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A DECOMPOSITION OF INTEGER VECTORS. II

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In this paper we shall consider integer vectors $\mathbf{n}=[n_1, n_2, \ldots, n_k]$ and write for such vectors: $h(\mathbf{n})=\max|n_i|$, $l(\mathbf{n})=\sqrt{n_1^2+n_2^2+\cdots+n_k^2}$. One of us has recently proved [3] that for every non-zero vector $\mathbf{n}\in\mathbf{Z}^k$ $(k>1)$ there is a decomposition: $\mathbf{n}=u\mathbf{p}+v\mathbf{q}$, $u, v\in\mathbf{Z}$, where $\mathbf{p}, \mathbf{q}\in\mathbf{Z}^k$ are linearly independent and

$$h(\mathbf{p})\,h(\mathbf{q})\leq 2h(\mathbf{n})^{(k-2)/(k-1)}.$$

The exponent $(k-2)/(k-1)$ cannot be improved (see [2], Remark after Lemma 1). It is natural to ask for the best value of the coefficient. We chall answer this question for $k=3$ by proving the following two theorems.

Theorem 1. For every non-zero vector $\mathbf{n}\in\mathbf{Z}^3$ there exist linearly independent vectors $\mathbf{p}, \mathbf{q}\in\mathbf{Z}^3$, such that $\mathbf{n}=u\mathbf{p}+v\mathbf{q}$, $u, v\in\mathbf{Z}$ and

$$h(\mathbf{p})\,h(\mathbf{q})<\sqrt{\frac{4}{3}h(\mathbf{n})}.$$

Theorem 2. For every $\varepsilon>0$ there exists a non-zero vector $\mathbf{n}\in\mathbf{Z}^3$, such that for all non-zero vectors $\mathbf{p}, \mathbf{q}\in\mathbf{Z}^3$ and all $u, v\in\mathbf{Q}$ $\mathbf{n}=u\mathbf{p}+v\mathbf{q}$ implies

$$h(\mathbf{p})\,h(\mathbf{q})>\sqrt{\left(\frac{4}{3}-\varepsilon\right)h(\mathbf{n})}.$$

Originally, in the proof of Theorem 1 some computer calculations were used which were kindly performed by Dr. T. Regińska. We thank her for the help.

The proof of Theorem 1 will be based on geometry of numbers. The inner product of two vectors $\mathbf{n}, \mathbf{m}$ will be denoted by $\mathbf{nm}$, their exterior product by $\mathbf{n}\times\mathbf{m}$, the area of a plane domain $\mathbf{D}$ by $A(\mathbf{D})$.

Lemma 1. Let a_i, b_i be real numbers $(i=1,2,3)$ and M_1, M_2, M_3 the three minors of order two of the matrix $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$ not all equal to 0. The area of the domain $\mathbf{H}$: $|a_ix+b_iy|\leq 1$ $(i=1, 2, 3)$ equals

$$\frac{2|M_1M_2|+2|M_1M_3|+2|M_2M_3|-M_1^2-M_2^2-M_3^2}{M_1M_2M_3},$$

if each of the numbers $|M_1|$, $|M_2|$, $|M_3|$ is less that the sum of the two others, and $4/\max\{|M_1|, |M_2|, |M_3|\}$ otherwise.

Proof. We may assume without loss of generality that

$$|M_1|=\operatorname{abs}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}>0, \quad |M_1|\geq|M_2|=\operatorname{abs}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix},$$

$$|M_1|\geq|M_3|=\operatorname{abs}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}.$$

The affine transformation $a_1x+b_1y=X$, $a_2x+b_2y=Y$ transforms the domain **H** into the domain

$$\mathbf{H}': \quad |X|\le 1,\ |Y|\le 1;\quad \left|\frac{M_2}{M_1}X-\frac{M_3}{M_1}Y\right|\le 1.$$

If $|M_2|+|M_3|>|M_1|$, the domain **H′** is obtained from the square $|X|\le 1$, $|Y|\le 1$ by subtracting two rectangular triangles, symmetric to each other with respect to (0, 0), with the vertices

$$\pm\left(1,\ -\operatorname{sgn}\frac{M_2}{M_3}\,\frac{|M_1|-|M_2|}{|M_3|}\right),\quad \pm\left(1,\ -\operatorname{sgn}\frac{M_2}{M_3}\right),$$

$$\pm\left(\frac{|M_1|-|M_3|}{|M_2|},\ -\operatorname{sgn}\frac{M_2}{M_3}\right).$$

Hence,

$$A(\mathbf{H}')=4-\frac{(|M_2|+|M_3|-|M_1|)^2}{|M_2||M_3|}.$$

If $|M_2|+|M_3|\le|M_1|$, then **H′** coincides with the square $|X|\le 1$, $|Y|\le 1$ and $A(\mathbf{H}')=4$. Since $A(\mathbf{H})=A(\mathbf{H}')/|M_1|$, the lemma follows.

L e m m a 2. If $0\le a\le b<1$, then the domain

$$\mathbf{D}: \quad |x|\le 1,\quad |y|\le 1,\quad |ax+by|\le 1,\quad x^2+y^2+(ax+by)^2\le\frac{3}{2}$$

contains an ellipse **E** with

$$A(\mathbf{E})>\pi\sqrt{\frac{3}{4}}. \tag{1}$$

P r o o f. We take

$$\mathbf{E}: \quad f(x,y)=x^2+c\left(\frac{ab}{b^2+1}x+y\right)^2\le 1,$$

where

$$c=\max\left\{\frac{2}{3}(b^2+1),\ \frac{(b^2+1)^2}{(b^2+1)^2-a^2b^2}\right\}. \tag{2}$$

In order to see that $|x|\le 1$, $|y|\le 1$ for $(x,y)\in\mathbf{E}$, we notice that by (2)

$$\min_y f(x,y)=x^2,\quad \min_x f(x,y)=\frac{c}{c\frac{a^2b^2}{b^2+1}+1}y^2\ge y^2. \tag{3}$$

Moreover, for $(x,y)\in\mathbf{E}$ we have by (2)

$$x^2+y^2+(ax+by)^2\le\frac{3}{2}\Big(\frac{2}{3}\,\frac{a^2+b^2+1}{b^2+1}x^2 \tag{4}$$

$$+\frac{2}{3}(b^2+1)\Big(\frac{ab}{b^2+1}x+y\Big)^2\Big)\le\frac{3}{2}f(x,y)\le\frac{3}{2}.$$

If for $(x,y)\in\mathbf{E}$ we had $|ax+by|>1$, it would follow

$$x^2+y^2<\frac{1}{2}, \tag{5}$$

hence, by Cauchy-Schwarz inequality

$$(ax+by)^2\le(a^2+b^2)(x^2+y^2)<2\cdot\frac{1}{2}=1, \tag{6}$$

a contradiction. Thus, for $(x, y) \in \mathbf{E}$ we have

(7) $$|ax+by| \leqq 1.$$

Finally, $A(\mathbf{E}) = \pi/\sqrt{c}$ and since by (2) $c<4/3$, (1) follows.

Lemma 3. Let $\mathbf{n} \in \mathbf{Z}^3 \setminus \{[0, 0, 0]\}$. The lattice of integer vectors. $\mathbf{m} \in \mathbf{Z}^3$ such that $\mathbf{nm}=0$ has a basis $\mathbf{a}=[a_1, a_2, a_3]$, $\mathbf{b}=[b_1, b_2, b_3]$, such that

(8) $$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \frac{n_3}{(n_1, n_2, n_3)}, \quad \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = \frac{n_1}{(n_1, n_2, n_3)},$$
$$\begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} = \frac{n_2}{(n_1, n_2, n_3)}.$$

Proof. Since $\mathbf{na}=\mathbf{nb}=0$ and $\mathbf{a}$, $\mathbf{b}$ are linearly independent, we have
$$\mathbf{n}=c(\mathbf{a}\times\mathbf{b})$$
for a certain $c \in \mathbf{Q}$. However, the numbers $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$, $\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}$ and $\begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}$ are relatively prime (see e. g. [1, p. 53]); hence, the formulae (8) hold with $\pm$ sign on the right-hand side. Changing if necessary the order of $\mathbf{a}$, $\mathbf{b}$, we get the lemma.

Lemma 4. For every vector $\mathbf{n} \in \mathbf{Z}^3$ different from $[0,0, 0]$ and $[\pm 1, \pm 1, \pm 1]$ for any choice of signs, there exists a vector $\mathbf{m} \in \mathbf{Z}^3$ such that

(9) $$\mathbf{mn}=0,$$

(10) $$0<h(\mathbf{m})<\sqrt{\frac{4}{3}h(\mathbf{n})}$$

and

(11) $$l(\mathbf{m})<\sqrt{2h(\mathbf{n})}.$$

Proof. Without loss of generality we may assume that

(12) $$0 \leqq n_1 \leqq n_2 \leqq n_3 > 0.$$

If $n_2=n_3$ we take
$$\mathbf{m}=\begin{cases} [1, 0, 0] & \text{if } n_1=0, \\ [0, 1, -1] & \text{if } n_1 \neq 0, \end{cases}$$
and we find (9)-(11) satisfied, unless $n_1=n_2=n_3=1$. Therefore, we may assume besides (12) that $n_2<n_3$.

In virtue of Lemma 2 the domain
$$\mathbf{D}:\quad |X| \leqq 1,\ |Y| \leqq 1,\ \left|\frac{n_1}{n_3}X+\frac{n_2}{n_3}Y\right| \leqq 1,\ X^2+Y^2+\left(\frac{n_1}{n_3}X+\frac{n_2}{n_3}Y\right)^2 \leqq \frac{3}{2}$$
contains an ellipse $\mathbf{E}$ with $A(\mathbf{E}) > \pi\sqrt{3/4}$.

Let $\mathbf{a}$, $\mathbf{b}$ be a basis, the existence of which is asserted by Lemma 3. The substitution
$$X=\frac{a_1x+b_1y}{\sqrt{\frac{4}{3}n_3}}, \quad Y=\frac{a_2x+b_2y}{\sqrt{\frac{4}{3}n_3}}$$
transforms $\mathbf{D}$ into the domain
$$\mathbf{D}':\quad |a_ix+b_iy| \leqq \sqrt{\frac{4}{3}n_3} \quad (i=1, 2, 3), \quad \sum_{i=1}^{3}(a_ix+b_iy)^2 \leqq 2n_3.$$

Hence, $\mathbf{D}'$ contains an ellipse $\mathbf{E}'$ with

$$A(\mathbf{E}')=\frac{4}{3}n_3\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^{-1}A(\mathbf{E})>\pi\sqrt{\frac{4}{3}}(n_1, n_2, n_3)\geq\pi\sqrt{\frac{4}{3}},$$

by (8). Since the packing constant for ellipses is $\pi/\sqrt{12}$, it follows that $\mathbf{E}'$ and, hence, $\mathbf{D}'$ contains in its interior a point $(x_0, y_0)\in\mathbf{Z}^2$ different from (0, 0). Putting $\mathbf{m}=x_0\mathbf{a}+y_0\mathbf{b}$, we get the assertion of the Lemma.

Lemma 5. If $0\leq a\leq 1$, $0\leq b\leq 1$ and $a+b>1$, the area of the hexagon $|x|\leq 1$, $|y|\leq 1$, $|ax+by|\leq 1$ is greater than $[24/(a^2+b^2+1)]^{1/2}$.

Proof. In virtue of Lemma 1 the area in question equals

$$(2ab+2a+2b-a^2-b^2-1)/ab,$$

thus, it remains to prove that for (a, b) in the domain

$$\mathbf{G}:\ 0\leq a\leq 1,\ 0\leq b\leq 1,\ a+b>1$$

the following inequality holds

$$f(a, b)=(2ab+2a+2b-a^2-b^2-1)^2(a^2+b^2+1)-24a^2b^2>0.$$

We have $\partial\mathbf{G}=\mathbf{L}_1\cup\mathbf{L}_2\cup\mathbf{L}_3$, where

$$\mathbf{L}_1=\{(a, 1): 0\leq a\leq 1\},\ \mathbf{L}_2=\{(1, b): 0\leq b\leq 1\},\ \mathbf{L}_3=\{(a, 1-a): 0\leq a\leq 1\}.$$

We find $f(a, 1)=a^2(a-1)^3(a-5)+3a^2$, but for $a\leq 1$ $a^2(a-1)^3(a-5)\geq 0$, hence $f(a, 1)\geq 3a^2\geq 0$. In view of symmetry between a and b, $f(1, b)\geq 3b^2\geq 0$. Moreover, $f(a, 1-a)=8a^2(1-a)^2(2a-1)^2\geq 0$. Hence, for $(a, b)\in\partial\mathbf{G}$ we have $f(a, b)\geq 0$ with the equality attained only if $(a, b)\notin\mathbf{G}$ It suffices to show that in the interior of $\mathbf{G}$ the function $f(a, b)$ has no local extremum.

Indeed, putting $g(a, b)=2ab+2a-a^2-b^2-1$, we find

$$\frac{\partial f}{\partial a}=2ag^2+2(2b+2-2a)(a^2+b^2+1)g-48ab^2,$$

$$\frac{\partial f}{\partial b}=2bg^2+2(2a+2-2b)(a^2+b^2+1)g-48a^2b,$$

hence,

$$a\frac{\partial f}{\partial a}-b\frac{\partial f}{\partial b}=2(a-b)[(a+b)g+(a^2+b^2+1)(2-2a-2b)],$$

$$b\frac{\partial f}{\partial a}-a\frac{\partial f}{\partial b}=4(b-a)[(a+b+1)(a^2+b^2+1)g-12ab(a+b)].$$

The equations $\partial f/\partial a=\partial f/\partial b=0$ imply $a=b$ or

(13) $$(a+b)g+(a^2+b^2+1)(2-2a-2b)=0,$$
$$(a+b+1)(a^2+b^2+1)g-12ab(a+b)=0.$$

Eliminating g from the above equations we obtain

(14) $$2(a^2+b^2+1)[(a+b)^2-1]-12ab(a+b)^2=0.$$

The left-hand sides of the equations (13) and (14) are symmetric functions of a, b. Expressing them in terms of $s=a+b$ and $p=ab$, then eliminating p, we get

$$s(s-1)(2s-1)(4s^2-s+1)=0.$$

For $s=x+y>1$ this is clearly impossible, there remains the possibility $a=b$. However, in that case

$$\frac{\partial f}{\partial a}=16a^3-24a^2+18a-4=2(2a-1)^3+3(2a-1)+1>1.$$

Lemma 6. For every nonzero vector $\mathbf{n}=[n_1, n_2, n_3]\in \mathbf{Z}^3$ there exist lineary independent vectors $\mathbf{p}, \mathbf{q}\in \mathbf{Z}^3$ such that $\mathbf{pn}=\mathbf{qn}=0$, and

$h(\mathbf{p})\,h(\mathbf{q})<\sqrt{\frac{2}{3}}\,l(\mathbf{n})$, if each of the numbers $|n_1|, |n_2|, |n_3|$ is less than the sum of the two others;

$$h(\mathbf{p})\,h(\mathbf{q})\le h(\mathbf{n}),\ \text{otherwise}.$$

Proof. We may assume without loss of generality that $0\le n_1\le n_2\le n_3>0$. In virtue of Lemmata 1 and 5 the area $A(\mathbf{K})$ of the domain

$$\mathbf{K}:\ |X|\le 1,\ |Y|\le 1,\ \left|\frac{n_1}{n_3}X-\frac{n_2}{n_3}Y\right|\le 1$$

satisfies

$$\text{(15)}\qquad \begin{cases} A(\mathbf{K})>\sqrt{\dfrac{24}{n_1^2+n_2^2+n_3^2}}\,n_3, & \text{if}\quad n_1+n_2>n_3,\\ A(\mathbf{K})=4, & \text{otherwise.}\end{cases}$$

Let a, b be a basis, the existence of which is asserted in Lemma 3. The affine transformation $X=a_1x+b_1y$, $Y=a_2x+b_2y$ transforms the domain $\mathbf{K}$ into the domain

$$\mathbf{K}':\quad |a_ix+b_iy|\le 1\quad (i=1, 2, 3)$$

satisfying

$$\text{(16)}\qquad A(\mathbf{K}')=A(\mathbf{K})\frac{(n_1, n_2, n_3)}{n_3}.$$

In virtue of Minkowski's second theorem there exist two linearly independent integer vectors $[x_1, y_1]$ and $[x_2, y_2]$ such that

$$\text{(17)}\qquad |a_ix_j+b_iy_j|\le\lambda_j\quad (i=1, 2, 3;\ j=1, 2)$$

and

$$\text{(18)}\qquad \lambda_1\lambda_2A(\mathbf{K}')\le 4.$$

Putting $\mathbf{p}=\mathbf{a}x_1+\mathbf{b}y_1$, $\mathbf{q}=\mathbf{a}x_2+\mathbf{b}y_2$, we infer that $\mathbf{p}, \mathbf{q}$ are linearly independent, satisfy $\mathbf{pn}=\mathbf{qn}=0$ and in virtue of (15), (18)

$$h(\mathbf{p})\,h(\mathbf{q})\le\lambda_1\lambda_2\begin{cases}<\sqrt{\dfrac{2}{3}}\,l(\mathbf{n}), & \text{if}\quad n_1+n_2>n_3,\\ \le n_3, & \text{otherwise.}\end{cases}$$

Proof of Theorem 1. If $\mathbf{n}=[\varepsilon_1, \varepsilon_2, \varepsilon_3]$, where $\varepsilon_i\in\{1, -1\}$, it suffices to take $\mathbf{p}=[\varepsilon_1, \varepsilon_2, 0]$, $\mathbf{q}=[0, 0, \varepsilon_3]$. If $\mathbf{n}\neq[\varepsilon_1, \varepsilon_2, \varepsilon_3]$ for every choice of $\varepsilon_1, \varepsilon_2, \varepsilon_3$, then by Lemma 4 there exists a vector $\mathbf{m}\in \mathbf{Z}^3$ satisfying the conditions

$$\text{(19)}\qquad \mathbf{mn}=0,$$

$$\text{(20)}\qquad 0<h(\mathbf{m})<\sqrt{\frac{4}{3}}\,h(\mathbf{n}),\quad 0<l(\mathbf{m})<\sqrt{2h(\mathbf{n})}.$$

Now, by Lemma 6 applied with $\mathbf{n}$ replaced by $\mathbf{m}$ there exist vectors $\mathbf{p}, \mathbf{q}\in \mathbf{Z}^3$ such that

$$\text{(21)}\qquad \mathbf{pm}=\mathbf{qm}=0,\quad \dim(\mathbf{p}, \mathbf{q})=2$$

and

(22) $$h(\mathbf{p})\,h(\mathbf{q})<\max\{\sqrt{\tfrac{2}{3}}\,l(\mathbf{m}),\ h(\mathbf{m})\}.$$

The equations (20) and (22) imply that $\mathbf{n}=u\mathbf{p}+v\mathbf{q}$; $u,\ v\in\mathbf{Q}$, while the inequalities (20) and (22) imply that $h(\mathbf{p})\,h(\mathbf{q})<[(4/3)\,h(\mathbf{n})]^{1/2}$.

It follows that the number $c_0(3)$ defined in [5] by the formula

$$c_0(k)=\sup_{\substack{\mathbf{n}\in Z^k\\ \mathbf{n}\neq 0}}\ \inf_{\substack{\mathbf{p},\,\mathbf{q}\in Z^k\\ \dim(\mathbf{p},\,\mathbf{q})=2\\ n=u\mathbf{p}+v\mathbf{q},\ u,\ v\in\mathbf{Q}}} h(\mathbf{p})\,h(\mathbf{q})\,h(\mathbf{n})^{\frac{k-2}{k-1}}$$

satisfies $c_0(3)\leq\sqrt{4/3}$ and if $c_0(3)=\sqrt{4/3}$, the supremum occurring in the definition of $c_0(k)$ is not attained. By Theorem 2 of [5] there exist vectors $\mathbf{p}_0$, $\mathbf{q}_0\in\mathbf{Z}^3$ linearly independent and such that $\mathbf{n}=u_0\mathbf{p}_0+v_0\mathbf{q}_0$, $u_0,\ v_0\in\mathbf{Z}$, and $h(\mathbf{p}_0)h(\mathbf{q}_0)<[(4/3)h(\mathbf{n})]^{1/2}$. The proof of Theorem 1 is complete.

The proof of Theorem 2 is again based on several lemmata. We shall set for $t=1,\ 2,\ 3,\ \ldots$

$$\mathbf{n}_t=[(2t^2+2t)(6t^2+4t-1),\ (2t^2+2t)(6t^2+6t-1),$$
$$(4t^2+4t)^2-(2t^2-1)(2t^2+2t-1)],$$

and for vectors $\mathbf{m}$, $\mathbf{p}$,... we shall denote the ν-th coordinate by m_ν, p_ν respectively.

Lemma 7. If $\mathbf{n}_t\mathbf{m}=0$, $\mathbf{m}\in\mathbf{Z}^3$, $0<h(\mathbf{m})\leq 8t^2+8t-2$, then we have $\mathbf{m}=\mathbf{m}_i$ for an $i\leq 6$, where

$$\mathbf{m}_1=[6t^2+6t-1,\ -(6t^2+4t-1),\ 0],\quad \mathbf{m}_2=[2t^2+2t-1,\ -(4t^2+4t),\ 2t^2+2t],$$
$$\mathbf{m}_3=[4t^2+4t,\ -(2t^2-1),\ -(2t^2+2t)],\quad \mathbf{m}_4=[2t^2+2t+1,\ 2t^2+4t+1,\ -(4t^2+4t)],$$
$$\mathbf{m}_5=[2,6t^2+8t+1,\ -(6t^2+6t)]\ (t\neq1),\quad \mathbf{m}_6=[6t^2+6t+1,\ 4t+2,\ -(6t^2+6t)].$$

Proof. The vectors $\mathbf{m}_i$ $(1\leq i\leq 6)$ all satisfy the equation $nm_i=0$. Since the vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ are linearly independent, every vector $\mathbf{m}\in\mathbf{Z}^3$ satisfying $\mathbf{nm}=0$ is of the form $u\mathbf{m}_1+v\mathbf{m}_2$, $u,\ v\in\mathbf{Q}$.

Let $u=a/c$, $v=b/c$, $a,b,c\in\mathbf{Z}$, $(a,b,c)=1$, $c>0$. It follows from $c\mid am_{1i}+bm_{2i}$, $c\mid am_{1j}+bm_{2j}$ that $c\mid(a,\ b)(m_{1i}m_{2j}-m_{2i}m_{1j})$, hence, $c\mid m_{1i}m_{2j}-m_{2i}m_{1j}$ $(1\leq i<j\leq 3)$.

But $(m_{11}m_{23}-m_{21}m_{13},\ m_{12}m_{23}-m_{22}m_{13})=m_{23}(m_{11},\ m_{12})=m_{23}$ and $(m_{23},\ m_{11},\ m_{22}-m_{21},\ m_{12})=(m_{23},\ m_{21},\ m_{12})=1$, hence, $c=1$ and we get $\mathbf{m}=a\mathbf{m}_1+b\mathbf{m}_2$. Considering the third coordinate, we find $|b|(2t^2+2t)\leq 8t^2+8t-2$, hence, $|b|\leq 3$.

Considering the first coordinate, we get

$$|a(6t^2+6t-1)+b(2t^2+2t-1)|\leq 8t^2+8t-2;$$
$$|a|(6t^2+6t-1)\leq 8t^2+8t-2+|b|(2t^2+2t-1)\leq 14t^2+14t-15,$$

hence, $|a|\leq 1$ or $a=\pm2$, $b=3$. For $a=0$ we get $\mathbf{m}=b[2t^2+2t-1,\ -(4t^2+4t),\ 2t^2+2t]=\pm\mathbf{m}_2$. For $|a|=1$ the inequality for the second coordinate

$$|a(6t^2+4t-1)+b(4t^2+4t)|\leq 8t^2+8t-2$$

gives $b=0$ or $ab<0$. For $a=\pm1$, $b=0$ we get $\mathbf{m}=\pm\mathbf{m}_1$; for $a=\pm1$, $b=\mp1$ we get $\mathbf{m}=\pm\mathbf{m}_3$; for $a=\pm1$, $b=\mp2$ we get $\mathbf{m}=\pm\mathbf{m}_4$; for $a=\pm1$, $b=\mp3$ we get $\mathbf{m}=\pm\mathbf{m}_5$; for $a=\pm2$, $b=\mp3$ we get $\mathbf{m}=\pm\mathbf{m}_6$.

Lemma 8. If $\mathbf{p}$, $\mathbf{q}\in\mathbf{Z}^3$ are linearly independent and $\mathbf{pm}_1=\mathbf{qm}_1=0$, then

$$h(\mathbf{p})h(\mathbf{q})>4t^2+4t.$$

Proof. $\mathbf{pm}_1=0$ implies $p_1\equiv 0 \bmod 6t^2+4t-1, p_2\equiv 0 \bmod 6t^2+6t-1$. Hence, $p_1=p_2=0$ or $|p_2|\geq 6t^2+6t-1$. Similarly, $q_1=q_2=0$ or $|q_2|\geq 6t^2+6t-1$. Since $\mathbf{p}$, $\mathbf{q}$ are linearly independent, $h(\mathbf{p})h(\mathbf{q})\geq 6t^2+6t-1>4t^2+4t$.

Lemma 9. If $\mathbf{p}, \mathbf{q}\in \mathbf{Z}^3$ are linearly independent and

$$\mathbf{pm}_2=\mathbf{qm}_2=0,$$

then

$$h(\mathbf{p})h(\mathbf{q})\geq 4t^2+4t.$$

Proof. The equation

$$\mathbf{pm}_2=(2t^2+2t-1)p_1-(4t^2+4t)p_2+(2t^2+2t)p_3=0$$

gives $p_1\equiv 0 \bmod 2t^2+2t-1$, hence, $p_1=0$ or $|p_1|\geq 2t^2+2t$. The former possibility gives $|p_3|\geq 2$. Similarly, $q_1=0$, $|q_3|\geq 2$ or $|q_1|\geq 2t^2+2t$. Since $\mathbf{p}$, $\mathbf{q}$ are linearly independent, $p_1=q_1=0$ is excluded, hence,

$$h(\mathbf{p})h(\mathbf{q})\geq \min\{2(2t^2+2t), (2t^2+2t)^2\}\geq 4t^2+4t.$$

Lemma 10. If $\mathbf{p}, \mathbf{q}\in \mathbf{Z}^3$ are linearly independent and $\mathbf{pm}_3=\mathbf{qm}_3=0$, then $h(\mathbf{p})h(\mathbf{q})\geq 4t^2+4t$.

Proof. The equation

$$\mathbf{pm}_3=(4t^2+4t)p_1-(2t^2-1)p_2-(2t^2+2t)p_3=0$$

gives $p_2\equiv 0 \bmod 2t^2+2t$, hence $p_2=0$ or $|p_2|\geq 2t^2+2t$. The further proof is similar to that of Lemma 9.

Lemma 11. If $\mathbf{p}\in \mathbf{Z}^3$, $\mathbf{pm}_4=0$, then either $\mathbf{p}=0$ or $h(\mathbf{p})\geq 2t+1$.

Proof. The equation

$$\mathbf{pm}_4=(2t^2+2t+1)p_1+(2t^2+4t+1)p_2-(4t^2+4t)p_3=0$$

gives

$$(2t^2+2t)(p_1+p_2-2p_3)+p_1+(2t+1)p_2=0. \tag{24}$$

If $p_1+p_2-2p_3=0$, then $p_1+(2t+1)p_2=0$ and either $p_1=0$ or $|p_1|\geq 2t+1$. If $p_1+p_2-2p_3\neq 0$, then since by (24) $p_1\equiv p_2 \bmod 2$, we obtain

$$p_1+p_2-2p_3=2s,\ s\in \mathbf{Z}\setminus\{0\},\quad p_1+(2t+1)p_2=-(4t^2+4t)s.$$

Hence, $p_3+tp_2=-(2t^2+2t+1)s$ and

$$\max\{|p_2|, |p_3|\}\geq \frac{2t^2+2t+1}{t+1}>2t,$$

thus $h(\mathbf{p})\geq 2t+1$.

Lemma 12. If $\mathbf{p}, \mathbf{q}\in \mathbf{Z}^3$ are linearly independent and $\mathbf{pm}_5=\mathbf{qm}_5=0$, then $h(\mathbf{p})h(\mathbf{q})>4t^2+4t$ $(t\neq 1)$.

Proof. The equation

$$\mathbf{pm}_5=2p_1+(6t^2+8t+1)p_2-(6t^2+6t)p_3=0$$

gives

$$2p_1+(2t+1)p_2+(6t^2+6t)(p_2-p_3)=0.$$

If $p_2=p_3$, we get $p_1\equiv 0 \bmod 2t+1$, hence, $|p_1|\geq 2t+1$. If $p_2\neq p_3$, we get $(2t+3)\max\{|p_1|, |p_2|\}\geq 6t^2+6t$, hence,

$$\max\{|p_1|, |p_2|\}\geq \frac{6t^2+6t}{2t+3}>3t-2$$

and $h(\mathbf{p})\geq 3t-1$. Similarly, $q_2=q_3$ and $|q_1|\geq 2t+1$ or $h(\mathbf{q})\geq 3t-1$. Since $\mathbf{p}$, $\mathbf{q}$ are linearly independent, $p_2=p_3$, $q_2=q_3$ is excluded and we get for $t\neq 1$

$$h(\mathbf{p})\,h(\mathbf{q})\geq\min\{(2t+1)(3t-1),\ (3t-1)^2\}\geq(2t+1)(3t-1).$$

Lemma 13. If $\mathbf{p}, \mathbf{q}\in \mathbf{Z}^3$ are linearly independent and $\mathbf{p}\mathbf{m}_6=\mathbf{q}\mathbf{m}_6=0$, then

$$h(\mathbf{p})\,h(\mathbf{q})\geq 4t^2+4t.$$

Proof. The equation

$$\mathbf{p}\mathbf{m}_6=(6t^2+6t+1)p_1+(4t+2)p_2-(6t^2+6t)p_3=0$$

gives

$$(6t^2+6t)(p_1-p_3)+p_1+(4t+2)p_2=0.$$

If $p_1-p_3=0$, we get $p_1\equiv 0 \bmod 4t+2$, hence, $|p_1|\geq 4t+2$.
If $|p_1-p_3|\geq 2$, we get

$$(4t+3)\max\{|p_1|,\ |p_2|\}\geq 2(6t^2+6t),$$

hence,

$$\max\{|p_1|,\ |p_2|\}\geq\frac{12t^2+12t}{4t+3}>3t$$

and $h(\mathbf{p})\geq 3t+1$. If $p_1-p_3=\pm 1$, we get $p_1+(4t+2)p_2=(6t^2+6t)$, hence either

$$|p_1|\geq 4t+2 \quad\text{or}\quad p_2=\Big[\mp\frac{(6t^2+6t)}{4t+2}\Big] \quad\text{or}\quad p_2=\Big[\mp\frac{(6t^2+6t)}{4t+2}\Big]+1.$$

The last two formulae give the following possible values for $\mp[p_1, p_2]$:

$$[3t,\ \frac{3t}{2}],\quad [t-1,\ \frac{3t+1}{2}],\ [-t-2,\ \frac{3t+2}{2}],\ [-3t-3,\ \frac{3t+3}{2}].$$

Hence, either $h(\mathbf{p})\geq 3t+2\{t/2\}$ or $p_1-p_3=\pm 1$ and $p_2=[(3t+2)/2]$. Similarly, either $h(\mathbf{q})\geq 3t+2\{t/2\}$ or $q_2-q_3=\pm 1$ and $q_2=[(3t+2)/2]$. Since $\mathbf{p}$, $\mathbf{q}$ are linearly independent it follows that

$$h(\mathbf{p})\,h(\mathbf{q})\geq(3t+2\{\tfrac{t}{2}\})\Big[\frac{3t+2}{2}\Big]\geq 4t^2+4t.$$

Proof of Theorem 2. Since

$$\lim_{t\to\infty}\frac{4t^2+4t}{\sqrt{(4t^2+4t)^2-(2t^2-1)(2t^2+2t-1)}}=\sqrt{\frac{4}{3}},$$

for every $\varepsilon>0$ there exist t, such that

$$4t^2+4t>\sqrt{(\tfrac{4}{3}-\varepsilon)\,h(\mathbf{n}_t)} \tag{2}$$

and we fix such a value of t.

If $\mathbf{n}_t=u\mathbf{p}+v\mathbf{q}$, $u, v\in\mathbf{Q}$ and $\mathbf{p}, \mathbf{q}\in\mathbf{Z}^3$ are linearly dependent, then since $(n_{t1}, n_{t2}, n_{t3})=1$, we have either $\mathbf{p}=0$ or $\mathbf{p}=sn_t$, $s\in\mathbf{Z}\setminus\{0\}$, thus $h(\mathbf{p})\geq h(\mathbf{n}_t)$, and similarly for q. It follows that for $\mathbf{p}\neq 0$, $\mathbf{q}\neq 0$

$$h(\mathbf{p})\,h(\mathbf{q})\geq h(\mathbf{n}_t)^2>\sqrt{(\tfrac{4}{3}-\varepsilon)\,h(\mathbf{n}_t)}.$$

If $\mathbf{p}$, $\mathbf{q}$ are linearly independent, then $\mathbf{p}\times\mathbf{q}\neq 0$ and $(\mathbf{p}\times\mathbf{q})\mathbf{n}_t=0$. On the other hand, either $h(\mathbf{p})\,h(\mathbf{q})\geq 4t^2+4t$ or $h(\mathbf{p}\times\mathbf{q})\leq 2h(\mathbf{p})\,h(\mathbf{q})\leq 2(4t^2+4t-14)$ $=8t^2+8t-2$. In the latter case in virtue of Lemma 7 we have $\mathbf{p}\times\mathbf{q}=\mathbf{m}_i$, for na $i\leq 6$. Hence, $p\mathbf{m}_i=q\mathbf{m}_i=0$ and from Lemmata 8-13 we obtain $h(\mathbf{p})h(\mathbf{q})\geq 4t^2+4t)$.

In view of (25) the theorem follows.

Remark. There exist decompositions $\mathbf{n}_t = u\mathbf{p} + v\mathbf{q}$ with $h(\mathbf{p})h(\mathbf{q}) = 4t^2 + 4t$, namely

$$\mathbf{n}_t = (6t^2+4t-1)[2t^2+2t,\ 0,\ -(2t^2+2t-1)] + (2t^2+2t)(6t^2+6t-1)\ [0,\ 1,\ 2]$$

or

$$\mathbf{n}_t = (2t^2+2t)(6t^2+4t-1)[1,\ 0,\ 2] + (6t^2+6t-1)[0,\ 2t^2+2t,\ 1-2t^2].$$

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Received 14. 11. 1986
Revised 13. 03. 1988