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ПЛИСКА

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POSITIVE PRECOMPACT OPERATORS

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In this paper precompact and semiprecompact operators between ordered vector spaces endowed with linear topologies are considered. Such operators are defined on directed vector spaces which are not supposed to be vector lattices. The terminology used is that of Cristescu (1977).

If X is an ordered linear space, then we denote by X_+ the subset of its positive elements and by $[a, b]$ the order segment determined by any two elements a, b of X.

The vector topologies considered in this paper will be supposed Hausdorff topologies.

If X and Y are ordered vector spaces endowed with vector topologies, then we denote by $\mathscr{R}(X, Y)$ the set of all regular operators mapping X into Y and we denote by $\mathscr{L}(X, Y)$ the set of linear operators which are continuous with respect to the topologies given on the spaces X and Y.

Definition 1. *One calls a* space of type $(\mathscr{R})$ *any directed vector space which satisfies the Riesz decomposition condition.*

Lemma 1. *Let Y be a space of type $(\mathscr{R})$ endowed with a locally solid topology. If B is a precompact subset whose elements are positive, then for any neighbourhood W of the origin there exists $y_0 \in Y_+$ such that*

$$B \subset [0, y_0] + W \cap Y_+. \tag{1}$$

Proof. Let W be an arbitrary neighbourhood of the origin and W_1 a solid neighbourhood of the origin such that $W_1 \subset W$. Denoting $W_0 = W_1 \cap X_+ - W_1 \cap X_+$, the set W_0 is a neighbourhood of the origin. Let B_0 be a finite subset of B such that $B \subset B_0 + W$, and let y_0 be an upper bound of B_0. If $y \in B$, then there exist $b \in [0, y_0]$ and $v_i \in W_1 \cap X_+$ $(i=1, 2)$, such that $y = b + v_1 - v_2$. From $0 \leqq v_2 \leqq b + v_1$ follows $v_2 = b' + v'$ with $0 \leq b' \leq b$ and $0 \leq v' \leq v_1$. Therefore, $y = (b - b') + (v_1 - v')$ with $b - b' \in [0, y_0]$ and $v_1 - v' \in W_1$. Consequently, (1) is valid.

Lemma 2. *Let Y be a space of type $(\mathscr{R})$. Let Z be a complete vector lattice endowed with an (ω)-continuous locally solid topology. If $y_0 \in Y$, then the set $\{V \in \mathscr{R}(Y, Z) \mid V([0, y_0])$ precompact$\}$ is a band of the complete vector lattice $\mathscr{R}(Y, Z)$.*

This lemma given in [1] for a vector lattice Y, holds too if Y is only a space of type $(\mathscr{R})$.

Definition 2. Let X and Y be ordered vector spaces endowed with vector topologies and let $U: X \to Y$ be a linear operator. The operator U is said to be semiprecompact, if for any topologically bounded subset A of X the set $U(A)$ is precompact with respect to the topology of Y. The operator U is said to be precompact, if there exists a neighbourhood W of the origin in X, such that $U(W)$ is a precompact subset of Y.

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The following proposition is a generalization of Theorem 2.1 in [1] and proposition 1 in [3].

Proposition 1. *Let X and Y be spaces of type ($\mathscr{R}$) and Z be a complete vector lattice. Suppose that X, Y, Z are endowed with locally solid topologies and that the topology of Z is (ω)-continuous. Let $U_i: X \to Y$ and $V_i: Y \to Z$ ($i=1, 2$) be linear operators, such that $0 \leq U_1 \leq U_2$ and $0 \leq V_1 \leq V_2$. If the operators U_2 and V_2 are precompact, then the operator V_1U_1 is precompact.*

Proof. Let B be a solid neighbourhood of the origin in Y, such that $V_2(B)$ be a precompact set in Z. From $0 \leq V_1 \leq V_2$ and from the topological boundedness of $V_2(B \cap Y_+)$ it follows that the set $V_1(B \cap Y_+)$ is also topologically bounded [4, 6.1.1]. Let W and W_0 be neighbourhoods of the origin in Z, such that $W_0 + W_0 \subset W$. There exists a number $\varepsilon > 0$ such that

(2) $$\varepsilon V_1(B \cap Y_+) \subset W_0.$$

Let now A be a solid neighbourhood of the origin in X such that $U_2(A)$ be a precompact set in Y. By Lemma 1 there exists an element $y_0 \in Y_+$ such that

(3) $$U_2(A \cap X_+) \subset [0, y_0] + (\varepsilon B) \cap Y_+.$$

Since Y is a space of type ($\mathscr{R}$), from (3) and from $0 \leq U_1 \leq U_2$ it follows

(4) $$U_1(A \cap X_+) \subset [0, y_0] + \varepsilon (B \cap Y_+).$$

On the other hand, the set $V_2([0, y_0])$ is precompact and from $0 \leq V_1 \leq V_2$ it follows, with lemma 1, that $V_1([0, y_0])$ is also a precompact set. Let E be a finite subset of Z such that

(5) $$V_1([0, y_0]) \subset E + W_0.$$

From (2), (4) and (5) it follows

$$V_1U_1(A \cap X_+) \subset E + W;$$

therefore, $V_1U_1(A \cap X_+)$ is a precompact set.

Finally, since the topology of the space X is locally solid, the set

$$A_0 = A \cap X_+ - A \cap X_+$$

is a neighbourhood of the origin in X and the set $V_1U_1(A_0)$ is precompact.

Definition 3. *Let X be a directed vector space endowed with a vector topology. A subset A of X is said to be* (τo)-bounded, *if for any neighbourhood W of the origin there exists $x_0 \in X_+$ such that $A \subset [-x_0, x]_0 + W$.*

Proposition 2. *Let X be a directed vector space endowed with a vector topology with the property (S) [4, 6.1.5] and let Y be a space of type ($\mathscr{R}$) endowed with a locally solid topology. Let $U_i \in \mathscr{L}(X, Y)$ ($i=1, 2$), with $0 \leq U_1 \leq U_2$ and assume U_2 be semiprecompact. Then:*

(i) *For any topologically bounded subset $A \subset X$, the set $U_1(A)$ is* (τo)-bounded,

(ii) *If any order segment of Y is a precompact set, then U_1 is a semi-precompact operator.*

Proof. (i) Let $A_0 \subset X_+$ be a topologically bounded set. The set $U_2(A_0)$ is a precompact subset of Y, hence, by lemma 1 for any solid neighbourhood W_0 of the origin in Y there exists $y_0 \in Y_+$ such that

(6) $$U_2(A_0) \subset [0, y_0] + W_0 \cap Y_+.$$

Since Y is a space of type ($\mathscr{R}$), from $0 \leq U_1 \leq U_2$ and from (6) it follows

(7) $$U_1(A_0) \subset [0, y_0] + W_0.$$

Let now A be any topologically bounded subset of X and W any neighbourhood of the origin in Y. Let A_0 be a topologically bounded subset of X_+, such that $A \subset A_0 - A_0$, and let W_0 be a solid neighbourhood of the origin in Y, such that $W_0 + W_0 \subset W$. Let $y_0 \in Y_1$, such that (7) be valid. Then

$$U_1(A) \subset [-y_0, y_0] + W;$$

therefore, $U_1(B)$ is (τo)-bounded.

(ii) Let $A \subset X_+$ be topologically bounded and let W be an arbitrary neighbourhood of the origin in Y. Let W_0 be a solid neighbourhood of the origin in Y, such that $W_0 + W_0 \subset W$. Since $U_2(A)$ is a precompact set, there exists $y_0 \in Y_+$ such that (6) be valid. Consequently, (7) also holds. On the other hand, there exists by hypothesis a finite subset H of Y such that $[0, y_0] \subset H + W_0$. By (7) we have $U_1(A) \subset H + W$, therefore, $U_1(A)$ is a precompact set. Since the space X has the property (S), the proof is complete.

P r o p o s i t i o n 3. *If X is a space of type ($\mathscr{R}$) endowed with a locally full topology, and Y is a locally convex lattice of type (M), which is topologically complete, then every precompact linear operator mapping X into Y is a regular operator.*

P r o o f. Let $U: X \to Y$ be a linear precompact operator and S be a full neighourhbood of the origin in the space X such that $U(S)$ be precompact. If $0 \leq x \in S$, then $U([0, x]) \subset U(S)$. By [4, 7.1.9, proposition 5], for all $x \in S \cap X_+$ there exists $V_0(x) = \sup U([0, x])$.

Since S is an absorbing set, $V_0(x)$ exists for all $x \in X_+$. Putting $V(x' - x'') = V_0(x') - V_0(x'')$, $(x', x'' \in X_+)$, we obtain an additive and positive operator $V: X \to Y$ and we have $U \leq V$. Therefore, U is a regular operator.

R e m a r k. It results from [4, 8.1.4, Lemma 2] that if X is a space of type ($\mathscr{R}$) endowed with a locally solid topology and if Y is a Banach lattice in which there exist axial elements, then the set of precompact operators mapping X in Y is a vector sublattice of the space of regular operators.

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