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CONCERNING TRIVIAL MAXIMAL ABELIAN SUBALGEBRAS OF $B(X)$

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To the memory of Y. A. Tagamlitzki

We call a complex or real Banach algebra trivial, if it is either a Banach space with trivial (zero) multiplication or it is the unitization of such an algebra. Thus a trivial algebra is always commutative and in the case of an algebra with unit element it is a local ring, i. e. it has exactly one maximal ideal equal to its radical. In this paper we prove that for any real or complex Banach space X the algebra $B(X)$ of all its continuous endomorphisms has always a trivial maximal Abelian subalgebra and we give description of all such subalgebras.

Let X be a real or complex Banach space. For a non-void subset S of $B(X)$ denote by S' its commutant, i. e. the set

$$S' = \{T \in B(X) : TA = AT \text{ for all } A \text{ in } S\}.$$

It is a closed subalgebra of $B(X)$ containing its unity I , and in case when S consists of mutually commuting operators we have

$$S' = \bigcup \{ \mathcal{A} : \mathcal{A} \text{ is a maximal Abelian subalgebra of } B(X) \text{ with } S \subset \mathcal{A} \}.$$

This implies that S' is a maximal Abelian subalgebra of $B(X)$, provided it is commutative. This simple remark will be used in the proof of our theorem.

In the sequel we denote by X^* the conjugate space of a Banach space X and by T^* the conjugate operator of an element T in $B(X)$. We put also $\text{rad } \mathcal{A}$ for the radical of a commutative Banach algebra \mathcal{A} . Thus in case of a trivial algebra \mathcal{A} with unit element we have $\mathcal{A} = \text{rad } \mathcal{A} \oplus KI$, where K is the field of scalars ($K = \mathbb{C}$ or $K = \mathbb{R}$) and KI is the one-dimensional subspace of \mathcal{A} spanned by the unit element I .

Since for $\dim X \leq 1$ the whole algebra $B(X)$ is commutative and trivial, we assume in our result that $\dim X > 1$. In this case we say that a closed linear subspace X_0 of X is proper, if $\{0\} \neq X_0 \neq X$. For an operator A in $B(X)$ denote by $\ker A$ its kernel and by $\text{im } A$ its range, i. e. the sets $\ker A = \{x \in X : Ax = 0\}$ and $\text{im } A = \{Ax : x \in X\}$. Our result reads as follows

Theorem. *Let X be a real or complex Banach space with $\dim X > 1$ and let X_0 be a proper closed linear subspace of X . Then the set*

$$(1) \quad \{A \in B(X) : \text{im } A \subset X_0 \text{ and } X_0 \subset \ker A\}$$

is a trivial Abelian subalgebra of $B(X)$ and its unitization \mathcal{A} is a trivial maximal Abelian subalgebra of $B(X)$.

Conversely, if \mathcal{A} is a trivial maximal Abelian subalgebra of $B(X)$, then its radical $\text{rad } \mathcal{A}$ is of the form (1), where

$$(2) \quad X_0 = \bigcap \{ \ker A : A \in \text{rad } \mathcal{A} \}.$$

Proof. Denote by M the set (1). Obviously, it is a trivial subalgebra of $B(X)$. Let T be an operator in the commutant M' . For a functional f in X^*

with $X_0 \subset \ker f$ and for an element z in X_0 denote by $A(f, z)$ the one-dimensional operator given by $A(f, z)x = f(x)z$, this operator is clearly in M and so it commutes with T . Thus, for all x in X , all z in X_0 and all f in X^* with $X_0 \subset \ker f$ we have

$$(3) \quad f(Tx)z = f(x)Tz.$$

Choosing $f_0 \neq 0$ with $X_0 \subset \ker f_0$ and substituting for x in (3) an element x_0 in X with $f_0(x_0) = 1$ we obtain

$$Tz = \alpha_T z$$

for all z in X_0 , where α_T is the scalar given by $\alpha_T = f_0(Tx_0)$. Put $T_1 = T - \alpha_T I$. We have $T_1 \in M$ and $X_0 \subset \ker T_1$. We shall show that the operator T_1 is in M , i. e. $\text{im } T_1 \subset X_0$. If not, then there is an element u_0 in X with $T_1 u_0 \notin X_0$ and we can find an element A in M with $AT_1 u_0 \neq 0$ (A can be chosen to be of the form $A(f, z)$). But this is impossible, since $AT_1 u_0 = T_1 A u_0$ and $A u_0 \in X_0 \subset \ker T_1$. Thus, T_1 is in M and so T is in its unitization \mathcal{A} which is a commutative algebra and thus a maximal Abelian subalgebra of $B(X)$ since it equals M .

Conversely, suppose that \mathcal{A} is a trivial maximal Abelian subalgebra of $B(X)$ and put $M = \text{rad } \mathcal{A}$. For any two operators T_1 and T_2 in M we have $\text{im } T_1 \subset \ker T_2$, and so $\text{im } T_1 \subset X_0$, where X_0 is given by (2). Since $X_0 \subset \ker T_1$ and T_1 is an arbitrary element of M , it follows that M is contained in the set (1). By the maximality of \mathcal{A} M equals to this set, and so $\text{rad } \mathcal{A}$ is of the form (1). The conclusion follows.

Corollary 1. Any subset S of $B(X)$ consisting of mutually annihilating operators (i. e. $T_1 T_2 = 0$ for all T_i in S , $i = 1, 2$) is contained in some trivial maximal Abelian subalgebra of $B(X)$. In particular, any trivial subalgebra of $B(X)$ is contained in a trivial maximal Abelian subalgebra of $B(X)$.

Denote by $\mathcal{A}(X_0)$ the trivial maximal Abelian subalgebra of $B(X)$ whose radical is (1).

Corollary 2. The algebra $\mathcal{A}(X_0)$ is isomorphic as a Banach space to the space $B(X/X_0, X_0) \oplus K$, where $B(U, V)$ denotes the Banach space of all continuous linear operators from a Banach space U to a Banach space V and K is the field of scalars (the one-dimensional Banach space).

Examples. Taking as X_0 any subspace of X of codimension one, we obtain a trivial maximal Abelian subalgebra $\mathcal{A}(X_0)$ isomorphic as a Banach space to the space X . Its radical consists of one-dimensional operators of the form $A(f_0, z)$, where f_0 is a fixed functional in X with $\ker f_0 = X_0$ and $z \in X_0$. The isomorphism between $\mathcal{A}(X_0)$ and X is given by

$$A(f_0, z) + \lambda I \leftrightarrow z + \lambda e_0,$$

where e_0 is a fixed element in X with $f_0(e_0) = 1$.

Similarly, taking as X_0 a linear subspace of X of dimension one $X_0 = Kx_0$ with $x_0 \in X$ and $\|x_0\| = 1$, we obtain an algebra $\mathcal{A}(X_0)$ isomorphic as a Banach space to the conjugate space X^* . It consists of all operators of the form $A(f, x_0) + \lambda I$, where $f \in X^*$ with $x_0 \in \ker f$ and $\lambda \in K$. The Banach space isomorphism between $\mathcal{A}(x_0)$ and X^* is given by

$$A(f, x_0) + \lambda I \leftrightarrow f + \lambda f_0,$$

where f_0 is a fixed element in X^* with $f_0(x_0) = 1$.

In case when the space X has a direct sum decomposition $X = X_0 \oplus X_1$, where X_1 is isomorphic to X_0 , then the algebra $\mathcal{A}(X_0)$ is isomorphic as a Banach space to the space $B(X_0)$. In particular when $X = H$ — an infinite-

dimensional Hilbert space, then H can be orthogonally decomposed as $H = H_0 \oplus H_1$, where H_0 and H_1 are isometrically isomorphic to H . In this case the algebra $\mathcal{A}(H_0)$ is isomorphic as a Banach space to the space $B(H)$. It can be proved that in this case the operators in $\mathcal{A}(H_0)$ are of the following form. Let R be a partial isometry on H , which maps H_1 isometrically onto H_0 and maps H_0 onto $\{0\}$. Then

$$\mathcal{A}(H_0) = \{RA + AR : A \in B(H) \text{ and } RAR = \alpha(A)R\},$$

where $\alpha(A)$ is a scalar depending upon A . It can be shown that if $RA + AR = RA_1 + A_1R$, then $\alpha(A) = \alpha(A_1)$ and so it defines on $\mathcal{A}(H_0)$ a functional f given by $f(RA + AR) = \alpha(A)$. It is a multiplicative linear functional on $\mathcal{A}(H_0)$ and its kernel equals to $\text{rad } \mathcal{A}(H_0)$ (we have $\alpha(R^*) = 1$ and $R^*R + RR^* = I$).

If $\dim X = n < \infty$, then by Corollary 2 the possible dimensions of algebras $\mathcal{A}(X_0)$ are $(n-k) \cdot k + 1$, $k = 1, 2, \dots, n-1$, and so there are $[\frac{n}{2}]$ non-isomorphic trivial maximal Abelian subalgebras of $B(X)$, where $[r]$ is the integral part of a number r . The largest possible dimension of $\mathcal{A}(X_0)$ is in this case $[\frac{n^2}{4}] + 1$ and the smallest dimension is n . All these results in the case of finite dimensional spaces are well known even for more general scalars (cf. [2, Chapt. 2, § 3]), however, in the case of real or complex scalars our reasoning seems to be shorter. In case when X is a Hilbert space the maximal Abelian subalgebras of $B(X)$ which are local rings are known in the literature (cf. [1], or [3, p. 81, proposition 4.4]), however, the existence of such trivial algebras seems to be new and somewhat surprising. We finish this paper with some simple results on invariant subspaces for algebras $\mathcal{A}(X_0)$. In the sequel we denote by $\text{lin}(X)$ the family of all closed linear subspaces of a Banach space X , and for a subset S of $B(X)$ we denote by $\text{lat}(S)$ the set (it has a structure of a lattice) of all subspaces in $\text{lin}(X)$ which are invariant with respect to all operators in S . In case when S consists of a single operator T we simply write $\text{lat}(T)$.

Proposition 1. *Let X be a Banach space with $\dim X > 1$. Then*

$$(4) \quad \text{lat}(\mathcal{A}(X_0)) = \{Y \in \text{lin}(X) : \text{either } X_0 \subset Y, \text{ or } Y \subset X_0\},$$

where X_0 is a proper linear subspace of X .

Proof. It is clear that all subspaces in the family (4) are invariant with respect to all operators in $\mathcal{A}(X_0)$. On the other hand, if Y is a closed linear subspace of X which contains some element $x_0 \notin X_0$ and does not contain some element $z_0 \in X_0$, then it cannot be invariant with respect to all elements in $\mathcal{A}(X_0)$, since there always exists an operator of the form $A(f, z_0)$ which sends x_0 to z_0 . The conclusion follows.

A subalgebra \mathcal{A} of $B(X)$ is said to be reflexive (sf. [3]), if the condition $\text{lat}(\mathcal{A}) \subset \text{lat}(T)$ implies $T \in \mathcal{A}$.

Proposition 2. *Let H be a Hilbert space, $\dim H > 1$, then no trivial maximal Abelian subalgebra of $B(H)$ is reflexive.*

Proof. For a closed proper linear subspace H_0 of H denote by $P(H_0)$ the orthogonal projection of H onto H_0 . Clearly, we have $\text{lat}(\mathcal{A}(H_0)) \subset \text{lat}(P(H_0))$ and $P(H_0) \notin \mathcal{A}(H_0)$. The conclusion follows.

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