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EXPONENTIAL APPROXIMATION IN THE NORMS AND SEMI-NORMS

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The deviations of some entire functions of exponential type from real-valued functions and their derivatives are estimated. As approximation metrics we use the L^p-norms and power variations on R. Theorems presented here correspond to the Ganelius and Popov results concerning the one-sided trigonometric approximation of periodic functions (see [4, 5 and 8]). Some related facts were announced in [2, 3, 6 and 7].

1. Notation. Given a number $p \geq 1$, let $L^p(a, b)$ be the space of all complex-valued functions Lebesgue-integrable with p-th power on the interval (a, b). Denote by $L^\infty(a, b)$ the space of all measurable functions essentially bounded on (a, b). As usually, the norm of the function $f \in L^p(a, b)$ is defined by

$$\|f\|_{L^p(a,b)} = \begin{cases} \left(\int_a^b |f(x)|^p dx\right)^{1/p} & \text{if } p < \infty, \\ \operatorname*{ess\,sup}_{x \in (a,b)} |f(x)| & \text{if } p = \infty. \end{cases}$$

We write L^p instead of $L^p(-\infty, \infty)$. Moreover, by convention, $L \equiv L^1$.

Let L^p_{loc} be the class of all complex-valued functions belonging to every space $L^p(a, b)$, with finite a, b $(a < b)$. Denote by AC^m_{loc} the class of complex-valued functions f having the derivative $f^{(m)}$ absolutely continuous on each finite interval $\langle a, b\rangle$.

For any function $f \in L^p_{\mathrm{loc}}$, the limit

$$\lim_{-a,\, b \to \infty} \|f\|_{L^p(a,b)} = \|f\|_p$$

is finite or infinite. In the case of $f \in L^p$,

$$\|f\|_p = \|f\|_{L^p} < \infty.$$

Consider a (complex-valued) function f defined on the interval $I \equiv \langle a, b\rangle$. Write

$$V_p(f; I) \equiv \sup_\pi \left\{ \sum_{j=1}^m |f(x_j) - f(x_{j-1})|^p \right\}^{1/p} \quad (0 < p < \infty),$$

where the supremum is taken over all finite systems π of the intervals $\langle x_0, x_1\rangle$, $\langle x_1, x_2\rangle, \ldots, \langle x_{m-1}, x_m\rangle$ $(x_0 = a,\ x_m = b;\ m = 1, 2, \ldots)$. This quantity is often called the p-th power variation of f on I. If f is defined on $\mathrm{R} \equiv (-\infty, \infty)$, we can also introduce the p-th (power) variation

$$V_p(f) \equiv V_p(f; \mathrm{R}) \equiv \sup_I V_p(f; I) \quad (I \subset \mathrm{R}).$$

We assume, additionally, that

$$V_\infty(f) \equiv V_\infty(f; \mathrm{R}) \equiv \sup_{s,\, t \in \mathrm{R}} |f(s) - f(t)|.$$

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As well known, $V_p(f) \geqq V_q(f)$, if $0<p<q\leqq\infty$.

Denote by BV^p [resp. BV^p_{loc}] the class of all complex-valued functions φ with finite p-th variation $V_p(\varphi;\ \mathrm{R})$ [$V_p(\varphi;\ I)$ for each finite interval I]. Obviously, an arbitrary function $f\in BV^p$ [resp. $f\in BV^p_{\mathrm{loc}}$] is bounded on R [on finite intervals I]. Moreover, any f of class BV^p_{loc} $(0<p<\infty)$ has one at most enumerable set of discontinuity points x at which the one-sided limits $f(x\pm 0)$ exist. The class BV^p $(p\geqq 1)$ with non-negative functional $V_p(\varphi)$ is a certain semi-normed space.

Let E_σ be the class of all entire functions of exponential type, of order σ at most. Denote by $B_{\sigma,p}$ $(0<\sigma<\infty,\ 1\leqq p\leqq\infty)$ the set of functions $F\in E_\sigma$ which belong to L^p (on R). Write $B_\sigma = B_{\sigma,\infty}$. As well known [10, p. 248], $B_{\sigma,p}\subset B_{\sigma,q}$ if $1\leqq p<q\leqq\infty$.

Suppose that f is a fixed function of class L^p_{loc} [resp. BV^p_{loc}] $(p\geqq 1)$. Denote by $H_{\sigma,p}(f)$ [resp. $D_{\sigma,p}(f)$] the set of all functions $G\in E_\sigma$ such that $f-G\in L^p$ [$f-G\in BV^p$]. Introduce the quantities

$$A_\sigma(f)_p = \begin{cases} \inf\limits_{S\in H_{\sigma,p}(f)} \|f-S\|_p, & \text{if } H_{\sigma,p}(f) \text{ is not empty} \\ \infty & \text{otherwise} \end{cases}$$

and

$$\nabla_\sigma(f)_p = \begin{cases} \inf\limits_{S\in D_{\sigma,p}(f)} V_p(f-S), & \text{if } D_{\sigma,p}(f) \text{ is not empty} \\ \infty & \text{otherwise.} \end{cases}$$

The first [resp. the second] of them is called the best exponential approximation of f by entire functions of class E_σ, in L^p-norm [in BV^p-semi-norm].

We will write W^rBV^p for the class consisting of all functions $\varphi\in AC^{r-1}_{\mathrm{loc}}$ such that $\varphi^{(r)}\in BV^p$ $(r\in\mathrm{N},\ p\geqq 1)$. The symbols c_k [resp. $c_l(r,\dots)$] $(k,\ l\in\mathrm{N})$ will mean some positive absolute constants [positive numbers depending only on the indicated parameters $r,\dots$].

2. Fundamental lemmas. Let us begin with an analogue of the well-known Bernstein inequality.

Lemma 1. *If* $G\in B_\sigma$ $(0<\sigma<\infty)$, *then*

$$V_p(G')\leqq\sigma V_p(G) \text{ for each } p\geqq 1. \tag{1}$$

Proof. Putting

$$u_k = \frac{2k+1}{2\sigma}\pi \quad (k=0,\ \pm 1,\ \pm 2,\dots),$$

we have

$$G'(t) = \frac{1}{\sigma}\sum_{k=-\infty}^{\infty}\frac{(-1)^k}{u_k^2}G(t+u_k) \tag{2}$$

for all real t (see e. g. [10, p. 216]).

Consider an arbitrary partition

$$\{a=x_0<x_1<\dots<x_{m-1}<x_m=b\}$$

of a finite interval $\langle a,\ b\rangle$. By the identity (2) and Minkowski's inequality, for every finite $p\geqq 1$,

$$\{\sum_{j=1}^{m}|G'(x_j)-G'(x_{j-1})|^p\}^{1/p}$$

$$\leq\frac{1}{\sigma}\sum_{k=-\infty}^{\infty}\frac{1}{u_k^2}\{\sum_{j=1}^{m}|G(t_j+u_k)-G(t_{j-1}+u_k)|^p\}^{1/p}$$

$$\leq\frac{1}{\sigma}\sum_{k=-\infty}^{\infty}\frac{1}{u_k^2}V_p(G;\ \mathrm{R})=\frac{8\sigma}{\pi^2}V_p(G;\ \mathrm{R})\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}.$$

This gives (1) for finite $p\geq 1$. If $p=\infty$, the proof is trivial.

Consider now functions φ belonging to the space L^q $(1\leq q\leq\infty)$. Introduce the singular integral

$$W[\varphi](z)=\frac{1}{\pi}\int_{-\infty}^{\infty}\varphi(t)K_\sigma(z-t)\,dt\quad(z=x+iy), \tag{3}$$

with

$$K_\sigma(\zeta)=(\cos\sigma\zeta-\cos 2\sigma\zeta)/(\sigma\zeta^2)\quad(0<\sigma<\infty).$$

Clearly, $K_\sigma\in B_{2\sigma,1}$.

As well known, $W[\varphi]\in B_{2\sigma}$ and in the case of $\varphi\in B_{\sigma,q}$

$$W[\varphi](x)=\varphi(x)\quad(x\in\mathrm{R}).$$

Further, $\|K_\sigma\|_1\leq c_1\pi$ $(c_1\leq 2+4\pi^{-2}\log 3)$. Consequently, $\|W[\varphi]\|_q\leq c_1\|\varphi\|_q$, i. e., $W[\varphi]\in L^q$ (see [1, Sect. 106]).

An easy calculation leads to

Lemma 2. *Let* $\varphi\in BV^p$ $(1\leq p\leq\infty)$. *Then*

$$V_p(W[\varphi])\leq c_1V_p(\varphi).$$

Given a positive number c and a positive integer r, let ρ be an even real-valued function continuous with its derivatives ρ', ρ'' on R, satisfying the conditions

1° $\rho(0)=\rho'(0)=0$,
2° $\rho'(t)=o(t^{r+1})$ and $\rho''(t)=O(t^r)$ as $t\to 0+$,
3° $\rho(t)=1$ for all $t\geq c$.

Consider the Bernoulli type function

$$\Phi_r(x)=\frac{1}{2\pi}\lim_{\alpha\to\infty}\int_{-\alpha}^{\alpha}\frac{\rho(t)}{(it)^r}e^{itx}\,dt\quad(x\in\mathrm{R}).$$

As well known, Φ_r is real-valued bounded and Lebesgue-integrable on R. In the case of $r\geq 2$, it is continuous everywhere ([1, Sect. 101]).

Lemma 3. *If* $\sigma\geq c$, *then there exist entire functions* $P_{\sigma,r}$, $Q_{\sigma,r}\in B_{\sigma,1}$ *such that*

1° $P_{\sigma,r}(x)\geq\Phi_r(x)$, $Q_{\sigma,r}(x)\leq\Phi_r(x)$ *for all* $x\in\mathrm{R}$,

2° $\|P_{\sigma,r}^{(\nu)}-\Phi_r^{(\nu)}\|_1\leq\frac{c_2(r,\nu)}{\sigma^{r-\nu}}$, $\|\Phi_r^{(\nu)}-Q_{\sigma,r}^{(\nu)}\|_1\leq\frac{c_2(r,\nu)}{\sigma^{r-\nu}}$ *for* $\nu=0,1,\dots,r$.

The proof is given in [9] (see also [6, Sect. 2]).

Finally, we will present the following supplementary

Lemma 4. *Let* $f\in BV_{\mathrm{loc}}^p$ $(1\leq p\leq\infty)$ *and let* $\nabla_\sigma(f)_p=0$ *for some finite* $\sigma>0$. *Then there exists an entire function* $F\in E_\sigma$ *such that* $F(x)=f(x)$ *for all real* x.

P r o o f. Consider a function $f\in BV^p$ $(1\leq p\leq\infty)$. By the assumption for every $\nu\in\mathbf{N}$, there are entire functions $F_\nu\in E_\sigma$ satisfying the condition

$$\sup_{u,\,v\in\mathbf{R}}|f(u)-F_\nu(u)-f(v)+F_\nu(v)|\leq\frac{1}{\nu}. \tag{4}$$

Without loss of generality, we may suppose that $f(0)=F_\nu(0)=0$.

From (4) it follows that $|f(u)-F_\nu(u)|\leq\nu^{-1}$ $(\nu=1, 2,\dots)$, uniformly in $u\in\mathbf{R}$. Consequently, $\lim\limits_{\nu\to\infty}F_\nu(u)=f(u)$, uniformly on $\mathbf{R}$, and $\sup\limits_{u\in\mathbf{R}}|F_\nu(u)|\leq M$ for $\nu=1, 2,\dots$, where $M=1+\sup\limits_{u\in\mathbf{R}}|f(u)|<\infty$.

Further, if $z=x+iy$ is an arbitrary complex number, the Bernstein inequality leads to

$$|F_\nu(z)|\leq Me^{\sigma|y|}\quad(\nu=1, 2,\dots)$$

and

$$|F_\nu(z)-F_\mu(z)|\leq Me^{\sigma|y|}\sup_{u\in\mathbf{R}}|F_\nu(u)-F_\mu(u)|\quad(\mu,\ \nu\in\mathbf{N})$$

(see [1, Sect. 83]). Hence, in view of the well-known Weierstrass Theorem, the limit $\lim\limits_{\nu\to\infty}F_\nu(z)=F(z)$ is finite for every complex z, $F\in E_\sigma$ and $F(x)=f(x)$ on $\mathbf{R}$.

In the general case, when $f\in BV^p_{\mathrm{loc}}$, the starting point is similar to that of Theorem 1 in Sect. 107 of [1].

3. Main results. Now, some approximation theorems will be given.

T h e o r e m 1. *Let f be a real-valued function of class AC^{r-1}_{loc} $(r\in\mathbf{N})$, having the derivative $f^{(r)}\in BV^p_{\mathrm{loc}}$ $(1\leq p\leq\infty)$, and let $\nabla_c(f^{(r)})_p<\infty$ for some positive number c. Then for every $\sigma\geq c$, there exists an entire function $T_\sigma\in E_\sigma$ such that*

1^0 *$T_\sigma(x)\geq f(x)$ for all $x\in\mathbf{R}$,*

2^0 *$V_p(T_\sigma-f)\leq c_3(r)\,\sigma^{-r}\,\nabla_\sigma(f^{(r)})_p$.*

P r o o f. Given any $\lambda>1$, let us choose an entire function $g_\sigma\in E_\sigma$, real-valued on $\mathbf{R}$ such that

$$V_p(f^{(r)}-g_\sigma^{(r)})\leq\lambda\nabla_\sigma(f^{(r)})_p\quad(\sigma\geq c). \tag{5}$$

Retain the symbols Φ_r, $P_{\sigma,r}$, $Q_{\sigma,r}$ used in Lemma 3.

By the well-known theorem ([1, Sect. 101]), for all real x,

$$f(x)-g_\sigma(x)=\Omega_c(x)+\int_{-\infty}^{\infty}\{f^{(r)}(t)-g_\sigma^{(r)}(t)\}\,\Phi_r(x-t)\,dt,$$

where Ω_c denotes some entire function of class E_c, real-valued on $\mathbf{R}$. Therefore, putting

$$\Lambda(z)=g_\sigma(z)+\Omega_c(z)\quad(z=x+iy)$$

and

$$h^+(t)=\frac{1}{2}\{|f^{(r)}(t)-g_\sigma^{(r)}(t)|+f^{(r)}(t)-g_\sigma^{(r)}(t)\},$$

$$h^-(t)=\frac{1}{2}\{|f^{(r)}(t)-g_\sigma^{(r)}(t)|-f^{(r)}(t)+g_\sigma^{(r)}(t)\},$$

we can write

$$f(x)=\Lambda(x)+\int_{-\infty}^{\infty} h^+(t)\,\Phi_r(x-t)\,dt-\int_{-\infty}^{\infty} h^-(t)\,\Phi_r(x-t)\,dt \quad (x\in \mathsf{R}).$$

Introduce the function of a complex variable z:

$$T_\sigma(z)=\Lambda(z)+\int_{-\infty}^{\infty} h^+(t)\,P_{\sigma,r}(z-t)\,dt-\int_{-\infty}^{\infty} h^-(t)\,Q_{\sigma,r}(z-t)\,dt.$$

It is easy to show that $T_\sigma\in E_\sigma$ (see the proof of Lemma 4).

The identity

$$T_\sigma(x)-f(x)=\int_{-\infty}^{\infty} h^+(t)\{P_{\sigma,r}(x-t)-\Phi_r(x-t)\}\,dt+\int_{-\infty}^{\infty} h^-(t)\{\Phi_r(x-t)-Q_{\sigma,r}(x-t)\}\,dt$$

ensures that $T_\sigma(x)\geqq f(x)$ for all real x. Furthermore, by Minkowski's inequality, (5) and Lemma 3,

$$\begin{aligned}V_p(T_\sigma-f)&\leqq V_p(h^+)\,\|P_{\sigma,r}-\Phi_r\|_1+V_p(h^-)\,\|\Phi_r-Q_{\sigma,r}\|_1\\&\leqq V_p(f^{(r)}-g_\sigma^{(r)})\,\{\|P_{\sigma,r}-\Phi_r\|_1+\|\Phi_r+Q_{\sigma,r}\|_1\}\\&\leqq\lambda\,\nabla_\sigma(f^{(r)})_p\,.\,2c_2(r,\,0)\,\sigma^{-r}.\end{aligned}$$

Thus, the proof is completed.

The following related result can be obtained parallelly (cf. Ths 3.2, 3.3 of [6], Th. 4.5 of [7] and Ths 3, 4 of [3]).

Theorem 1'. *Let f be a real-valued function of class AC_{loc}^{r-1}, with $f^{(r)}\in L_{\mathrm{loc}}^p$ $(1\leqq p<\infty)$, and let $A_c(f^{(r)})_p<\infty$ for some positive number c. Then for every $\sigma\geqq c$, there exists an entire function $\tilde{T}_\sigma\in E_\sigma$ such that*

1° $\tilde{T}_\sigma(x)\geqq f(x)$ *for all* $x\in\mathsf{R}$,

2° $\|\tilde{T}_\sigma-f\|_p\leqq c_4(r)\,\sigma^{-r}A_\sigma(f^{(r)})_p$.

Remark. Theorems 1, 1' in which the conditions 1° are dropped remain also valid for complex-valued functions f.

Proposition 1. *Let $\psi\in L$ and let $\psi'\in BV^p$ $(1\leqq p\leqq\infty)$. Suppose that for some entire function G of class E_σ $(0<\sigma<\infty)$ the estimate*

$$V_p(\psi-G)\leqq c_5\,\sigma^{-1}\,V_p(\psi') \tag{6}$$

holds. Then

$$V_p(\psi'-G')\leqq c_\sigma\,V_p(\psi'). \tag{7}$$

Proof. It can easily be observed that the function ψ is uniformly contionuous and bounded on R; whence $\psi\in L^\alpha$ for each $\alpha\geqq 1$. From (6) it follows that $G\in B_\sigma$.

Consider the operator W defined by (3). Since $W[\psi]\in B_{2\sigma,1}$, we have $W'[\psi]\in B_{2\sigma,1}$. Therefore, $W[\psi]\in BV^p$ and in view of Lemma 2,

$$\nabla_\sigma(W[\psi])_p\leqq V_p(W[\psi]-G)=V_p(W[\psi-G])\leqq c_1\,V_p(\psi-G),$$

i. e.

$$\nabla_\sigma(W[\psi])_p\leqq c_1c_5\,\sigma^{-1}\,V_p(\psi'). \tag{8}$$

Given any $\lambda>1$, let $S[W[\psi]]$ be an entire function of class B_σ such that

$$V_p(W[\psi]-S[W[\psi]])\leqq\lambda\,\nabla_\sigma(W[\psi])_p. \tag{9}$$

By subadditivity of p-th variation,

$$V_p(\psi'-G')\leq V_p(\psi'-W[\psi'])+V_p(S'[W[\psi]]-G')$$
$$+V_p(W[\psi']-S'[W[\psi]])=N_1+N_2+N_3,$$

and (see Lemma 2)

$$N_1\leq V_p(\psi')+V_p(W[\psi'])\leq(1+c_1)V_p(\psi').$$

From Lemma 1, (9), (8) and (6) it follows that

$$N_2\leq\sigma V_p(S[W[\psi]]-G)\leq\sigma\{V_p(S[W[\psi]]-W[\psi])$$
$$+V_p(W[\psi]-G)\}\leq\sigma\{\lambda\nabla_\sigma(W[\psi])_p+c_1V_p(\psi-G)\}$$
$$\leq\sigma\{\lambda c_1c_5\sigma^{-1}V_p(\psi')+c_1c_5\sigma^{-1}V_p(\psi')\}=(\lambda+1)c_1c_5V_p(\psi').$$

Since $W[\psi']=W'[\psi]$ $(W[\psi]\in B_{2\sigma})$, we have

$$N_3=V_p(W'[\psi]-S'[W[\psi]])$$
$$\leq 2\sigma V_p[W[\psi]-S[W[\psi]])\leq 2\sigma\lambda\nabla_\sigma(W[\psi])_p,$$

by Lemma 1 and (9). Applying (8), we get $N_3\leq 2\lambda c_1c_5V_p(\psi')$.

Thus,

$$V_p(\psi'-G')\leq(1+c_1)V_p(\psi')+(\lambda+1)c_1c_5V_p(\psi)'+2\lambda c_1c_5V_p(\psi'),$$

and passing to limit as $\lambda\to 1+$, we conclude that $V_p(\psi'-G')\leq(1+c_1+4c_1c_5)V_p(\psi')$.

This gives (7). Analogously, the following implication can also be proved (see the estimates (1.1), (2.3) and propos. 2.7 of [7]; cf. propos. of [9]).

Proposition 1′. *Let ψ be as in Proposition 1 with a finite $p\geq 1$. Suppose that for some entire function G of class E_σ $(0-\sigma<\infty)$,*

$$\|\psi-G\|_p\leq c_7\sigma^{-1-1/p}V_p(\psi').$$

Then

$$\|\psi'-G'\|_p\leq c_8\sigma^{-1/p}V_p(\psi').$$

Theorem 2. *Suppose that f is a real-valued function of class BV^p $(1\leq p<\infty)$. Then for every finite $\sigma>0$ there exists an entire function $T^*_\sigma\in B_\sigma$ satisfying the conditions:*

1° *$T^*_\sigma(x)\geq f(x)$ for all real x,*

2° *$\|T^*_\sigma-f\|_p\leq c_9\sigma^{-1/p}V_p(f)$,*

3° *$V_p(T^*_\sigma-f)\leq c_{10}V_p(f)$.*

The proof is similar to that of Theorem 3 in [8].

Theorem 3. *Let f be a real-valued function of class W^rBV^p $(r\in\mathbb{N}$, $1\leq p<\infty)$. Then for every finite $\sigma>0$ there exists an entire function $T_\sigma\in E_\sigma$ such that*

1° *$T_\sigma(x)\geq f(x)$ for all real x,*

2° *$\|T^{(\nu)}_\sigma-f^{(\nu)}\|_p\leq\frac{c_{11}(r,\nu)}{\sigma^{r-\nu+1/p}}V_p(f^{(r)})$,*

3° *$V_p(T^{(\nu)}_\sigma-f^{(\nu)})\leq\frac{c_{12}(r,\nu)}{\sigma^{r-\nu}}V_p(f^{(r)})$,*

where $\nu=0,1,\dots,r-1$. Moreover, in the case when $f^{(r-1)}\in L$, the estimates in 2° and 3° also hold for $\nu=r$.

Proof. In view of Theorem 2, there is an entire function $T^*_{\sigma,r} \in B_\sigma$ $(\sigma > 0)$, real-valued on R, satisfying the inequalities

$$\text{(10)} \qquad \begin{cases} \| T^*_{\sigma,r} - f^{(r)} \|_p \leq c_9 \, \sigma^{-1/p} V_p(f^{(r)}), \\ V_p(T^*_{\sigma,r} - f^{(r)}) \leq c_{10} V_p(f^{(r)}). \end{cases}$$

Suppose further that $\sigma \geq c > 0$. Retain the symbols Φ_r, $P_{\sigma,r}$, $Q_{\sigma,r}$ defined in Section 2, and start with the identities

$$f(x) = F_c(x) + \int_{-\infty}^{\infty} f^{(r)}(t) \, \Phi_r(x-t) \, dt$$

$$= F_c(x) + J_\sigma(x) + \int_{-\infty}^{\infty} \{ f^{(r)}(t) - T^*_{\sigma,r}(t) \} \, \Phi_r(x-t) \, dt \quad (x \in \mathrm{R}),$$

where F_c means some entire function of class E_σ and

$$J_\sigma(z) \equiv \int_{-\infty}^{\infty} \Phi_r(u) \, T^*_{\sigma,r}(z-u) \, du \quad (z = x+iy, \quad x, y \in \mathrm{R})$$

(see [1, Sect. 101]). It is easily seen that $J_\sigma \in B_\sigma$.

Introduce the auxiliary function

$$g(x) \equiv f(x) - F_c(x) - J_\sigma(x) = \int_{-\infty}^{\infty} \{ f^{(r)}(t) - T^*_{\sigma,r}(t) \} \, \Phi_r(x-t) \, dt;$$

write

$$h^+(t) \equiv \frac{1}{2} \{ | f^{(r)}(t) - T^*_{\sigma,r}(t) | + f^{(r)}(t) - T^*_{\sigma,r}(t) \},$$

$$h^-(t) \equiv \frac{1}{2} \{ | f^{(r)}(t) - T^*_{\sigma,r}(t) | - f^{(r)}(t) + T^*_{\sigma,r}(t) \}.$$

Then

$$g(x) = \int_{-\infty}^{\infty} h^+(t) \, \Phi_r(x-t) \, dt - \int_{-\infty}^{\infty} h^-(t) \, \Phi_r(x-t) \, dt \quad (x \in \mathrm{R}).$$

Putting

$$Y_\sigma(z) \equiv \int_{-\infty}^{\infty} h^+(t) \, P_{\sigma,r}(z-t) \, dt - \int_{-\infty}^{\infty} h^-(t) \, Q_{\sigma,r}(z-t) \, dt \quad (z = x+iy),$$

we have

$$\text{(11)} \qquad Y_\sigma(x) - g(x) = \int_{-\infty}^{\infty} h^+(t) \{ P_{\sigma,r}(x-t) - \Phi_r(x-t) \} \, dt$$

$$+ \int_{-\infty}^{\infty} h^-(t) \{ \Phi_r(x-t) - Q_{\sigma,r}(x-t) \} \, dt.$$

Therefore, $Y_\sigma \in B_{\sigma,p}$ and $Y_\sigma(x) \geq g(x)$ for all $x \in \mathrm{R}$.

Taking the entire function T_σ with values

$$\text{(12)} \qquad T_\sigma(z) \equiv F_c(z) + J_\sigma(z) + Y_\sigma(z),$$

we observe that

$$\text{(13)} \qquad T_\sigma(x) - f(x) = Y_\sigma(x) - g(x) \quad \text{for all } x \in \mathrm{R}.$$

Hence,

$$T_\sigma \in E_\sigma \text{ and } T_\sigma(x) \geqq f(x) \text{ on } \mathrm{R}.$$

From the identity (11) it follows that for each non-negative integer $\nu \leqq r-1$,

$$Y_\sigma^{(\nu)}(x) - g^{(\nu)}(x) = \int_{-\infty}^{\infty} h^+(t)\{P_{\sigma,r}^{(\nu)}(x-t) - \Phi_r^{(\nu)}(x-t)\}\,dt$$

$$+ \int_{-\infty}^{\infty} h^-(t)\{\Phi_r^{(\nu)}(x-t) - Q_{\sigma,r}^{(\nu)}(x-t)\}\,dt \quad (x \in \mathrm{R}).$$

Therefore, by Minkowski's inequalities and Lemma 3,

$$\| Y_\sigma^{(\nu)} - g^{(\nu)} \|_p \leqq \| h^+ \|_p \; \| P_{\sigma,r}^{(\nu)} - \Phi_r^{(\nu)} \|_1 + \| h^- \|_p \; \| \Phi_r^{(\nu)} - Q_{\sigma,r}^{(\nu)} \|_1$$

$$\leqq 2\, c_2(r, \nu)\, \sigma^{\nu-r} \| f^{(r)} - T_{\sigma,r}^* \|_p.$$

Consequently (see (13) and (10)),

$$\| T_\sigma^{(\nu)} - f^{(\nu)} \|_p = \| Y_\sigma^{(\nu)} - g^{(\nu)} \|_p \leqq 2\, c_2(r, \nu)\, c_9\, \sigma^{\nu-r-1/p}\, V_p(f^{(r)}).$$

Since

$$V_p(Y_\sigma^{(\nu)} - g^{(\nu)}) \leqq V_p(h^+) \| P_{\sigma,r}^{(\nu)} - \Phi_r^{(\nu)} \|_1$$

$$+ V_p(h^-) \| \Phi_r^{(\nu)} - Q_{\sigma,r}^{(\nu)} \|_1 \leqq 2\, c_2(r, \nu)\, \sigma^{\nu-r} V_p(f^{(r)} - T_{\sigma,r}^*),$$

we have

$$V_p(T_\sigma^{(\nu)} - f^{(\nu)}) = V_p(Y_\sigma^{(\nu)} - g^{(\nu)}) \leqq 2\, c_2(r, \nu)\, c_{10}\, \sigma^{\nu-r} V_p(f^{(r)}).$$

Thus, for T_σ defined by (12), the inequalities occuring in 1^0 and 2^0—3^0 (with non-negative $\nu \leqq r-1$) are proved.

Assuming that $f^{(r-1)} \in L$ and applying propositions 1 and 1′, we get at once the desired assertion for $\nu = r$.

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