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# ON THE STRONG ASYMPTOTIC BEHAVIOR OF ASYMPTOTICALLY NONEXPANSIVE SEMI-GROUPS IN BANACH SPACES

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*Dedicated to the late Y. A. Izrael*

**1. Introduction.** Throughout this paper  $X$  denotes a uniformly convex Banach space and  $C$  is a nonempty closed subset of  $X$ . Let  $J$  be an unbounded subset of  $[0, \infty)$  satisfying

$$(1.1) \quad t+s \in J \text{ for every } t, s \in J$$

and

$$(1.2) \quad t-s \in J \text{ for every } t, s \in J \text{ with } t > s.$$

A family  $\{T(t): t \in J\}$  of  $T(t): C \rightarrow C$  is called an asymptotically nonexpansive semi-group on  $C$ , if

$$(1.3) \quad T(t+s) = T(t)T(s) \text{ for every } t, s \in J$$

and there exists a function  $a(\cdot): J \rightarrow [0, \infty)$  with  $\lim_{t \rightarrow \infty} a(t) = 1$  such that

$$(1.4) \quad \|T(t)x - T(t)y\| \leq a(t) \|x - y\| \text{ for every } x, y \in C \text{ and } t \in J.$$

In particular, if  $a(t) = 1$ , then  $\{T(t): t \in J\}$  is called a nonexpansive semi-group on  $C$ . The set of fixed points of  $\{T(t): t \in J\}$  will be denoted by  $F$  i. e.  $F = \{x \in C: T(t)x = x \text{ for every } t \in J\}$ .

The purpose of this note is to investigate the strong convergence of  $T(t)x$  ( $x \in C$ ) as  $t \rightarrow \infty$ . The main result is stated as follows.

**Theorem 1.** *Let  $\{T(t): t \in J\}$  be an asymptotically nonexpansive semi-group on  $C$  and let  $x \in C$ . Then the conditions (i) and (ii) are equivalent:*

(i)  $T(t)x, t \in J$ , converges strongly as  $t \rightarrow \infty$ .

(ii)  $\lim_{t \rightarrow \infty} \|T(t+h)x - T(t)x\| = 0$  for every  $h \in J$  and there exists a  $x_0 \in F$  such that

$$(1.5) \quad \liminf_{t,s \rightarrow \infty} (\liminf_{h \rightarrow \infty} \|T(t+h)x + T(s+h)x - 2x_0\|) \\ \leq \liminf_{t,s \rightarrow \infty} \|T(t)x + T(s)x - 2x_0\|.$$

As a direct consequence of Theorem 1 we obtain the following theorem which has been proved by Taniguti, Takahashi and Abe [3].

**Theorem 2.** *Let  $\{T(t): t \in J\}$  be an asymptotically nonexpansive semi-group on  $C$  and let  $x \in C$ . Suppose that*

(a<sub>1</sub>) *there exist  $x_0 \in F$ ,  $\varphi \in C_{11}[0, \infty)$ ,  $\psi \in C[0, \infty)$  and a nonnegative function  $b$  defined on  $J$  with  $\lim_{h \rightarrow \infty} b(h) = 1$  such that*

$$\varphi(\|T(h)u + T(h)v - 2x_0\|) \leq \varphi(b(h) \|u + v - 2x_0\|) \\ + [\psi(b(h) \|u - x_0\|) - \psi(\|T(h)u - x_0\|) + \psi(b(h) \|v - x_0\|) - \psi(\|T(h)v - x_0\|)]$$

for every  $u, v \in \{T(t)x : t \in J\}$  and  $h \in J$ , where  $C_{11}[0, \infty)$  denotes the set of increasing continuous functions defined on  $[0, \infty)$ , and

$$(a_2) \quad \lim_{t \rightarrow \infty} \|T(t+h)x - T(t)x\| = 0 \text{ for every } h \in J.$$

Then  $T(t)x, t \in J$ , converges strongly as  $t \rightarrow \infty$  (to an element of  $F$ ).

Remarks. 1) Suppose that  $T: C \rightarrow C$  is nonexpansive,  $0 \in C, T0=0$  and there is a  $c \geq 0$  such that

$$(1.6) \quad \|Tu + Tv\|^2 \leq \|u + v\|^2 + c(\|u\|^2 - \|Tu\|^2 + \|v\|^2 - \|Tv\|^2)$$

for every  $u, v \in C$ . Then, the nonexpansive semi-group  $\{T^n : n=1, 2, \dots\}$  satisfies condition  $(a_1)$  in Theorem 2 with  $x_0=0, \varphi(t)=t^2, \psi(t)=ct^2$  and  $b(n) \equiv 1$ . Condition (1.6) has been considered in [2], and note that (1.6) with  $c=0$  is satisfied, if  $T$  is odd, i. e.  $C=-C$  and  $T(-u)=-Tu$  for  $u \in C$ .

2) See [3] for applications of Theorem 2.

**2. Proofs of the Theorems.** We start with the

**Proof of Theorem 1.** First, suppose that (i) holds. Let  $\lim_{t \rightarrow \infty} T(t)x = u$ . By  $\|T(t+s)x - T(s)u\| \leq a(s)\|T(t)x - u\|$ , we obtain that  $T(s)u = u$  for every  $s \in J$ , i. e.  $u \in F$  so that  $F \neq \emptyset$ . Clearly,  $\lim_{t \rightarrow \infty} \|T(t+h)x - T(t)x\| = 0$  for every  $h \in J$  and (1.5) is satisfied for every  $x_0 \in F$ .

Next, suppose that (ii) holds. We note that  $\|T(t)x - x_0\|$  is convergent as  $t \rightarrow \infty$ . In fact,  $\|T(t+h)x - x_0\| = \|T(h)T(t)x - T(h)x_0\| \leq a(h)\|T(t)x - x_0\|$ . Letting  $h \rightarrow \infty$ , we have

$$\limsup_{t \rightarrow \infty} \|T(t)x - x_0\| = \limsup_{h \rightarrow \infty} \|T(t+h)x - x_0\| \leq \|T(t)x - x_0\|$$

for every  $t \in J$  and hence,  $\limsup_{t \rightarrow \infty} \|T(t)x - x_0\| \leq \liminf_{t \rightarrow \infty} \|T(t)x - x_0\|$ .

Set  $d = \lim_{t \rightarrow \infty} \|T(t)x - x_0\|$ . Now

$$(2.1) \quad 2\|T(t+h)x - x_0\| \leq \|T(t+h)x - T(s+h)x\| + \|T(t+h)x + T(s+h)x - 2x_0\|$$

for every  $t, s$  and  $h$  in  $J$ . Since

$$\lim_{h \rightarrow \infty} \|T(t+h)x - T(s+h)x\| = 0 \text{ for every } t, s \in J$$

(by the assumption  $\lim_{t \rightarrow \infty} \|T(t+h)x - T(t)x\| = 0$  for  $h \in J$ ), by taking the  $\liminf$  as  $h \rightarrow \infty$  in (2.1) we obtain

$$2d \leq \liminf_{h \rightarrow \infty} \|T(t+h)x + T(s+h)x - 2x_0\| \text{ for every } t, s \in J.$$

Hence, by (1.5)

$$2d \leq \liminf_{t,s \rightarrow \infty} \|T(t)x + T(s)x - 2x_0\| \leq \limsup_{t,s \rightarrow \infty} \|T(t)x + T(s)x - 2x_0\| \\ \leq \lim_{t,s \rightarrow \infty} [\|T(t)x - x_0\| + \|T(s)x - x_0\|] = 2d.$$

Consequently,

$$\lim_{t,s \rightarrow \infty} \|(T(t)x - x_0) + (T(s)x - x_0)\| = 2d.$$

So, by uniform convexity of  $X$  and  $\lim_{t \rightarrow \infty} \|T(t)x - x_0\| = d$ , we have

$$\lim_{t,s \rightarrow \infty} \|T(t)x - T(s)x\| = \lim_{t,s \rightarrow \infty} \|(T(t)x - x_0) - (T(s)x - x_0)\| = 0.$$

Hence,  $T(t)x$  converges strongly as  $t \rightarrow \infty$ . This completes the proof.

**Proof of Theorem 2.** It is sufficient to show that (1.5) is satisfied. It follows from  $(a_1)$  that

$$\varphi(\|T(t+h)x + T(s+h)x - 2x_0\|) \leq \varphi(b(h)\|T(t)x + T(s)x - 2x_0\|)$$

$$+ [\psi(b(h) \| T(t)x - x_0 \|) - \psi(\| T(t+h)x - x_0 \|) + \psi(b(h) \| T(s)x - x_0 \|) - \psi(\| T(s+h)x - x_0 \|)]$$

for every  $t, s$  and  $h$  in  $J$ . As shown in the proof of Theorem 1,  $\| T(t)x - x_0 \|$  is convergent as  $t \rightarrow \infty$ . Set  $\lim_{t \rightarrow \infty} \| T(t)x - x_0 \| = d$ . Letting  $h \rightarrow \infty$  in the inequality above, we have

$$(2.2) \quad \varphi(\liminf_{h \rightarrow \infty} \| T(t+h)x + T(s+h)x - 2x_0 \|) \leq \varphi(\| T(t)x + T(s)x - 2x_0 \|) + [\psi(\| T(t)x - x_0 \|) - \psi(d) + \psi(\| T(s)x - x_0 \|) - \psi(d)]$$

for every  $t, s \in J$ . Now, by taking the  $\liminf$  as  $t, s \rightarrow \infty$  in (2.2), we obtain

$$\begin{aligned} & \varphi(\liminf_{t, s \rightarrow \infty} (\liminf_{h \rightarrow \infty} \| T(t+h)x + T(s+h)x - 2x_0 \|)) \\ & \leq \liminf_{t, s \rightarrow \infty} \varphi(\| T(t)x + T(s)x - 2x_0 \|) = \varphi(\liminf_{t, s \rightarrow \infty} \| T(t)x + T(s)x - 2x_0 \|). \end{aligned}$$

Therefore,

$$\begin{aligned} & \liminf_{t, s \rightarrow \infty} (\liminf_{h \rightarrow \infty} \| T(t+h)x + T(s+h)x - 2x_0 \|) \\ & \leq \liminf_{t, s \rightarrow \infty} \| T(t)x + T(s)x - 2x_0 \|, \end{aligned}$$

e. (1.5) is satisfied. This completes the proof.

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Received 28. 7. 1986