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ПЛИСКА
БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

ON LINEAR OPERATORS ACTING IN SPACES OF ANALYTIC FUNCTIONS AND COMMUTING WITH EULER'S OPERATOR

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In memory of our teacher Y. A. Tagamlitzki

1. Preliminary notes. Let G be a bounded domain in the complex plane C and $A(G)$ denote the space of functions $f(z)$ which are analytic in G. Let us denote the space of polynomials in C by S and assume that $A(G)$ is endowed with the topology of uniform convergence on the compacts of G.

In paper [1] the general form of the operators $L: S \to S$ commuting with the operator of differentiation $\mathcal{D}=d/dz$ was found, and in [2] A. V. Bratishchev and Yu. F. Korobeinik proved that it is the same as for the linear operators $L: A(G) \to A(G)$ continuous in some weak sense and commuting with the operator $\mathcal{D}$. (They suppose that the domain G is simply-connected.)

In the present paper a similar result is obtained for operators in $A(G)$ commuting with the Euler operator $E=a_0 z\mathcal{D}+a_1 I$, where $a_0 \neq 0$ and a_1 are complex constants and I is the identity in $A(G)$. This result generalizes the results of [3] in the same sense in which Bratishchev and Korobeinik generalized the results of [1]. With its help the question of the minimal commutativity of the Euler operator in the algebra of the linear operators $L: A(G) \to A(G)$ is settled.

The results of the present paper were annouced in [4]. Here the same results are given in detail and complete proofs.

2. Description of the structures and two definitions. Let M be a C-linear set (for instance in $A(G)$) and A and B be linear operators acting from M to M. We denote by $F(M)$ the algebra whose elements are all linear operators $L: M \to M$. The algebraic operations in $F(M)$ are the usual ones with operators $(AB)y := A(By)$ and so on. Let a convergence h^* be introduced in a subalgebra $Z \subseteq F(M)$ in such a way that $B_n \xrightarrow{h^*} B$ implies $PB_n \xrightarrow{h^*} PB$ and $B_nQ \xrightarrow{h^*} BQ$ for arbitrary operators P and Q of the algebra Z. Obviously, in such a case, if the operators B_n commute with a given operator A, i. e. $B_nA=AB_n$ and $B_n \xrightarrow{h^*} B$, then the limit operator B commutes with A too, i. e. $BA=AB$. In addition, in this case every operator of the type

$$B=(h^*)\sum_{k=0}^{\infty} r_k(A), \tag{1}$$

where $A \in Z$ and $r_k(A)$, $k=0, 1, 2, \ldots$, are polynomials of A, commutes with the operator A. Indeed, every operator B of type (1) is h^*-limit of the partial sums $S_n=\sum_{k=0}^{n} r_k(A)$, i. e. $S_n \xrightarrow{h^*} B$ and $BA=AB$ follows immediately from the obvious relation $S_nA=AS_n$.

The operators of type (1) are polynomially generated by A. The operators of a given algebra Z whose commutants are composed by their corresponding polynomially-generated operators only are of a special interest. We introduce the following

Definition 1. *An operator $A \in Z$ is called a minimally commuting element of the algebra Z, if its commutant in Z includes operators of type* (1) *only.*

Before giving the next definition, let us denote by h the convergence generated by the topology of the space $A(G)$; we will write $y = (h-\lim_{n\to\infty}) y_n$ or $y^n \xrightarrow{h} y$, if the sequence $\{y_n\}_{n=1}^{\infty}$, $y_n \in A(G)$ is h-convergent to the function $y \in A(G)$, i. e. if this sequence is uniformly convergent to y on every compact $K \subseteq G$. We will denote by $[S]_{A(G)}$ the set of functions $y \in A(G)$, which are h-limits of sequences of polynomials in $A(G)$. According to the Runge approximation theorem (c. f. [5]), if G is a simply connected domain in $\mathbb{C}$, $[S]_{A(G)} = A(G)$ holds. This circumstance explains the great interest in the space $[S]_{A(G)}$.

Definition 2. *An operator $L \in F(M)$, $M \supseteq S$ is called continuous in the sense of Bratishchev and Korobeinik or m-continuous operator, if the equality*

(2)
$$(Ly)(z) = \lim_{n\to\infty} (Ly_n)(z), \quad z \in G,$$

holds for every function $y \in [S]_{A(G)}$ and for every sequence $\{y_n\}_{n=1}^{\infty}$, $y_n \in S$ such that $y = (h-\lim_{n\to\infty}) y_n$.

3. A property of the operators $L: S \to S$ commuting with the Euler operator and having m-continuous extension in the space $A(G)$. We have proved in [3] that an operator $L: S \to S$ commutes with the Euler operator, if it admits a representation of the type

(3)
$$(Ly)(z) = \sum_{k=0}^{\infty} b_k z^k y^{(k)}(z), \quad \forall z \in \mathbb{C}, \ \forall y \in S,$$

where $\{b_k\}_{k=0}^{\infty}$ is a sequence of complex constants.

We shall establish here that if an operator of type (3) admits a m-continuous extension in the space $A(G)$, then its corresponding sequence is convergent of some order to zero.

Theorem 1. *Let G be a bounded domain in $\mathbb{C}$ and $0 \notin \overline{\operatorname{conv}}(G)$. If $L: A(G) \to A(G)$ is a m-continuous linear operator, which acts in S according to the formula*

(4)
$$(Ly)(t) = \sum_{k=0}^{\infty} d_k t^k y^{(k)}(t), \quad \forall t \in G, \ \forall y \in S,$$

where $\{d_k\}_{k=0}^{\infty}$ is a sequence of complex constants, then the asymptotic equality

(5)
$$|d_k|^{1/k} = O(k^{-1}), \quad k \to \infty,$$

holds ($\overline{\operatorname{conv}}(G)$ is the closed convex hull of G).

Lemma 1. *Let G be a bounded domain in $\mathbb{C}$ and $0 \notin \overline{\operatorname{conv}}(G)$. Then for every complex number $c \neq 0$ there exists a point t^c such that $t^c \in G$ and $(c+1)t^c \notin \overline{\operatorname{conv}}(G)$.*

Proof. Suppose the opposite holds: there exists a number $c = c_0 \neq 0$ such that $(c_0+1)G \subseteq \overline{\operatorname{conv}}(G)$. Then $\overline{\operatorname{conv}}[(c_0+1)G] \subseteq \overline{\operatorname{conv}}[\overline{\operatorname{conv}}(G)]$, i. e.

(6)
$$(c_0+1)\overline{\operatorname{conv}}(G) \subseteq \overline{\operatorname{conv}}(G).$$

Applying (6) n—times, we obtain the inclusion

(7) $$(c_0+1)^n\overline{\operatorname{conv}}(G)\subseteq\overline{\operatorname{conv}}(G),\quad n\in N.$$

Now, because of (7), for $x\in\overline{\operatorname{conv}}(G)$ is fulfiled $(c_0+1)^n x\in\overline{\operatorname{conv}}(G)$. If $|c_0+1|<1$, letting $n\to\infty$, we obtain the contradiction $0\in\overline{\operatorname{conv}}(G)$. Similarly, if $|c_0+1|>1$, letting $n\to\infty$, we find that G is not bounded, which is another contradiction. If $|c_0+1|=1$, by using the assumption $0\notin\overline{\operatorname{conv}}(G)$, we obtain the contradiction $c_0=0$. Thus Lemma 1 is proved.

Proof of Theorem 1. We denote by $U(p;\,q)$ the disc of centre p and radius q. Now, if $z_0\in G$ $(z_0\neq 0)$, let us consider the disc $U(z_0;\,\theta|z_0|)$, where the positive number θ is such that $G\subseteq U(z_0;\,\theta|z_0|)$. Then $|z/z_0-1|<\theta$, $\forall z\in G$ and the series $\bar{y}(z)=\sum_{k=0}^{\infty}1/(z_0^k\theta^k)(z-z_0)^k$ is h-convergent in the disc $U(z_0;\,\theta|z_0|)$, i. e.

$$\bar{y}(z)=(h-\lim_{n\to\infty})P_n(z),\quad P_n(z)=\sum_{k=0}^{n}1/(\theta^k z_0^k)(z-z_0)^k\in S.$$

Hence, since the operator L is m-continuous, it follows

(8) $$(L\bar{y})(z)=\lim_{n\to\infty}(LP_n)(z),\quad \forall z\in G.$$

From (8), according to (4), we have

$$(L\bar{y})(z_0)=\lim_{n\to\infty}\sum_{k=0}^{\infty}d_k z_0^k P_n^{(k)}(z_0)$$

$$=\lim_{n\to\infty}\sum_{k=0}^{n}d_k z_0^k k!/(\theta^k z_0^k)=\lim_{n\to\infty}\sum_{k=0}^{n}d_k k!/\theta^k.$$

Consequently, the series $\sum_{k=0}^{\infty}d_k k!/\theta^k$ converges to $(L\bar{y})(z_0)$ and the inequality

(9) $$\overline{\lim_{k\to\infty}}|d_k k!|^{1/k}\leq\theta$$

holds. Because of the inequality (9), the series $\sum_{k=0}^{\infty}d_k k!/z^{k+1}$ determines a function

(10) $$B(z)=\sum_{k=0}^{\infty}d_k k!/z^{k+1},$$

which is analytic in the domain $\{z:\,\theta<|z|\leq\infty\}$. We shall prove that it is possible to extend this function analytically in the domain $\{z:\,0<|z|\leq\infty\}$. It is sufficient to establish that for every $c\in\mathrm{C}$, $0<|c|\leq\theta$ there exist numbers α and r and a function $T_c(Z)$ such that the following propositions hold:

a) $T_c(z)$ is analytic in the domain $\{z:\,|z-\alpha|>r\}$;

b) $|c-\alpha|>r$;

c) $T_c(z)=B(z)$, if $|z|$ is sufficiently large.

Indeed, let c be a fixed number such that $0<|c|\leq\theta$. According to Lemma 1, there exists a point t^c such that $t^c\in G$ and $(c+1)t^c\notin\overline{\operatorname{conv}}(G)$. Let us consider a disc $U(a;\,\lambda)$ such that

(11) $$G\subseteq U(a;\,\lambda),\ \overline{\operatorname{conv}}(G)\subseteq\overline{U(a;\,\lambda)},\ (c+1)t^c\notin U(a;\,\lambda).$$

Now we put $\alpha=a/t^c-1$, $r=\lambda/|t^c|$

$$T_c(z)=\sum_{k=0}^{\infty} b_k/(z-\alpha)^{k+1},$$

where the right-hand side is Laurent's series of the function $B(z)$ in the domain $\{z\colon \theta+|\alpha|<|z-\alpha|<\infty\}$ (it is not difficult to prove that this series doesn't contain non-negative powers of $z-\alpha$). The proposition c) is obvious, whereas the proposition b) is equivalent to the inequality $|(c+1)t^c-a|>\lambda$, which is true according to (11).

In order to prove a), let us take $R>|\alpha|+\theta$ and calculate

$$b_k=1/(2\pi i)\int_{|\alpha-z|=R} B(z)(z-\alpha)^k dz.$$

According to (10), we obtain

$$b_k=1/(2\pi i)\int_{|\alpha-z|=R}\Big(\sum_{\nu=0}^{\infty}\nu!\,d_\nu/z^{\nu+1}\Big)\Big(\sum_{s=0}^{k}\binom{k}{s}z^s(-\alpha)^{k-s}\Big)dz$$

$$=\sum_{\nu=0}^{\infty}\sum_{s=0}^{k}\nu!d_\nu\binom{k}{s}(-\alpha)^{k-s}1/(2\pi i)\int_{|\alpha-z|=R} z^s/z^{\nu+1}dz.$$

Thus, because of

$$\int_{|\alpha-z|=R} z^s/z^{\nu+1}dz=\begin{cases}2\pi i, & \nu=s,\\ 0, & \nu\neq s,\end{cases}$$

we obtain the equality

$$b_k=\sum_{\nu=0}^{k}\nu!\,d_\nu(-\alpha)^{k-\nu}\binom{k}{\nu}. \tag{12}$$

On the other hand, because of (4)

$$\Big(L\Big[\sum_{s=0}^{k}(z-t)^s/t^s\binom{k}{s}(-\alpha)^{k-s}\Big]\Big)(t)=\sum_{\nu=0}^{\infty}d_\nu t^\nu\Big[\sum_{s=0}^{k}(z-t)^s/t^s\binom{k}{s}(-\alpha)^{k-s}\Big]^{(\nu)}_{z=t}$$

$$=\sum_{\nu+0}^{k}d_\nu t^\nu(\nu!/t^\nu)\binom{k}{\nu}(-\alpha)^{k-\nu}=\sum_{\nu=0}^{k}d_\nu\nu!\binom{k}{\nu}(-\alpha)^{k-\nu}.$$

From this and (12) for $t=t^c$ we obtain

$$b_k=(L[((z-t^c)/t^c-\alpha)^k])(t^c)=(L[(z/t^c-1-(a/t^c-1))^k])(t^c).$$

$$b_k=(L[((z-a)/t^c)^k])(t^c). \tag{13}$$

Now having (13) and the fact that L is m-continuous, we prove that the series

$$\sum_{k=0}^{\infty} b_k/r^{k+1}=1/r\sum_{k=0}^{\infty}(L[((z-a)/(rt^c))^k])(t^c) \tag{14}$$

is convergent. In fact, the n-th partial sum of the series (14) is

$$\sum_{k=0}^{n}(L[((z-a)/(rt^c))^k])(t^c)=\Big(\sum_{k=0}^{n}L[((z-a)/(rt^c))^k]\Big)(t^c)=\Big(L\Big[\sum_{k=0}^{n}((z-a)/(rt^c))^k\Big]\Big)(t^c).$$

The inequality $|(z-a)/(rt^c)|<1$ holds in the disc $U(a;\lambda)$ and, consequently, in the domain G. The sequence of the polynomials $y_n(z)=\sum_{k=0}^{n}((z-a)/(rt^c))^k$ is h-convergent to the function $\varphi(z)=\sum_{k=0}^{\infty}((z-a)/(rt^c))^k$. As the operator L is m-

continuous, the limit $\lim_{n\to\infty}(Ly_n)(z)=(L\varphi)(z)$, $\forall z\in G$, exists and the series (14) is convergent. So a) is proved too. So we have proved that the series (10) can be analytically extended in the domain $\{0<|z|\leq\infty\}$. Consequently, the equality

(15) $$\lim_{k\to\infty}(|d_k|\,k!)^{1/k}=0$$

holds.

From (15), applying Stirling's formula $k!=(2\pi k)^{1/2}(k/e)^k e^{\theta/12}$ $\theta\in(0,1)$ we obtain the equality (5). Theorem 1 is proved.

The following theorem will be of further use.

Theorem 2. *If a sequence $\{d_k\}_{k=1}^{\infty}$, $d_k\in C$ satisfies the condition* (5), *then the series $\sum_{k=0}^{\infty} d_k z^k y^{(k)}(z)$ is convergent for every $z\in G$ and every function $y(z)$ from $A(G)$. In this case the operator $\Lambda: A(G)\to A(G)$, acting according to the formula*

(16) $$(\Lambda y)(z)=\sum_{k=0}^{\infty} d_k z^k y^{(k)}(z),\quad \forall y\in A(G),\ \forall z\in G.$$

is (h, h)-continuous extension of the operator (3).

Proof. Let $y(z)$ be an arbitrary function from $A(G)$ and $z_0\in G$. Let us consider the circumference Γ with centre z_0 and small enough radius b. Applying Cauchy's integral formula and denoting by M_i, $i=1,2$, large enough constants, we obtain the estimate

$$|d_k z_0^k y^{(k)}(z_0)|\leq|d_k||z_0|^k|k!/(2\pi i)\int_\Gamma y(\tau)/(\tau-z_0)^{k+1}d\tau|$$

$$\leq|d_k||z_0|^k k!/(2\pi)\max_\Gamma|y(z)|/b^{k+1}2\pi b\leq|d_k|\,k!M_1^kM_2,$$

which proves the first part of Theorem 2, because with the help of Stirling's formula we can easily obtain that

$$\lim_{k\to\infty}(|d_k|\,k!\,M_1^k)^{1/k}=\lim_{k\to\infty}(|d_k|^{1/k}/k^{-1})k^{-1}(2\pi k)^{1/(2k)}k/e\,e^{\theta/(12k)}M_1=0$$

In order to prove that the operator (16) is (h, h)-continuous, let us choose an arbitrary sequence $\{y_n\}_{n=1}^{\infty}$, $y_n\in A(G)$, which is h-convergent to a function $y\in A(G)$. Fixing some compact $K\subseteq G$, consider the sequence

(17) $$\lambda_n=\max_{z\in K}|(\Lambda y_n)(z)-(\Lambda y)(z)|.$$

It is enough to prove that $\lim_{n\to\infty}\lambda_n=0$ if $K\subseteq G$. Fixing some other compact K_1 such that $K\subset\overset{\circ}{K_1}$, $K_1\subset G$ and applying Cauchy's integral formula to the function $y_n(z)-y(z)$, we obtain the estimate

(18) $$\max_{z\in K}|y_n^{(k)}(z)-y^{(k)}(z)|\leq k!/b^kA\max_{z\in K_1}|y_n(z)-y(z)|,$$

where A and b are constants independent on n and K.

From (17) and (16), according to estimate (18), we obtain

(19) $$\lambda_n=\max_{z\in K}|\sum_{k=0}^{\infty} d_k z^k(y_n^{(k)}(z)-y^{(k)}(z))|\leq\max_{z\in K}(\sum_{k=0}^{\infty}|d_k||z|^k|y_n^{(k)}(z)-y^{(k)}(z)|)$$

$$\leq\sum_{k=0}^{\infty}|d_k|r^k\max_{z\in K}|y_n^{(k)}(z)-y^{(k)}(z)|\leq\sum_{k=0}^{\infty}|d_k|r^k(k!A)/b^k\max_{z\in K_1}|y_n(z)-y(z)|$$

$$\leq A\max_{z\in K_1}|y_n(z)-y(z)|\sum_{k=0}^{\infty}|d_k|\,k!\,(r/b)^k\ (r=\sup_G|z|,\ b=\tfrac{1}{2}\operatorname{dist}(K,\partial K_1)).$$

When proving the first part of this theorem, it became clear that this last series is absolutely convergent. Denoting its sum by σ, from (19) we obtain the estimate

$$\lambda_n \leqq A\sigma \max_{z \in K_1} |y_n(z) - y(z)|. \tag{20}$$

Now, from (20) we obtain $\lim\limits_{n\to\infty} \lambda_n = 0$, because the h-convergency $y_n \to y$ implies that $\lim\limits_{n\to\infty} \max\limits_{z \in K_1} |y_n(z) - y(z)| = 0$ for every compact $K_1 \subseteqq G$. Theorem 2 is proved.

Corollary 1. *Under the assumptions of Theorem 2 the spaces S and $[S]_{A(G)}$ are invariant subspaces of the operator Λ.*

The invariance of the space S is obvious, and the invariance of the space $[S]_{A(G)}$ is directly implied by the (h, h)-continuity of the operator Λ.

4. General formula of the m-continuous linear operators acting from $[S]_{A(G)}$ to $A(G)$ and commuting with the Euler operator. Let Q be again a bounded domain in $\mathbb{C}$ and $0 \notin \overline{\mathrm{conv}}(G)$. Let us consider the Euler operator E: $A(G) \to A(G)$, which acts according to the formula

$$(Ey)(t) = a_0 t y'(t) + a_1 y(t), \quad \forall y \in A(G), \ \forall t \in G, \tag{21}$$

where $a_0 \neq 0$ and a_1 are arbitrary complex numbers.

Theorem 3. *Let L: $[S]_{A(G)} \to A(G)$ be a m-continuous linear operator and $ELy = LEy$, $\forall y \in S$. Then there exists a sequence $\{d_k\}_{k=0}^{\infty}$, $d_k \in \mathbb{C}$ such that the equality* (5) *and the representation*

$$(Ly)(t) = \sum_{k=0}^{\infty} d_k t^k y^{(k)}(t), \quad \forall y \in [S]_{A(G)}$$

hold.

Proof. First we shall prove that S is an invariant subspace of the operator L. It is enough to establish that $\varphi_k(z) :\equiv (Lz^k)(z) \in S$, $\forall k = 0, 1, 2, \ldots$. The equality $ELz^k = LEz^k$ implies at once that $\varphi_k(z)$ satisfies the differential equation

$$k\varphi_k(z) = z\varphi_k'(z), \quad k = 0, 1, 2, \ldots,$$

which we can rewrite as follows

$$(\varphi_k(z)/z^k)' = 0, \quad k = 0, 1, 2, \ldots. \tag{22}$$

From (22), because of the fact that the domain G is connected, we obtain $\varphi_k(z) = c_k z^k$, $c_k = \mathrm{const}$, $k = 0, 1, 2, \ldots$. Consequently, $L(S) \subseteqq S$. So, considering the operator L over S only, we can claim that a linear operator acts from S into S and commutes with the Euler operator. According to Theorem 1 from our paper [3] the operator L acts over S according to the formula (4), in which $\{d_k\}$ is a sequence of complex numbers. From here, in view of the fact that the operator L is m-continuous and applying Theorem 1 (from the present paper), we obtain the asymptotic equality (5).

According to (5) and Theorem 2, we conclude that $\Lambda: A(G) \to A(G)$ (see (16)) is (h, h)-continuous and the equality

$$(Ly)(z) = (\Lambda y)(z), \quad \forall z \in G, \ \forall y \in S \tag{23}$$

holds. Now we have still to prove that (23) holds for $y \in [S]_{A(G)}$ too.

Let $y \in [S]_{A(G)}$ and the sequence $\{y_n\}_{n=1}^{\infty}$, $y_n \in S$ be h-convergent to y. Applying the m-continuity of the operator L, equality (23) and (h, h)-continuity of the operator Λ, we obtain

$$(Ly)(z) = \lim_{n\to\infty} (Ly_n)(z) = \lim_{n\to\infty} (\Lambda y_n)(z) = (\Lambda y)(z), \quad \forall z \in G.$$

Theorem 3 is proved.

The following theorem is inverse to Theorem 3 in some sense.

Theorem 4. *Let G be a domain in $\mathbb{C}$, M a subspace of the space $A(G)$, for example $M=[S]_{A(G)}$, $M=A(G)$. Let $E^{-1}(M)=\{y\in A(G): Ey\in M\}$, where $E=a_0t\mathscr{D}+a_1I$ is the Euler operator. If the operator $L: M\to A(G)$ is defined by the equality*

$$(24)\qquad (Ly)(z)=\sum_{k=0}^{\infty} d_k z^k y^{(k)}(z),\quad \forall z\in G,\ \forall y\in M,\ |d_k|^{1/k}=0(k^{-1}),\ k\to\infty,$$

then $LEy=ELy$, $\forall y\in M_1:=M\cap E^{-1}(M)$.

Proof. In view of the above conditions we conclude that we may differentiate series (24) for every $y\in M$ (even for $\forall y\in A(G)$). So we end the proof by a direct comparison of the representations of LEy and ELy.

Let us now assume that $E\in Z$, where Z is a certain algebra of m-continuous linear operators $L: [S]_{A(G)}\to[S]_{A(G)}$ such that $L(S)\subseteq S$. Further we introduce h^*-convergency of a sequence $\{L_n\}_{n=1}^{\infty}\subseteq Z$; such a sequence we call h^*-convergent to an operator $L\in Z$, if $Ly=(h-\lim_{n\to\infty})(L_n y)$, $\forall y\in[S]_{A(G)}$.

Theorem 5. *Let the hypotheses of Theorem 3 hold for a domain G Then the Euler operator E is a h^*-minimally commuting element of the algebra Z.*

The proof immediately follows from the proposition that the operators $E_k: [S]_{A(G)}\to[S]_{A(G)}$ $(E_k y)(t)=t^k y^{(k)}(t)$, $t\in G$, $k\geq 0$, are polynominals of the operator E. We obtain the last fact from the equalities $E_{k+1}=E_1E_k-kE_k$, $k=1, 2, \ldots$

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