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DIRECT AND CONVERSE THEOREMS FOR SPLINE APPROXIMATION WITH FREE KNOTS

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A new modulus is introduced by means of which it is possible to obtain direct and converse theorems for spline approximation by spline functions with free knots, in the uniform metric.

In this paper we give the complete proofs of the propositions announced in [10]. We shall consider the spline approximations of functions in the uniform metric. Let us denote by $S(k, n)$ the class of all spline functions in the interval $[0, 1]$ of degree k with $n+1$ knots, i. e., $s\in S(k, n)$ if $s\in C^{k-1}[0, 1]$ ($C^r[a, b]$ denotes as usually the set of all functions which have r continuous derivatives in the interval $[a, b]$) and there exist $n+1$ points x_i, $i=0, \dots, n$, $0=x_0<x_1<\cdots<x_n=1$, such that in each interval $[x_{i-1}, x_i]$, $i=1, \dots, n$, s is an algebraic polynomial of degree at most k. In the case $k=0$ $S(0, n)$ coincides with the class of all step functions with $n-1$ jumps. We then suppose that s is continuous either on the right, or on the left. The spline functions were introduced in the theory of approximation by I. J. Schoenberg [1]. There exists an extensive bibliography for spline approximation [2, 11].

We shall consider also spline functions with a defect. These spline functions of degree k and $n+1$ knots are piecewise polynomial functions s without the restriction $s\in C^{k-1}[0, 1]$. The class of all spline functions with a defect of degree k and with $n+1$ knots will be denoted by $\tilde{S}(k, n)$, i. e. $s\in\tilde{S}(k, n)$ if there exist $n+1$ points x_i, $i=0, \dots, n$, $0=x_0<x_1<\cdots<x_n=1$, such that in each interval $[x_{i-1}, x_i]$, $i=1, \dots, n$, s is an algebraic polynomial of degree k. In the case $k=0$ we have $\tilde{S}(0, n)=S(0, n)$.

We shall consider the best uniform approximation $E_n^k(f)$ of the function f in the interval $[0, 1]$ by means of spline functions of $S(k, n)$:

$$E_n^k(f)=\inf_{s\in S(k,n)}\ \sup_{x\in[0,1]}|f(x)-s(x)|$$

and the best uniform approximation $\tilde{E}_n^k(f)$ of the function f by means of spline functions of $\tilde{S}(k, n)$:

$$\tilde{E}_n(f)=\inf_{s\in\tilde{S}(k,n)}\ \sup_{x\in[0,1]}|f(x)-s(x)|.$$

It is easy to see [3] that the following lemma is valid:

Lemma 1. *Let $f\in C[0,1]$. Then there exists a constant $c(k)$ depending only on k such that*

(1) $$\tilde{E}^k_m(f) \leq E^k_m(f) \leq c(k)\tilde{E}^k_n(f),$$

where $m=(n-1)k+n$.

Obviously for every n, $\tilde{E}^0_n(f)=E^0_n(f)$.

Many authors consider the problem to find the classes of functions which can be characterized by their best uniform approximation by means of spline functions, i. e., the theorems of Bernstein-Jackson's type (see for example [3—8]). But in all these cases either the direct theorems do not coincide with the converses or the theorems are obtained with some restrictions (for example on the knots of the spline functions).

It is well known now that the spline approximations with free knots cannot be characterized by the usual moduli of continuity. In this paper we introduce new moduli by means of which it is possible to characterize the corresponding classes by the best uniform approximation $E^k_n(f)$.

1. Denote by V the class of all functions with bounded variation in the interval $[0, 1]$ with a variation ≤ 1 which are continuous either on the right, or on the left.

For every function f defined on the interval $[0, 1]$ we define the modulus $\nu_k(f, \delta)$ as follows:

$$\nu_k(f, \delta) = \inf_{\varphi \in V} \sup_{|\varphi(x+kh)-\varphi(x)| \leq \delta} |\Delta^k_h f(x)|,$$

where as usual $\Delta^k_h f(x)$ denotes the kth difference of the function f with a step h at the point x:

$$\Delta^k_h f(x) = \sum_{m=0}^{k} (-1)^{k+m} \binom{k}{m} f(x+mh)$$

and the sup is over all x and h, for which $|\varphi(x+kh)-\varphi(x)| \leq \delta$, $x, x+kh \in [0, 1]$.

Let us mention some of the properties of the moduli $\nu_k(f, \delta)$:

1.1 $\nu_k(f, \delta)$ is a monotone increasing function of δ, i. e., $\nu_k(f, \delta_1) \leq \nu_k(f, \delta_2)$ if $\delta_1 \leq \delta_2$.

1.2. If f is a continuous function then the inf in the definition of $\nu_k(f, \delta)$ can be taken only on the continuous functions belonging to V.

1.3. $\nu_k(f, \delta) \leq \omega_k(f, \delta)$, where

$$\omega_k(f, \delta) = \sup_{x,\, x+kh \in [0, 1]} |\Delta^k_h f(x)|$$

is the kth modulus of continuity of the function f. This property follows immediately from the fact that the function $\varphi(x)=x$ belongs to V. Properties 1.1 and 1.2 are evident.

1.4. The inf in the definition of $\nu_k(f, \delta)$ can be taken only on the monotone increasing functions belonging to V.

P r o o f. Let $\varphi \in V$. For the function $\psi(x) = V^x_0\varphi \in V$, where $V^x_0\varphi$ denote the variation of the function φ in the interval $[0, x]$ we have:

$$\sup_{|\varphi(x+kh)-\varphi(x)| \leq \delta} |\Delta^k_h f(x)| \geq \sup_{V^{x+kh}_x \varphi \leq \delta} |\Delta^k_h f(x)| \geq \sup_{|\psi(x+kh)-\psi(x)| \leq \delta} |\Delta^k_h f(x)|.$$

From this inequality follows the proposition.

1.5. If $k>r$ then $\nu_k(f,\delta)\leq 2^{k-r}\nu_r(f,\delta)$.

Proof. Let $\varphi\in V$ be monotone increasing function in the interval [0, 1]. We have:

$$\sup_{\varphi(x+kh)-\varphi(x)\leq\delta}|\Delta_h^k f(x)|=\sup_{\varphi(x+kh)-\varphi(x)\leq\delta}\left|\sum_{l=0}^{k-r}(-1)^{l+k-r}\binom{k-r}{l}\Delta_h^r f(x+lh)\right|$$

$$\leq\sum_{l=0}^{k-r}\binom{k-r}{l}\sup_{\varphi(x+kh)-\varphi(x)\leq\delta}|\Delta_h^r f(x+lh)|$$

$$\leq\sum_{l=0}^{k-r}\binom{k-r}{l}\sup_{\varphi(x+(l+r)h)-\varphi(x)\leq\delta}|\Delta_h^r f(x+lh)|$$

$$\leq\sum_{l=0}^{k-r}\binom{k-r}{l}\sup_{\varphi(x+rh)-\varphi(x)\leq\delta}|\Delta_h^r f(x)|=2^{k-r}\sup_{\varphi(x+rh)-\varphi(x)\leq\delta}|\Delta_h^r f(x)|.$$

Using property 1.4 from here we obtain

$$\nu_k(f,\delta)=\inf_{\varphi\in V}\sup_{|\varphi(x+kh)-\varphi(x)|\leq\delta}|\Delta_h^k f(x)|$$

$$\leq 2^{k-r}\inf_{\varphi\in V}\sup_{|\varphi(x+rh)-\varphi(x)|\leq\delta}|\Delta_h^r f(x)|=2^{k-r}\nu_r(f,\delta)$$

1.6. If there exists the derivative $f^{(r)}$ and $k>r$ then

$$\nu_k(f,\delta)\leq(2\delta/k)^r\nu_{k-r}(f^{(r)},2\delta).$$

Proof. Let $\varepsilon>0$ be arbitrary δ be given and $\varphi\in V$ be monotone increasing in [0, 1], such that

$$\sup_{\varphi(x+(k-r)h)-\varphi(x)\leq 2\delta}|\Delta_h^{k-r}f^{(r)}(x)|\leq\nu_{k-r}(f^{(r)},2\delta)+\varepsilon.$$

Since the function $(\varphi(x)+x)/2\in V$ and

$$\Delta_h^k f(x)=\int_0^h\ldots\int_0^h\Delta_h^{k-r}f^{(r)}(x+t_1+\cdots+t_r)dt_1\ldots dt_r$$

we have

$$\nu_k(f,\delta)\leq\sup{}^*|\Delta_h^k f(x)|\leq\sup{}^*\int_0^h\ldots\int_0^h|\Delta_h^{k-r}f^{(r)}(x+t_1+\ldots+t_r)|dt_1\ldots dt_r$$

$$\leq\sup{}^*\int_0^h\ldots\int_0^h\sup{}^{**}|\Delta_h^{k-r}f^{(r)}(x+t_1+\cdots+t_r)|dt_1\ldots dt_r$$

$$\leq\left(\frac{2\delta}{k}\right)^r(\nu_{k-r}(f^{(r)};2\delta)+\varepsilon),$$

where the sup* is over $(\varphi(x+kh)-\varphi(x)+kh)/2\leq\delta$, and sup** over $\varphi(x+t_1+\cdots+t_r+(k-r)h)-\varphi(x+t_1+\cdots+t_r)\leq 2\delta$.

Since $\varepsilon>0$ is arbitrary, the property 1.6 follows.

Lemma 2. *If the function f has bounded variation in the interval $[0, 1]$, $V_0^1 f\leq V_f$, then $\nu_1(f, \delta)\leq(V_f)\delta$.*

Proof. Let us set $\varphi(x)=V_0^x f/V_0^1 f\in V$. We have

$$\nu_1(f, \delta)=\inf_{\varphi\in V}\sup_{|\varphi(x+h)-\varphi(x)|\leq\delta}|f(x+h)-f(x)|$$

$$\leq\sup_{\psi(x+h)-\psi(x)\leq\delta}|f(x+h)-f(x)|=\sup_{V_x^{x+h}f/V_0^1 f\leq\delta}|f(x+h)-f(x)|\leq(V_f)\delta.$$

Lemma 3. *Let the function f have kth derivative $f^{(k)}$ with bounded variation, $V_0^1 f^{(k)}\leq V_{f^{(k)}}$. Then*

$$\nu_{k+1}(f, \delta)\leq 2^{k+1}(k+1)^{-k}(V_{f^{(k)}})\delta^{k+1}.$$

Proof. From the property 1.6 and Lemma 2 follows

$$\nu_{k+1}(f, \delta)\leq\left(\frac{2\delta}{k+1}\right)^k\nu_1(f^{(k)}, 2\delta)$$

$$\leq\left(\frac{2\delta}{k+1}\right)^k(V_{f^{(k)}})2\delta=2_{k+1}(k+1)^{-k}(V_{f^{(k}})\delta^{k+1}.$$

Let us notice that there exist functions f with unbounded variation such that $\nu_1(f, \delta)=O(\delta)$. For example the function f defined in the interval $[0, 1]$ by

$$f(x)=\begin{cases}1/n & \text{for } x\in((2n+1)/n(n+1), 1/n),\\ 0 & \text{for } x\in(1/(n+1), (2n+1)/n(n+1)],\\ 0 & \text{for } x=0\end{cases}$$

(n — positive integer number).

But if f is continuous, then from $\nu_1(f, \delta)=O(\delta)$ follows that f has bounded variation (see [12]).

2. Now we shall prove the basic theorem.

Theorem 1. *Let $f\in C[0, 1]$. Then there exists a constant $N(k)$ depending only on k such that for all natural numbers $k>0$, $n>0$ the following inequalities are valid:*

$$\nu_{k+1}(f, 1/n)\leq 2^{k+1}E_n^k(f), \tag{2}$$

$$E_n^k(f)\leq N(k)\nu_{k+1}(f, (k+1)/n). \tag{3}$$

For every function f we have

$$\nu_1(f, 1/n)=2E_n^0(f). \tag{4}$$

Proof. In view of Lemma 1 in order to prove (2), (3) it is sufficient to prove

$$\nu_{k+1}(f, 1/n)\leq 2^{k+1}\widetilde{E}_n^k(f), \tag{5}$$

$$\widetilde{E}_n^k(f)\leq M(k)\nu_{k+1}(f, 1/n), \tag{6}$$

where $M(k)$ is a constant depending only on k.

Let us first prove (5). Let $\varepsilon>0$ be arbitrary, $k\geq 1$ and $s\in\tilde{S}(k, n)$ is such that

$$\sup_{x\in[0,1]}|f(x)-s(x)|\leq\tilde{E}_n^k(f)+\varepsilon. \tag{7}$$

Let the knots of s be x_i, $i=x_0, 1, \ldots, n$, $0=x_0<\cdots<x_n=1$. In each interval $[x_{i-1}, x_i]$ s is an algebraic polynomial of degree k. We define the function $\varphi\in V$ in the following way: φ is a monotone step function with jumps $1/(n-1)$ at the points x_i, $i=1, \ldots, n-1$.

Let us estimate $\nu_{k+1}(f, \delta)$ for $\delta<1/(n-1)$. We have:

$$\nu_{k+1}(f, \delta)\leq\sup_{|\varphi(x+(k+1)h)-\varphi(x)|\leq\delta}|\Delta_h^{k+1}f(x)|. \tag{8}$$

But in order to have $|\varphi(x+(k+1)h)-\varphi(x)|\leq\delta<1/(n-1)$ by the construction of φ it is necessary that the points x, $x+(k+1)h$ belong to one and the same interval $[x_{i-1}, x_i]$. Since s is in this interval an algebraic polynomial of degree k,

$$\Delta_h^{k+1}s(x)=0. \tag{9}$$

Therefore we obtain from (7)—(9) for $\delta<1/(n-1)$

$$\nu_{k+1}(f, \delta)\leq\sup_{|\varphi(x+(k+1)h)-\varphi(x)|\leq\delta}|\Delta_h^{k+1}f(x)|$$

$$\leq\sup_{|\varphi(x+(k+1)h)-\varphi(x)|\leq\delta}\{|\Delta_h^{k+1}(f(x)-s(x))+\Delta_h^{k+1}s(x)|\}\leq 2^{k+1}(E_n^k(f)+\varepsilon). \tag{10}$$

Since $\varepsilon>0$ is arbitrary, (10) proves (5).

Let us prove now (6). Let $\varepsilon>0$ be arbitrary, $k\geq 1$ and $\varphi\in V$ such that

$$\sup_{|\varphi(x+(k+1)h)-\varphi(x)|\leq 1/n}|\Delta_h^{k+1}f(x)|\leq\nu_{k+1}(f, 1/n)+\varepsilon.$$

In view of property 1.4 we may suppose that φ is monotone increasing, $\varphi(0)=0$.

Let us consider the sets A_i:

$$A_i=\{x: x\in[0, 1],\quad (i-1)/n\leq\varphi(x)\leq i/n\},\quad i=1, \ldots, n.$$

The set A_i may be empty, in every other case A_i is an interval (not necessary closed). Moreover, $\bigcup_{i=1}^{n}A_i=[0, 1]$.

If $x, x+(k+1)h\in A_i$, then $\varphi(x+(k+1)h)-\varphi(x)\leq 1/n$ and therefore

$$|\Delta_h^{k+1}f(x)|\leq\nu_{k+1}(f; 1/n)+\varepsilon. \tag{11}$$

Using the continuity of the function f, (11) holds for the closed interval $\bar{A}_i$.

Using one theorem of H. Whitney [9], from (11) follows that there exists a constant $M(k)$ depending only on k and an algebraic polynomial $s_i(x)$ of degree k, interpolating the function f at the ends of the interval A_i, such that

(12) $$\max_{x \in \bar{A}_i} |f(x)-s(x)| \leq M(k)\,(\nu_{k+1}(f;\,1/n)+\varepsilon).$$

Let us define $s \in \tilde{S}(k, n)$ as follows:

$$s(x)=s_i(x) \quad \text{for} \quad x \in \bar{A}_i.$$

From (12) we obtain

(13) $$\max_{x \in [0,\,1]} |f(x)-s(x)| \leq M(k)\,(\nu_{k+1}(f,\,1/n)+\varepsilon).$$

Since $\varepsilon>0$ is arbitrary, from (13) follows (6).

Let us eventually obtain (4). The proof of the inequality

$$\nu_1\left(f,\,1/n\right) \leq 2E_n^0(f)$$

is the same as the proof of (2).

To prove

$$E_n^0(f) \leq \frac{1}{2}\,\nu_1(f,\,1/n)$$

using the above construction of the intervals A_i, we set

$$s_i(x)=(\sup_{x \in A_i} f(x)+\inf_{x \in A_i} f(x))/2 \quad \text{for} \quad x \in A_i.$$

Obviously the function $s \in S(0, n)$ defined by $s(x)=s_i(x)$ for $x \in A_i$ satisfies

$$\sup_{x \in [0,\,1]} |f(x)-s(x)| \leq \frac{1}{2}\,\nu_2(f,\,1/n).$$

Bl. Sendov called my attention to the fact that from Theorem 1 and Lemma 2 follows easily one result of G. Freud and the author [13]: $E_n^k(f) \leq c(k)V_0^1 f^{(k)}/n^{k+1}$, where the constant $c(k)$ depends only on k.

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