will show here which of them are gauge invariant. The gauge group can be abstractly defined as the dynamic symmetries which transform the field-space at each point into itself. (Applied to the Maxwellian system, this comes down to the addition of a gradient to the "4"-vector potential). We show that a gauge invariant current is equivalent to the sum of two currents Φ^i and T^i, where the former is Noetherian and corresponds to an infinitessimal translation plus Lorentz transformation in M, and the latter form is dynamic independently of the dynamics: a topological current [3].

2. The gauge group. The field is given by the electromagnetic potentials $A_1,\dots,A_4$. Thus, field space is also R^4, and we may use the A's as Cartesian coordinates there. Let $P=M\times R^4$. Let $t_1,\dots,t_4$ be a Lorentz coordinate system in M. We use $t_1,\dots,t_4$, $A_1,\dots,A_4$ as coordinates in P. Let a vector field

$$U=B_i\partial/\partial A_i \tag{2.1}$$

be given in P, with vanishing t-components. This infinitessimal transformation in P induces an infinitessimal transformation V in the bundle B of first-order jets of sections in P. Here the coordinates are those already listed, plus the A_{ij} corresponding to the ∂_jA_i, where ∂_j means $\partial/\partial t^j$. The t and A components are the same as for U, but additional terms are needed for the A_{ij}, and the induced infinitessimal transformation is $V=U+C_{ij}\partial/\partial A_{ij}$, where $C_{ij}=\partial_jB_i+A_{kj}\partial B_i/\partial A_k$.

For U as in (2.1), one may [1, sec. 6] take $D_VL=0$ as the condition for a gauge symmetry, where D_VL is the Lie derivative with respect to V of the Lagrangean density. For us

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$$L = (A_{ij} - A_{ji})(A^{ij} - A^{ji})/4.$$

Hence, $D_V L = (C_{ij} - C_{ji})(A^{ij} - A^{ji})/2$. The terms of the first degree in the A_{ki} yield $\partial_i B_j = \partial_j B_i$. The quadratic terms can be manipulated to show that the B's must be independent of the A_i. More precisely, (2.1) is an infinitessimal gauge symmetry, if and only if

(2.2) $$B_i = \partial \Lambda / \partial t^i,$$

where Λ depends only on the space-time coordinates.

3. Gauge invariant quantities. Let $\Lambda = t^k$. Then, $U = \partial / \partial A_k$ and V is the same. Let F be a gauge invariant function defined on B. Then, $D_V F = 0$ says that $\partial F / \partial A_k = 0$. Next, take $\Lambda = t^1 t^2$. Then, $V = t^1 \partial^2 - t^2 \partial^1 + \partial / \partial A_{12} - \partial / \partial A_{21}$. Hence, we also have $\partial F / \partial A_{ij} + \partial F / \partial A_{ji} = 0$. These two sets of conditions also suffice to ensure $D_V F = 0$, when (2.2) holds. Consequently, a quantity is gauge invariant, if and only if it can be expressed in terms of the t's and the field-components $F_{ij} = A_{ij} - A_{ji}$ alone.

4. Gauge invariant currents. A tensor ε is gauge invariant, if $D_V \varepsilon = 0$, when (2.2) holds. For a current (1.1), this just requires each J_i to be gauge invariant. If in addition it is dynamic, we may consult [2, 3/8 and 4/1]:

(4.1) $$\partial J^i / \partial A_{kj} + \partial J^j / \partial A_{ki} = -M^i g^{kj} + 2M^k g^{ij} - M^j g^{ki},$$

(4.2) $$M^i = R^i + S_k A^{ki} - B_k A^{ik}.$$

We know that the J's depend only on the F^{ij} (and on the t's). We may express the M's in terms of the F^{ij} and the $A_{ij} + A_{ji}$, and then set the latter variables to 0, and the M's will still work for (4.1). But now the (partial derivative) S_i in (4.2) will have to be the same as B_i.

As in [2, 4/2], we construct a competing current

$$L^k = A_{mn} F^{mk} A^n + (1/2) A_{mn} F^{mn} (R_m - A_{mn} A^n) = (1/4) F_{mn} A^k + R_m F^{mk}.$$

This formula was devised to make L satisfy (4.1) with J replaced by L. We define $Z^i = J^i - L^i$. Then Z satisfies (4.1), with the M's equal to 0. The general form of such Z is known [2,533]. We can, therefore, assert for the sum $Z + L = J$ that

(4.3) $$J^k = N^k + {}^a N^{km} F_{am} + R_m F^{mk} + {}^{ab} N^{kmn} F_{am} F_{bn}$$
$$+ (1/4) B^k F_{mn} F^{mn} + {}^{abc} N^{kmnp} F_{am} F_{bn} F_{cp}.$$

These N's are skew-symmetric in each set of indices. For example, ${}^{abc}N^{kmnp}$ is skew-symmetric in ab, and skew-symmetric in $kmnp$. They can be used to define a system of linear differential forms

$${}^{abc}\varphi = {}^{abc} N^{kmnp} d_{kmnp},$$

where d_{kmnp} is just d^{kmnp} with its indices lowered. (The notation d^1, d^{12}, d^{123} etc., is that used in (1.1); ${}^{ab}\varphi$, ${}^a\varphi$ and φ are forms of degree 3, 2 and 1 defined analogously.)

We will consider the Hodge stars of these forms; for example, the Hodge star of ${}^{ab}\varphi$ is

$${}^{ab}\psi = {}^{ab}\varphi^{kmn} \varepsilon_{kmnp} d^p,$$

where ε_{kmnp} is the sign of the permutation $kmnp$.

The B's form the components of an infinitessimal conformal transformation of space-time M. The R's and the N's depend on the t's and possibly on the A_k.

We shall prove from gauge invariance that

$$J^k = N^k + {}^aN^{km}F_{am} + {}^{ab}N^{kmn}F_{am}F_{bn} + (1/4)B^kF_{mn}F^{mn} + {}^{abc}N^{kmnp}F_{am}F_{bn}F_{cp}, \tag{4.4}$$

where the B^k are the components of an infinitessimal isometry of the space-time structure, and the forms ${}^{abc}\psi$, ${}^{ab}\psi$, ${}^a\psi$, ψ are closed.

5. Proof of (4.4). Let ${}^aP^{km} = K^mg^{ak} - R^kg^{am}$. Then, ${}^aP^{km}F_{am} = R_mF^{km}$. Thus, R_mF^{mk} in (4.3) can be "absorbed" into the ${}^a\Lambda^{km}F_{am}$. This gives us that J^k has the form (4.4), and is thus a polynomial in the Fs. For any particular degree, the homogeneous part of that degree is uniquely determined by J. Thus, the ${}^{abc}N^{kmnp}$, ${}^aN^{km}$ and N are determined by J and so they must be independent of A_k. A little more algebra shows that also B and ${}^{ab}\Lambda^{kmn}$ are uniquely determined.

The fact that (1.1) is dynamic imposes the condition [2, (3.7)], which now comes down to $\partial_kJ^k = 0$. Since the coefficients on the right side of an equation like (4.4) are uniquely determined by the left side, we conclude that certain sums are 0, for example, $\partial k^{abc}N^{kmnp} = 0$. This particular equation shows that ${}^{abc}\psi$ is closed, which is to say $d^{abc}\psi = 0$.

We also have $\partial_kB^k = 0$. We already know that B^i is an infinitessimal conformal vector field and so looks like A^i in [2, middle of p. 536], and if the divergence is to be 0, it must be an infinitessimal *isometry*.

6. Topological forms. We will show first that

$$\psi^k = (A_j\partial B^j + A_{ij}B^j)F^{ki} \tag{6.1}$$

defines a topological form, meaning a form closed upon restriction to any section of the bundle B over M. A necessary and sufficient condition that a current with components J^k be topological is that

$$\partial_kJ^k + A_{jk}\partial J^k/\partial A_j = 0 \text{ and } \partial j^i/\partial A_{kj} + \partial J^i/\partial A_{ki} = 0. \tag{6.2}$$

This is seen as follows. Using the notation of [2,530; 2, (2.2)] J is topological, if and only if [2, (3.2)] holds, whenever $A_{ijk} = A_{ikj}$. This is equivalent to (6.2). Using the fact that $B^j = m^{ji}t_i + b^j$, where m^{ji} is skew-symmetric, one can deduce that (6.1) satisfies (6.2). It follows also that

$$T^k = N^k + {}^aN^{km}F_{am} + {}^{ab}N^{kmn}F_{am}F_{pn} + {}^{abc}N^{kmnp}F_{mn}F_{bn}F_{cp} - (A_j\partial B^j + A_{ij}B^j)F^{ki}$$

is topological, since

$$N^k + {}^aN^{km}F_{am} + {}^{ab}N^{kmn}F_{am}F_{pn} + {}^{abc}N^{kmnp}F_{am}F_{bn}F_{cp}$$

satisfies (6.2).

The differential form

$$\Phi^k = (1/4)B^kF_{mw}F^{mn} + \Psi^k$$

is the one corresponding to the infinitessimal space-time isometry with components B^k. (See for example [2, (2.41)].) Obviously, $J^k = \Phi^k + T^k$, so J^k has the decomposition claimed in Section 1.

7. Correction of misprints in [2]. (a) Line 12 of [2,537] should read $\partial_k(\partial^k f^h - \partial^h f^k) = 0$; (b) in line 13 of [2,533] replace $p^{km})S^n$ by $p^{kn})S^m$; (c) in line 15 of [2,541] remove the period after 'provides'; (d) in line 8 of [2,530] replace x by X.

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