PLISKA
STUDIA MATHEMATICA
BULGARICA

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ON GENERALIZED ORLICZ SEQUENCE SPACES OF FOURIER COEFFICIENTS FOR TRIGONOMETRIC GAP SERIES. I

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To the memory of
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We investigate the operator associating with a function $f\in L^p_{2\pi}$, $1<p\le 2$, the sequence of Fourier coefficients of f with respect to a trigonometric gap system, as well as an operator from a modular space $X_{\rho_s^{(\varphi)}}$ to the generalized Orlicz sequence space l^φ.

1. Let (n_k) be an increasing sequence of positive integers. We take an increasing function $l(x)$, $x\ge 0$ such that $l(k)=n_k$ for $k=1, 2, \dots$, and we denote by $m(x)$ the inverse function of l. We write $A_\nu=\{k\in \mathrm{N}: 2^{\nu-1}\pi\le n_k<2^\nu\pi\}$, $\nu=1, 2, 3, \dots$, and we put $k_0=[m(\pi)]+1$, where $[x]$ denotes the integer part of x. Then, n_{k_0} is the least integer in A_1. Let $|A_\nu|$ be the number of elements of A_ν; then, $|A_\nu|<[m(2^\nu\pi)-m(2^{\nu-1}\pi)]+1=N_\nu$ for $\nu\in\mathrm{N}$.

Let

$$\sum_{k=1}^{\infty}(a_k(f)\cos n_k x+b_k(f)\sin n_k x)$$

be the Fourier series of a function $f\in L^p_{2\pi}$, $1<p\le 2$, with respect to the trigonometric gap system $\cos n_1x$, $\sin n_1x$, $\cos n_2x$, $\sin n_2x, \dots$ in $\langle 0, 2\pi\rangle$. With every $f\in L^p_{2\pi}$ we associate the sequence $c(f)=a_{k_0}(f), b_{k_0}(f), a_{k_0+1}(f), b_{k_0+1}(f), \dots$ with some fixed index k_0. We shall investigate the linear operator $c: f\to c(f)$ as an operator from some modular space $X_{\rho_\varphi^{(s)}}$ to a generalized Orlicz sequence space l^φ, generated by a sequence $\varphi=(\varphi_n)_{n=1}^\infty$ of φ-functions φ_n (for the terminology, see [2]), i. e. the space of sequences $c=(c_k)_{k=k_0}^\infty$ such that $\rho(\lambda c)=\sum_n\varphi_n(\lambda|c_n|)<\infty$ for a $\lambda>0$.

The following assumptions on the sequence φ will be fundamental.

A.1. There exists a constant $C\ge 1$ and a sequence of integers $(m(\nu))$ with $m(\nu)\in A_\nu$ such that $\varphi_\nu(u)\le C\varphi_{m(\nu)}(u)$ for $u\ge 0$ and $\nu\in A_\nu$;

A.2. The functions $\bar\varphi_n(u)=\varphi_n(u^{1/q})$, $u\ge 0$, where $1/p+1/q=1$, are concave.

Let us remark that A.1 is certainly satisfied, if $(\varphi_n(u))_{n=1}^\infty$ is an increasing (decreasing) sequence for all $u\ge 0$. Moreover, it is easily observed that if φ satisfies A.2, then

$$(*)\qquad \varphi_n(2u)\le 2^{1/q}\varphi_n(u)\ \text{for}\ u\ge 0,\ n\in\mathrm{N}.$$

In the following, we denote by ω_p the p-th modulus of continuity of f in $L^p_{2\pi}$, i. e.

$$\omega_p(f, \delta)=\sup_{|h|\le\delta}\Big(\int_0^{2\pi}|f(x+h)-f(x)|^p\,dx\Big)^{1/p}.$$

2. We prove now the following:

Theorem 1. *Let* $\varphi=(\varphi_n)_{n=1}^{\infty}$, *satisfy* A.1 *and* A.2. *Then, for every* $f\in L_{2\pi}^p$, $1<p\leq 2$, *there holds the inequality*

$$\rho(c(f))\leq \sum_{k=1}^{\infty}\rho_k^{(\varphi)}(f)=\rho_s^{(\varphi)}(f),$$

where

$$\rho_\nu^{(\varphi)}(f)=2CN_\nu\,\overline{\varphi}_{m(\nu)}\{N^{-1/q}\,\omega_p(\tfrac{1}{4}f,\ \tfrac{1}{2^\nu})\}$$

or $\nu\in\mathrm{N}$, *with* $1/p+1/q=1$.

Proof. Applying the Hausdorff-Young inequality to the function $F_h(x)=f(x+h)-f(x-h)$ and taking into account the formulae

$$a_k(F_h)=2b_k(f)\sin n_k h,\quad b_k(F_h)=-2a_k(f)\sin n_k h,$$

we obtain the inequality

$$\{\sum_{k=1}^{\infty}(|a_k(f)|^q+|b_k(f)|^q)|\sin n_k h|^q\}^{1/q}\leq \frac{1}{2}\{\frac{1}{\pi}\int_0^{2\pi}|F_h(x)|^p\,dx\}^{1/p}.$$

Restricting the summation on the left-hand side to $k\in A_\nu$ and observing that $|\sin n_k 2^{-\nu-1}|\geq 2^{-1/2}$ for $k\in A_\nu$, we obtain

(**)
$$\{\sum_{k\in A_\nu}(|a_k(f)|^q+|b_k(f)|^q)\}^{1/q}$$

$$\leq\frac{1}{\sqrt{2}}\{\frac{1}{\pi}\int_0^{2\pi}|F_{2^{-\nu-1}}(x)|^p\,dx\}^{1/p}\leq\frac{1}{\sqrt{2}}\,\frac{1}{\pi^{1/p}}\,\omega_p(f,\ \frac{1}{2^\nu}).$$

Now, we have by Jensen's inequality for concave functions

$$\sum_{k\in A_\nu}(\varphi_k(|a_k(f)|)+\varphi_k(|b_k(f)|))$$

$$\leq C\sum_{k\in A_\nu}(\overline{\varphi}_{m(k)}(|a_k(f)|)+\overline{\varphi}_{m(k)}(|b_k(f)|))$$

$$\leq 2C|A_\nu|\overline{\varphi}_{m(\nu)}\{\frac{1}{2|A_\nu|}\sum_{k\in A_\nu}(|a_k(f)|^q+|b_k(f)|^q)\}$$

$$\leq 2C|A_\nu|\overline{\varphi}_{m(\nu)}\{\frac{1}{2|A_\nu|}\,\frac{1}{\sqrt{2}^q}\,\frac{1}{\pi^{q/p}}\,\omega_p^q(f,\ \frac{1}{2^\nu})\}$$

$$\leq 2C|A_\nu|\overline{\varphi}_{m(\nu)}\{\frac{1}{|A_\nu|}\,\omega_p^q(\frac{1}{4}f,\ \frac{1}{2^\nu})\}.$$

Since $\overline{\varphi}_{m(\nu)}$ are concave, then $\overline{\varphi}_{m(\nu)}(u)/u$ are nonincreasing. Hence,

$$\sum_{k\in A_\nu}(\varphi_k(|a_k(f)|)+\varphi_k(|b_k(f)|))\leq 2CN_\nu\,\overline{\varphi}_{m(\nu)}\{N^{-1/q}\,\omega_p(\frac{1}{4}f,\ \frac{1}{2^\nu})\}=\rho_\nu^{(\varphi)}(f).$$

This gives

$$\rho(c(f))=\sum_{\nu=1}^{\infty}\sum_{k\in A_\nu}(\varphi_k(|a_k(f)|)+\varphi_k(|b_k(f)|))\leq\sum_{\nu=1}^{\infty}\rho_\nu^{(\varphi)}(f)=\rho_s^{(\varphi)}(f).$$

Taking as a special case $\varphi_n(u)=n^\beta|u|^\gamma$ with any real β and for $0<\gamma\leq q$, we obtain from Theorem 1 the following

Corollary 1. *If $0<\gamma\leq q$, β real and*

$$\sum_{\nu=1}^{\infty} m(\nu)^\beta N_\nu^{1-\gamma/q}\,\omega_p^\gamma\left(f,\frac{1}{2^\nu}\right)<\infty,$$

then

$$\sum_{n=1}^{\infty} n^\beta\left(|a_n(f)|^\gamma+|b_n(f)|^\gamma\right)<\infty.$$

This Corollary generalizes a number of well-known results on Fourier series (see e. g. [4, Chapter VI, § 3]; also [1, p. 149, Theorem 3.1]).

Following [1], one may consider also special cases with $k^r=O(n_k)$ for an $r>0$ and $k\in\mathbf{N}$, or $n_{k+1}/n_k\geq\alpha>1$ for $k\in\mathbf{N}$.

3. We are going to apply Theorem 1 in order to investigate the continuity of the linear operator $c: f\rightarrow c(f)$. Obviously, $\rho_s^{(\varphi)}$ is a pseudomodular in the space $L_{2\pi}^p$, thus generating the modular space

$$X_{\rho_s^{(\varphi)}}=\{f\in L_{2\pi}^p:\ \rho_s^{(\varphi)}(\lambda f)\rightarrow 0 \text{ as } \lambda\rightarrow 0+\}$$

(see [2, Def. 1.4]).

The following results is obtained applying Theorem 1, immediately:

Theorem 2. *Under assumptions A.1 and A.2, $c: f\rightarrow c(f)$ is a linear operator, continuous from $X_{\rho_s^{(\varphi)}}$ to l^φ.*

Let us remark that due to the inequalities (*), modular convergence and norm convergence are equivalent in both spaces $X_{\rho_s^{(\varphi)}}$ and l^φ, so there is no need to distinguish between them.

Theorem 2 generalizes results of [3] concerning trigonometric Fourier series, if we put $n_k=k$.

4. Now, let $\varphi=(\varphi_n)_{n=1}^\infty$ and $\psi=(\psi_n)_{n=1}^\infty$ be two sequences of φ-functions satisfying A.1 with the same $m(\nu)$. Let us consider the following assumption (see [2, 8.1]):

A.3. There exist positive numbers δ, K_1, K_2 and a sequence (ε_k) with $\varepsilon_k\geq 0$, $\sum_1^\infty \varepsilon_k<\infty$ such that for every $u\geq 0$ and $k\in\mathbf{N}$ the inequality $\varphi_k(u)<\delta$ implies

$$\psi_k(u)\leq K_1\,\varphi_k(K_2 u).$$

Let us note that A.3 is the necessary and sufficient condition, in order that $l^\varphi\subset l^\psi$ continuously (see [2, Theorem 8.5]).

Theorem 3. *If A.3 holds, then $X_{\rho_s^{(\varphi)}}\subset X_{\rho_s^{(\psi)}}$, and this imbedding is continuous both with respect to the modular convergencies, as well as to norm convergencies.*

Proof. Let $f\in X_{\rho_s^{(\varphi)}}$, then $\rho_s^{(\varphi)}(\lambda f)\rightarrow 0$ as $\lambda\rightarrow 0+$, whence $\rho_s^{(\varphi)}(\lambda f)<\delta$ for $0<\lambda<\lambda_1$ with some $\lambda_1>0$. Hence, $\rho_\nu^{(\varphi)}(\lambda f)<\delta$ for $0<\lambda<\lambda_1$, $\nu\in\mathbf{N}$, and so

$$\varphi_{m(\nu)}\left\{N_\nu^{-1/q}\omega_p\left(\frac{1}{4}\lambda f,\frac{1}{2^\nu}\right)\right\}<\delta.$$

By A.3,

$$\psi_{m(\nu)}\{N_\nu^{-1/q}\,\omega_p(\tfrac{1}{4}\,\lambda f,\ \tfrac{1}{2^\nu})\}\leq K_1\,\varphi_{m(\nu)}\{K_2\,N_\nu^{-1/q}\,\omega_p(\tfrac{1}{4}\,\lambda f,\ \tfrac{1}{2^\nu})\}$$

for $\nu\in N$, $0<\lambda<\lambda_1$. Thus $\rho_s^{(\psi)}(\lambda f)\leq K_1\,\rho_s^{(\varphi)}(K_2\lambda f)$ for $0<\lambda<\lambda_1$, which shows that $f\in X_{\rho_s^{(\varphi)}}$. Now, let $f_n\in X_{\rho_s^{(\varphi)}}$, $f_n\to 0$ in $X_{\rho_s^{(\varphi)}}$ in the sense of modular convergence (resp. norm convergence). From $f_n\to 0$ it follows that $\rho_s^{(\varphi)}(K_2\lambda f_n)\to 0$ as $n\to\infty$ for some $\lambda>0$ (resp. for every $\lambda>0$). Taking such a $\lambda>0$ fixed, we choose an index N such that $\rho_s^{(\varphi)}(\lambda f_n)<\delta$ for $n\geq N$. Arguing as above, we obtain $\rho_s^{(\psi)}(\lambda f_n)\leq K_1\,\rho_s^{(\varphi)}(K_2\lambda f_n)$ for $n\geq N$. Hence, $\rho_s^{(\psi)}(\lambda f_n)\to 0$ as $n\to\infty$ for a $\lambda>0$ (resp. for all $\lambda>0$). This means that $f_n\to 0$ in $X_{\rho_s^{(\psi)}}$ in the sense of modular convergence (resp. norm convergence).

Remark 1. From Theorems 2 and 3 and from [2, Theorem 8.5], we may put our results together in the form of the following diagram:

$$\begin{array}{ccc} X_{\rho_s^{(\varphi)}} & \xrightarrow[c]{\text{A.1, A.2}} & l^{\varphi} \\ {\scriptstyle \text{A.3}}\Big\downarrow{\scriptstyle id} & & {\scriptstyle id}\Big\downarrow{\scriptstyle \text{A.3}} \\ X_{\rho_s^{(\psi)}} & \xrightarrow[\text{A.1, A.2}]{c} & l^{\psi} \end{array}$$

Remark 2. All the above results may be extended to the case of almost periodic functions, taking noninteger values of n_k (see [1]).

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Received 15. 10. 1986