PLISKA
STUDIA MATHEMATICA
BULGARICA

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A SOLUTION OF THE TRIGONOMETRIC MOMENT PROBLEM VIA TAGAMLITZKI'S "THEOREM OF THE CONES"

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In memory of my teacher
Professor Y. A. Tagamlitzki

In 1952 Y. Tagamlitzki gave an elegant proof of the classical Bochner's theorem on the positively definite functions [1]. Unfortunately, he never published his proof. In this paper we consider a related but simpler problem, the trigonometric moment problem, by using Tagamlitzki's approach.

Definition 1. *A sequence $\{c_\nu\}_{-\infty}^{+\infty}$ of complex numbers is a moment sequence, if there exists a nondecreasing function $\alpha\colon [0, 2\pi]\to\mathbf{R}$ such that the equalities*

$$c_\nu=\int_0^{2\pi} e^{i\nu t}\, d\alpha(t),\quad \nu=0,\ \pm 1,\ \pm 2,\dots, \tag{1}$$

hold.

The following result is classical.

Theorem 1. (F. Riesz [2]). *A sequence $\{c_\nu\}_{-\infty}^{+\infty}$ is a moment seqence, if and only if for any trigonometric polynomial $q(t)=\sum_{-n}^{n} a_\nu e^{i\nu t}$, non-negative on the real axis, we have*

$$\sum_{-n}^{n} c_\nu a_\nu\geq 0. \tag{2}$$

(The degree n of q is arbitrary).

We shall prove Theorem 1 via Tagamlitzki's "Theorem of the cones." Since this general result of Tagamlitzki published in Bulgarian is unpopular, we are giving a complete formulation. To this end, we begin with some definitions.

Let W be a linear space and $F=\{F_\nu\}_{-\infty}^{+\infty}$ be a sequence of linear functionals. We say that F is a coordinate system in W, if the equalities $F_\nu(f)=0$, $f\in W$, $\nu=0, \pm 1, \pm 2,\dots$ imply $f=0$.

Definition 2. *A set $K\subset W$ is said to be a cone, if it has the following properties:*

1. *If $f\in K$ and λ is a nonegative real number, then $\lambda f\in K$.*
2. *If $f\in K$, $d\in K$, then $f+g\in K$.*

Definition 3. *Let $K\subset W$ be a cone and P be a norm defined in K. An element $f\in K$, $f\neq 0$, is P-irreducible, if the equalities*

$$f=g+h,\ P(f)=P(g)+P(h),\ f\in K,\ h\in K, \tag{3}$$

are possible only if $g=\lambda f$, $h=\mu f$, $\lambda\geq 0$, $\mu\geq 0$, $\mu+\lambda=1$.

Definition 4. *Let F be a coordinate system in the linear space W and $K\subset W$ be a cone. Further, let P be a norm defined in K. The cone K*

is (F, P) compact, if for any sequence $\{x_n\}_0^\infty \subset S_K$, $S_K \overset{\text{def}}{=} \{x,\ x \in K,\ P(x) \leq 1\}$ *there exist an element* $a \in S_K$ *and a subsequence* $\{x_{n_k}\} \subset \{x_n\}$ *such that*

$$\lim_{n_k \to \infty} F_\nu(x_{n_k}) = F_\nu(a) \tag{4}$$

holds for any $F_\nu \in F$.

It is proved in [3] that every (F, P) compact cone contains P-irreducible elements.

Now we may state Tagamlitzki's result we need.

Theorem 2. (*Theorem of the cones* [3]). *Let* W *be a linear space with coordinate system* F. *Given the two cones* L *and* K, $L \subset K \subset W$, *suppose the following conditions are satisfied:*

1. *The cone* L *is* (F, Q) *compact, whereas* K *is* (F, P) *compact.* (Q *and* P *are norms defined in* L *and* K *respectively*).

2. *All the* P-*irreducible elements of* K *belong to* L *and for any* P-*irreducible* $f \in K$ *the inequality* $P(f) \geq Q(f)$ *holds.*

Then, $L = K$ and we have $P \geq Q$ in the whole K.

Remark. For our goal in this paper the earlier version of Theorem 2 published in [4] is quite sufficient.

In order to prove Theorem 1, we introduce the linear space W of all the complex sequences $\{a_\nu\}_{-\infty}^{+\infty}$ and set $F_\nu(a) = a_\nu$, $\nu = 0, \pm 1, \pm 2 \ldots$ for any $a = \{a_\nu\}_{-\infty}^{+\infty} \in W$. It is clear that $F = \{F_\nu\}_{-\infty}^{+\infty}$ is a coordinate system in W. Further, we define the cones L and K as follows.

Definition 5. *A sequence* $\{c_\nu\}_{-\infty}^{+\infty}$ *belongs to* K, *if and only if the Riesz condition* (2) *is satisfied. Finally* L *consists of all moment sequences*

$$c_\nu = \int_0^{2\pi} e^{i\nu t}\, d\alpha(t), \quad \nu = 0, \pm 1, \pm 2, \ldots, \tag{5}$$

where $\alpha\colon [0, 2\pi] \to \mathbf{R}$ *is nondecreasing,* $\alpha(0) = 0$ *and* $\alpha(t) = \alpha(t-0)$ *for* $0 < t \leq 2\pi$

It is well known and easily seen that under these conditions α is uniquely determined by its moments $\{c_\nu\}_{-\infty}^{+\infty}$.

The following lemma is obvious.

Lemma 1. *The inclusion* $L \subset K$ *holds.*

Proof. It $q(t) = \sum_{-n}^{n} a_\nu e^{i\nu t}$ is non-negative on the real axis and $\{c_\nu\}_{-\infty}^{+\infty} \subset L$, we have

$$\sum_{-n}^{n} c_\nu a_\nu = \int_0^{2\pi} \sum_{-n}^{n} a_\nu e^{i\nu t}\, d\alpha(t) = \int_0^{2\pi} q(t)\, d\alpha(t) \geq 0$$

and (2) is established.

Lemma 2. *Denote by* P *the linear functional* $a \to a_0$, *where* $a = \{a_\nu\}_{-\infty}^{+\infty}$. *Then* P *is a norm in* K.

Proof. Let $c = \{c_\nu\}_{-\infty}^{+\infty}$ be an element of K. Since the trigonometric polynomials $q_1(t) = 1$ and $q_2(t) = 2 + \xi e^{int} + \bar{\xi} e^{-int}$, $|\xi| = 1$ are non-negative on the real axis, taking into account (2) we get $c_0 \geq 0$ and $2c_0 + \xi c_n + \bar{\xi} c_{-n} \geq 0$. In turn, the second inequality implies that the number $D = \xi c_n + \bar{\xi} c_{-n}$ is real. Setting $\xi = x + iy$, $c_n = p + iq$, $c_{-n} = \delta + i\gamma$, we find $\operatorname{Im} D = (q + \gamma)x + (p - \delta)y = 0$, i. e. $p = \delta$, $q = -\gamma$, since $\xi = x + iy$ is an arbitrary point on the unite circle.

Thus, we have proved $c_{-n}=\overline{c}_n$ and the relation $2c_0+\xi c_n+\overline{\xi}c_{-n}\geqq 0$ takes the form $c_0+\mathrm{Re}\,(c_n\xi)\geqq 0$, i. e. $-\mathrm{Re}\,(c_n\xi)\leqq c_0$. Now choosing $\xi=-e^{i\varphi}$ with $\varphi=-\arg c_n$, we get $|c_n|\leqq c_0$, $n=0, \pm 1, \pm 2, \dots$, i. e. $|c_n|\leqq P(c)$ so that $P(c)=0$, $c\in K$, implies $c=0$. Since P is linear, it is a norm in K. Now, the inclusion $L\subset K$ shows that P is a norm also in L.

The following lemma is crucial in the whole proof.

L e m m a 3. *The P-irreducible elements in K have the form*

$$c=\{A\lambda^{\nu}\}_{-\infty}^{+\infty}, \tag{6}$$

where $A>0$ *and* $|\lambda|=1$.

P r o o f. Let $c=\{c_\nu\}_{-\infty}^{+\infty}$ be an element of K. Inspired by Tagamlitzki's proof of the Bochner theorem, we set

$$c=\frac{1}{4}A(\xi)+\frac{1}{4}A(-\xi),\quad A(\xi)=\{A_\nu(\xi)\}_{-\infty}^{+\infty},\quad |\xi|=1, \tag{7}$$

where $A_\nu(\xi)=2c_\nu+\xi c_{\nu+1}+\overline{\xi}c^*_{\nu-1}$, $\nu=0, \pm 1, \pm 2, \dots$ It is not diffcult to verify that $A(\xi)\in K$ for any complex ξ with $|\xi|=1$. Indeed, let the trigonometric polynomial $q(t)=\sum_{-n}^{n} a_\nu e^{i\nu t}$ be non-negative on the real axis. Then

$$\sum_{-n-1}^{n+1} b_\nu e^{i\nu t}=(2+\xi e^{it}+\overline{\xi}e^{-it})\,q(t) \tag{8}$$

has the same property. Thus, we have the inequality

$$\sum_{-n-1}^{n+1} b_\nu c_\nu\geqq 0, \tag{9}$$

which after a substitution of the explicit expressions of $\{b_\nu\}$ takes the form

$$\sum_{-n}^{n} a_\nu A_\nu(\xi)\geqq 0 \tag{10}$$

and shows that $A(\xi)\in K$. Since $-\xi$ is also on the unit circle, we conclude that $A(-\xi)\in K$, so (7) is a decomposition in K. Finally, P is linear and we have $P(c)=P(A(\xi)/4)+P(A(-\xi)/4)$. Now, we are ready to complete the proof.

Indeed, if $c\in K$ is P-irreducible, we obtain

$$4\lambda(\xi)c=A(\xi),\ \text{i. e.}\ 4\lambda(\xi)c_\nu=A_\nu(\xi),\quad \nu=0, \pm 1, \dots, \tag{11}$$

where $0\leqq\lambda(\xi)\leqq 1$. First, we shall solve (11) under the supposition that $c_0=1$. In this case we have $4\lambda(\xi)=2+\xi c_1+\overline{\xi}c_{-1}$ and (11) takes the form

$$(2+\xi c_1+\overline{\xi}c_{-1})\,c_\nu=2c_\nu+\xi c_{\nu+1}+\overline{\xi}c_{\nu-1}, \tag{12}$$

i. e.

$$(c_1c_\nu-c_{\nu+1})\,\xi+(c_{-1}c_\nu-c_{\nu-1})\overline{\xi}=0. \tag{13}$$

Since ξ is an arbitrary point on the unit circle, (13) implies

$$c_{\nu+1}=c_1c_\nu,\quad c_{\nu-1}=c_{-1}c_\nu,\quad \nu=\pm 1, \pm 2, \dots, \tag{14}$$

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* $\overline{\xi}$ is the conjugate number of ξ.

and by setting $\lambda=c_1$, $\mu=c_{-1}$ we easily get

(15) $$c_\nu=\lambda^\nu,\ c_{-\nu}=\mu^\nu,\quad \nu=0,\ 1,\ 2,\dots$$

Further, taking into account that $c_{-1}c_1=c_0=1$ and according to lemma 2 $c_{-1}=\overline{c_1}$, we get $\lambda\mu=1$, $\overline{\lambda}=\mu$, i. e. $\mu=\frac{1}{\lambda}$, $|\lambda|=1$. Now, (15) takes the form

(16) $$c_\nu=\lambda^\nu,\quad \nu=0,\ \pm1,\ \pm2,\dots$$

Finally, if $c\in K$ is an arbitrary P-irreducible element of K, we have $c\neq0$, i. e. $P(c)=c_0\neq0$, and by applying (16) to $\frac{c}{c_0}$, we obtain

(17) $$c=\{c_0\lambda^\nu\}_{-\infty}^{+\infty},\quad |\lambda|=1,\ c_0>0$$

and thus complete the proof.

Corollary. *All the P-irreducible elements of K belong to L.*

Proof. Let $c=\{A\lambda^\nu\}_{-\infty}^{+\infty}$, $A>0$ be P-irreducible. Since $|\lambda|=1$, there is a t_0, $0\leq t_0<2\pi$ such that $\lambda=e^{it_0}$, so $c=\{Ae^{i\nu t_0}\}_{-\infty}^{+\infty}$. Now define the function

$$\alpha(t)=\begin{cases}0, & 0\leqq t\leqq t_0,\\ A, & t_0<t\leqq 2\pi,\end{cases}$$

which is increasing because $A>0$. Since the equalities

$$c_\nu=\int_0^{2\pi} e^{i\nu t}\,d\alpha(t)$$

are obvious, the corollary is proved.

Lemma 4. *The cones K and L are (F, P) compact.*

Proof. First, let $\{c(m)\}_{-\infty}^{+\infty}\subset K$, $P(c(m))\leqq1$ be a sequence of elements of K. Since we have $|c_\nu(m)|\leqq P(c(m))\leqq1$, $\nu=0,\ \pm1,\ \pm2,\dots$, we may apply the Cantor diagonal process and select a subsequence $\{m_k\}$, such that $\lim\limits_{k\to\infty}c_\nu(m_k)$, $\nu=0,\ \pm1,\ \pm2,\dots$, exist. Setting $c_\nu=\lim\limits_{k\to\infty}c_\nu(m_k)$, we get a sequence $c=\{c_\nu\}_{-\infty}^{+\infty}\subset K$ with $P(c)\leqq1$ and such that $\lim\limits_{k\to\infty}F_\nu(c(m_k))=F_\nu(c)$ for any $F_\nu\in F$.

Thus, the (F, P) compactness of K is proved.
Now let $\{c(m)\}_{-\infty}^{+\infty}\subset L$, $P(c(m))\leqq1$ be an arbitrary sequence. In this case we have

$$P(c(m))=c_0(m)=\int_0^{2\pi}d\alpha_m(t)=\alpha_m(2\pi)-\alpha_m(0)=\alpha_m(2\pi)\leqq1$$

and by applying a well-known theorem of Helly [5], we select a subsequence $\{m_k\}$ such that $\lim\limits_{k\to\infty}\alpha_{m_k}(t)$ exists for every $t\in[0,\ 2\pi]$. Setting

$$\alpha(t)=\lim_{k\to\infty}\alpha_{m_k}(t),\quad c_\nu=\int_0^{2\pi}e^{i\nu t}\,d\alpha(t),\quad \nu=0,\ \pm1,\ \pm2,\dots,$$

by means of the second theorem of Helly [5], we get $c_\nu=\lim\limits_{k\to\infty}c_\nu(m_k)$. Since $c=\{c_\nu\}_{-\infty}^{+\infty}$ obviously belongs to L and satisfies the inequality $P(c)\leqq1$, the proof of Lemma 4 is completed.

It remains to summarize now. Since Lemma 1, the corollary of Lemma 3 and Lemma 4 permit us to apply Theorem 2 with $Q=P$, we conclude that $L=K$ and complete the proof of Theorem 1.

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Received 18. 11. 1986