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ON THE LOGARITHM OF THE DIFFERENTIAL OPERATOR

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To the memory of my late friend Y. A. Tagamlitzki

Two different proofs are given of the fact that $\ln s = -s\{\ln t + C\}$, where C is Euler's constant.

Introduction. In Operational Calculus, the exponential function $x(\lambda) = e^{\lambda w}$ (w operator) is defined as the solution of the differential equation $x'(\lambda) = wx(\lambda)$ such that $x(0) = 1$. In particular, we have $x(1) = e^{w}$ so that the operator w can be considered as the logarithm of the operator e^{w}, i. e. $w = \ln e^{w}$.

The exponential function satisfies the functional equation $e^{\lambda_1 w} . e^{\lambda_2 w} = e^{(\lambda_1 + \lambda_2)w}$. A similar equation is satisfied by the power s^{λ} of the differential operator s, $s^{\lambda_1} . s^{\lambda_2} = s^{\lambda_1 + \lambda_2}$. This suggests that s^{λ} can be considered as an exponential function $s^{\lambda} = e^{\lambda w}$, where w is the logarithm of s, i. e. $w = \ln s$. Then $(s^{\lambda})' = ws^{\lambda}$. Hence, we can find $w = (s^{\lambda})'/s^{\lambda}$, provided the fraction does not actually depend on λ. Indeed, we shall show that the following equation holds for all real λ

$$\frac{(s^{\lambda})'}{s^{\lambda}} = -s\{\ln t + C\} \quad (C \text{ — Euler's constant}),$$

so that we may write $\ln s = -s\{\ln t + C\}$. To prove this equation is the aim of this note. We are going to do it in two different ways.

1. Taking into account that $s = 1/l$ with $l = \{1\}$, we first make the following transformation

$$\frac{(s^{\lambda})'}{s^{\lambda}} = \left(\frac{1}{l^{\lambda}}\right)' l^{\lambda} = -l^{\lambda}\frac{(l^{\lambda})'}{l^{2\lambda}} = -\frac{(l^{\lambda})'}{l^{\lambda}} = -\frac{l^{1-\lambda}(l^{\lambda})'}{l} = -sl^{1-\lambda}(l^{\lambda})'.$$

Now we have for $\lambda > 0$

$$(l^{\lambda})' = \left\{\frac{t^{\lambda-1}}{\Gamma(\lambda)}\right\}' = \left\{\frac{t^{\lambda-1}}{\Gamma(\lambda)}\ln t - \frac{t^{\lambda-1}}{\Gamma(\lambda)} \cdot \frac{\Gamma'(\lambda)}{\Gamma(\lambda)}\right\},$$

where $\Gamma(\lambda)$ is the Euler gamma function. Hence, for $0 < \lambda < 1$,

$$l^{1-\lambda}(l^{\lambda})' = \left\{\int_0^1 \frac{(t-\tau)^{-\lambda}}{\Gamma(1-\lambda)}\left(\frac{\tau^{\lambda-1}}{\Gamma(\lambda)}\ln t - \frac{\tau^{\lambda-1}}{\Gamma(\lambda)} \cdot \frac{\Gamma'(\lambda)}{\Gamma(\lambda)}\right)d\tau\right\}.$$

Substituting $\tau = t\sigma$, we get

$$\begin{aligned} l^{1-\lambda}(l^{\lambda})' &= \left\{\int_0^1 \frac{(1-\sigma)^{-\lambda}}{\Gamma(1-\lambda)}\left(\frac{\sigma^{\lambda-1}}{\Gamma(\lambda)}(\ln t + \ln\sigma) - \frac{\sigma^{\lambda-1}}{\Gamma(\lambda)}\frac{\Gamma'(\lambda)}{\Gamma(\lambda)}\right)d\sigma\right\} \\ &= \left\{\ln t \cdot \frac{B(1-\lambda, \lambda)}{\Gamma(1-\lambda)\Gamma(\lambda)} + \int_0^1 \frac{(1-\sigma)^{-\lambda}\sigma^{\lambda-1}}{\Gamma(1-\lambda)\Gamma(\lambda)}\ln\sigma d\sigma - \frac{B(1-\lambda, \lambda)}{\Gamma(1-\lambda)\Gamma(\lambda)}\frac{\Gamma'(\lambda)}{\Gamma(\lambda)}\right\} \\ &= \left\{\ln t + \int_0^1 \frac{(1-\sigma)^{-\lambda}\sigma^{\lambda-1}}{\Gamma(1-\lambda)\Gamma(\lambda)}\ln\sigma d\sigma - \frac{\Gamma'(\lambda)}{\Gamma(\lambda)}\right\}, \end{aligned}$$

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because

$$\frac{B(1-\lambda, \lambda)}{\Gamma(1-\lambda)\Gamma(\lambda)} = 1.$$

To the remaining integral we apply the general formula

$$\int_0^1 \sigma^{p-1}(1-\sigma)^{q-1}\ln\sigma d\sigma = B(p, q)\left[\frac{\Gamma'(p)}{\Gamma(p)} - \frac{\Gamma'(p+q)}{\Gamma(p+q)}\right], \quad (p>0,\ q>0),$$

which is easily obtained by differentiating the formula

$$\int_0^1 \sigma^{p-1}(1-\sigma)^{q-1}d\sigma = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

with respect to p. So we obtain

$$\int_0^1 \frac{(1-\sigma)^{-\lambda}\sigma^{\lambda-1}}{\Gamma(1-\lambda)\Gamma(\lambda)}\ln\sigma d\sigma = \frac{\Gamma'(\lambda)}{\Gamma(\lambda)} - \frac{\Gamma'(1)}{\Gamma(1)}$$

and, taking account of $\Gamma'(1) = -C$,

$$l^{1-\lambda}(l^{\lambda})' = \{\ln t - \Gamma'(1)\} = \{\ln t + C\}.$$

This proves the required formula.

One may remark that formula (1) holds for every real number λ. Indeed, given any λ, one always can find such a λ_0 that $0<\lambda_0+\lambda<1$. Letting $\mu=\lambda_0+\lambda$, we have $w=(s^{\mu})'/s^{\mu}$ according to the result already obtained.

2. It is interesting that the following formula holds

$$\ln s = \lim_{\alpha\to 0}\frac{s^{\alpha}-1}{\alpha},$$

where α is a real variable and the limit is meant in the operational sense. In fact, we have

$$\frac{s^{\alpha}-1}{\alpha} = \frac{1-l^{\alpha}}{\alpha l^{\alpha}} = \frac{l-l^{\alpha+1}}{\alpha l^{\alpha+1}} = \frac{l^{2-\alpha}\dfrac{l-l^{\alpha+1}}{\alpha}}{l^3};$$

Since the denominator in the last fraction is constant, it suffices to determine the limit of the numerator. We have

$$l^{2-\alpha}\cdot\frac{l-l^{\alpha+1}}{\alpha} = \left\{\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} * \frac{1}{\alpha}\left(1-\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)\right\}.$$

If each factor of a convolution converges almost uniformly, then also the convolution converges almost uniformly to the convolution of the limits. Evidently, the first factor $\dfrac{t^{1-\alpha}}{\Gamma(2-\alpha)}$ converges almost uniformly to $\dfrac{t}{\Gamma(2)}$. To find the limit of the second factor, we write

$$\frac{1}{\alpha}\left(1-\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right) = \frac{1}{\alpha}(1-t^{\alpha}) + \frac{1}{\alpha}\left(t^{\alpha}-\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)$$
$$= -\frac{t^{\alpha}-1}{\alpha} + \frac{t^{\alpha}}{\Gamma(1+\alpha)}\,\frac{\Gamma(1+\alpha)-\Gamma(1)}{\alpha}.$$

Hence,

$$\lim_{\alpha\to 0}\frac{1}{\alpha}\left(1-\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right) = -\ln t + 1.\Gamma'(1) = -\ln t - C$$

and

$$\lim_{\alpha\to 0}\frac{s^{\alpha}-1}{\alpha}=\frac{l^{2}\{-\ln t-C\}}{l^{3}}=-s\{\ln t+C\}=\ln s.$$

Having this limit, we can show that $-s\{\ln t\}C=\ln s$. In fact, using the functional equation $l^{\lambda_1}.l^{\lambda_2}=l^{\lambda_1+\lambda_2}$, we may write

$$\frac{s^{\lambda+\alpha}-s^{\lambda}}{\alpha}=\frac{s^{\lambda}(s^{\alpha}-1)}{\alpha}=s^{\lambda}\frac{s^{\alpha}-1}{\alpha}$$

and hence,

$$(s^{\lambda})'=\lim_{\alpha\to 0}\frac{s^{\lambda+\alpha}-s^{\lambda}}{\alpha}=s^{\lambda}(-C-s\{\ln t\}),$$

which implies $-C-s\{\ln t\}=\ln s$, according to the general definition of a logarithm of an operator.

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