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SOME COVERING PROPERTIES OF LOCALLY UNIVALENT FUNCTIONS

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1. Introduction. In this note we study some aspects of the covering properties of functions f that are analytic and locally univalent in $U=\{|z|<1\}$, and at most p-valent in U but not univalent in U.

For each $t\in]0,1[$ greater than the radius of univalence of such a f there must exist two points z_t and z_t' on $\{|z|=t\}$, with $f(z_t)=f(z_t')(=w_t$, say) such that:

1) f is univalent on the anticlockwise-described arc $C(t)$ of $\{|z|=t\}$ between z_t and z_t',

2) z_t and z_t' are the initial and terminal points respectively of the directed arc $C(t)$, and

3) $f(C(t))$ is described clockwise relative to its inside.

Then, $\Gamma(t)=f(C(t))$ is a closed Jordan curve, analytic except at w_t. The preimage f^{-1} (Int $\Gamma(t)$) consists of a countable number of disjoint domains in U; let $D(t)$ denote that component which has $C(t)$ as part of its boundary. We call $D(t)$ the *adhering domain* to the *generating arc* $C(t)$. It was shown in [1] that $D(t)$ is simply-connected and goes to the boundary of U.

The question arises from [1(c), p. 97] as to whether

$$D(t)\subset\{|z|>t\} \tag{1}$$

for all t larger than the radius of univalence of f.
Here we show that (1) is not true in general, and we ask some further questions about the domains $D(t)$.

In addition we give an example that shows that the conformality condition in the following result cannot be removed:

Theorem A (Theorem 2 of [1]). Let $w=f(z)=z+a_2z^2+\ldots$ be analytic, locally univalent but not univalent in U, and strictly p-valent in U. Then, there exists some point w_0 in $\mathbb{C}_w$ such that $f(z)-w_0$ has at most (p-2) zeros in U.

2. Example 1. We now construct a Riemann surface $\mathcal{R}$ that shows that (1) cannot hold for all sufficiently large t. $\mathcal{R}$ will be a modification of another Riemann surface $\mathcal{R}_{\varepsilon_1}$ that we construct first, using the following domains in the w-plane:

$G_1=\{\mathrm{Re}\, w>-1\}$;
$G_2=\{\mathrm{Re}\, w<-1,\ \mathrm{Im}\, w<-1\}$;
$G_3=\{\mathrm{Re}\, w<-2,\ -1<\mathrm{Im}\, w<1\}$;
$G_4=\{\mathrm{Re}\, w<-1,\ \mathrm{Im}\, w>1\}$;
$G_5=G_1$; and

$G_6=$ the triangle in $\mathbb{C}_w$ with vertices $-1-\frac{7}{8}i, -1-\frac{5}{8}i$ and $-\frac{5}{4}-\frac{3}{4}i$.

Then, for each $\varepsilon \in [0,1]$, $\mathscr{R}_\varepsilon$ is the two-sheeted Riemann surface obtained by sewing G_k to G_{k+1}, $1 \leq k \leq 5$, along their common boundary, and slitting G_1 along the line segment $L_\varepsilon = [-1-i,\ \varepsilon-1-i]$.

Let the function

$$f_\varepsilon(z) = \sum_{n=1}^{\infty} a_n(\varepsilon) z^n$$

map U onto $\mathscr{R}_\varepsilon$ with $f_\varepsilon(0)$ lying on the portion G_1 of $\mathscr{R}_\varepsilon$.

For all sufficiently large $t \in]0,1[$, the level curve $f(\{|z|=t\})$ closely approximates $\partial\mathscr{R}_\varepsilon$, at least near to

$$S_\varepsilon = S \cup L_\varepsilon,$$

where S is the square

$$S = \partial(\mathrm{C}_w - f_\varepsilon(U)).$$

Let $\exp(i\theta_1)$ and $\exp(i\theta_2)$ denote the points of ∂U that are the preimages under f_0 of the point $w = -1-i$, arranged such that

$$f_0(e^{i\theta_1}) \in \partial G_2 \text{ and } f_0(e^{i\theta_2}) \in \partial G_5.$$

For $\varepsilon > 0$ and t sufficiently close to 1, the distance between the level curve $f_\varepsilon(\{|z|=t\})$ and S_ε is of the order of magnitude of $(1-t)$, except that where a corner of S_ε is also a corner of $\partial\mathscr{R}_\varepsilon$, the level curve is pulled in towards the corner. This is because near such a corner, $w' = f_\varepsilon(e^{i\theta'})$ say, we have

$$f(z) - w' \simeq (z - e^{i\theta'})^{3/2} A(w')$$

for some $A(w')$ independent of z.

Clearly, it is then possible to choose a particular pair (t_1, ε_1) with $t_1 \in]0,1[$ sufficiently large and $\varepsilon_1 > 0$ sufficiently small, such that there exist two points z_{t_1} and z'_{t_1} on $\{|z| = t_1\}$ near to $\exp(i\theta_1)$ and $\exp(i\theta_2)$ respectively, with the following properties:

(a) $C(t_1) = (z_{t_1}, z'_{t_1})$ is a generating arc on $\{|z| = t_1\}$, and

(b) $f_{\varepsilon_1}^{-1}(-1)$ lies in the adhering domain $D(t_1)$ generated by the arc $C(t_1)$.

This follows from the Carathéodory Kernel Theorem and the fact that -1 belongs to $\mathscr{R}_\varepsilon$ for each $\varepsilon \geq 0$.

We have to choose ε_1 sufficiently small and t_1 sufficiently large for (b) to hold, and ε_1 and t_1 sufficiently large so that the level curve has a double point near $w = -1-i$; this can be done by choosing first t_1 and then ε_1. In (a), z_{t_1} is chosen on $\{|z| = t_1\}$ such that $f(z_{t_1})$ is the 'last' double point on $f(\{|z| = t_1\})$ before the level curve sweeps round S to intersect the line segment $]-\infty, -2[$.

Then the point $w = -1$ must lie on $\mathscr{R}_{\varepsilon_1}$, inside the image under f_{ε_1} of the level curve $\{|z| = t_1\}$, so that $f_{\varepsilon_1}^{-1}(-1)$ lies inside $\{|z| < t_1\}$. It follows that

$$D_{t_1} \cap \{|z| > t_1\} \neq \emptyset.$$

Finally, the desired Riemann surface $\mathscr{R}$ is obtained from $\mathscr{R}_{\varepsilon_1}$ by slitting $\mathscr{R}_{\varepsilon_1}$ in G_1 along very small line segments $[-1-2_i^{-n},\ \varepsilon_{n+1}-1-2_i^{-n}]$, $n = 1, 2, \ldots$, where $\varepsilon_n \downarrow \to 0$, and by attaching small triangles inside S to G_1 midway between these slits. Similar arguments to those earlier applied inductively to the effect of each successive addition show that there exists a sequence $t_n \uparrow \to 1$ and a sequence of adhering domains $D(t_n)$ such that

$$D(t_n) \cap \{|z| < t_n\} \neq \varnothing .$$

3. Remark. It would be interesting to know if there exists a function f analytic in U and locally univalent in U, such that for some nested family of adhering domains, $D(t)$, we can have

$$D(t) \cap \{|z| < t\} \neq \varnothing$$

for all t sufficiently close to 1, or even perhaps for all t larger than the radius of univalence of f.

Also, the question arises as to whether, if f is assumed to be strictly p-valent in U with $f'(0)=1$, the number

$$T = \inf_f \{t : D(t) \cap \{|z| < t\} \neq \varnothing\}$$

is equal to R_u, the radius of univalence of the family of all such f, or whether $T > R_u$.

4. Example 2. We now construct a function f with the following properties: f is analytic and strictly p-valent in U, and the Riemann surface $\mathscr{R} = f(U)$ covers every point in the image plane al least $(p-1)$ times. This shows that the conformality condition in Theorem A cannot be removed.

Let $\mathscr{R}_1$ denote the image Riemann surface associated with the function

$$w = f_2(z) - 3 + i, \quad z \in U,$$

where f_2 is the function defined in Example 2 of [1, p. 99] with the choice

$$w_i = 1 + (i-1)/(p-2), \quad 1 \leq i \leq p-1.$$

Let $\mathscr{R}_2$ denote the Riemann surface associated with the function

$$w = z^2, \quad z \in U.$$

Now delete from $\mathscr{R}_1$ the copy of $\{|w| \leq 1\}$, whose interior lies in a single sheet of $\mathscr{R}_1$ and whose boundary meets $\partial\mathscr{R}_1$, and sew in its place a copy of $\mathscr{R}_2$ along $T = \{|w| = 1,\ w \neq i\}$; do this in such a way that adjacent points of $\partial\mathscr{R}_1$ on T are sewn to adjacent points (on the same sheet) of $\mathscr{R}_2$. Denote by $\mathscr{R}_3$ the resulting Riemann surface.

Next, to $\mathscr{R}_3$ sew a copy of

$$\mathscr{R}_4 = \{\operatorname{Re} w > -3, 0 < \operatorname{Im} w < 2\} - [\{|w| \leq 1\} \cup \{\operatorname{Im} w \leq 1, \operatorname{Re} w \geq 0\}]$$

along the connected copy of

$$\{w = e^{i\theta} : \tfrac{1}{2}\pi \leq \theta \leq \pi\} \cup \{\operatorname{Im} w = 1, \operatorname{Re} w \geq 0\}$$

on $\mathscr{R}_3$. Denote by $\mathscr{R}$ the resulting Riemann surface.

Then $\mathscr{R}$ has the desired properties. (Note too that f' has just one zero in U.)

Related questions will be discussed in [2].

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