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ON A THEOREM OF H. HOPF

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Single-valued and multi-valued mappings of finite dimensional sphere into Euclidean space are considered. The dimension of sets of points which are identified with respect to such mappings is estimated.

Given a mapping $f: S^{n-1} \to R^l$ and $0<\theta<\pi$, the question discussed in this paper is how large can be the set of points

$$B_\theta(f)=\{(x, y)\in S^{n-1}\times S^{n-1} \mid xy=\cos\theta,\quad f(x)=f(y)\}.$$

The following theorem follows from [7]:

Theorem H. *For $l\leq n-1$ the set $B_\theta(f)$ is not empty.*

We consider more generally the multi-valued mapping $F: S^{n-1}\to R^l$ and our purpose is to prove that

$$\dim\{(x, y)\in S^{n-1}\times S^{n-1} \mid xy=\cos\theta, F(x)\cap F(y)\neq\emptyset\}\geq 2(n-l)-3$$

for $l\leq n-2$.

If the mapping F is a single-valued one then the following inequality holds: $\dim B_\theta(f)\geq 2n-l-3$ for $l\leq n-2$.

1. Preliminaries

Definition 1. *The space X is called a free Z_2-space if a free involution $T: X\to X$ acts on X, i. e., there is a continuous single-valued mapping $T: X\to X$ such that:* a) *$Tx\neq x$ for every $x\in X$, and* b) *$T^2x=x$ for every $x\in X$.*

If R^n is the n-dimensional Euclidean space and $S^{n-1}=\{x=(x_1, \dots, x_n)\in R^n \mid \Sigma x_i^2=1\}$ is the unit sphere in R^n, then we shall consider S^{n-1} as a free Z_2-space with the involution $T: S^{n-1}\to S^{n-1}$ given by $T(x_1, \dots, x_n)=(-x_1, \dots, -x_n)$ for every $(x_1, \dots, x_n)\in S^{n-1}$.

Let X be a free Z_2-space. By $\tilde{X}$ we shall denote the orbit space of the involution $T: X\to X$, i. e., the space $\tilde{X}$ is the identification space of the space X with respect to the identification $x\sim y$ if and only if $Tx=y$. The natural projection $\pi: X\to\tilde{X}$ is the mapping $\pi(x)=\{x, Tx\}$ for $x\in X$.

Theorem 1 ([1], Ch. 3, § 4) *There is an exact sequence*

$$0\to H^0(\tilde{X})\xrightarrow{\pi^*} H^0(X)\xrightarrow{\sigma^0} H^0(\tilde{X})\xrightarrow{\delta^0(x)} H^1(\tilde{X})\to\cdots \tag{1}$$

$$\to H^i(\tilde{X})\xrightarrow{\pi^i} H^i(X)\xrightarrow{\sigma^i} H^i(\tilde{X})\xrightarrow{\delta^i(x)} H^{i+1}(\tilde{X})\to\cdots$$

for every free Z_2-space.

The sequence (1) is called Smith's exact sequence of the free Z_2-space X.

Here and later all cohomology will be Alexandroff—Čech cohomology with Z_2-coefficients [2, Ch. 8].

In the sequence (1) the homomorphisms $\pi^i: H^i(\tilde{X}) \to H^i(X)$ are induced by the projection $\pi: X \to \tilde{X}$; the homomorphisms $\sigma^i: H^i(X) \to H^i(\tilde{X})$ are the so-called transfer homomorphisms [1, Ch. 3, § 2].

Definition 2. *The single-valued continuous mapping $f: X_1 \to X_2$ of a free Z_2-space X_1 in a free Z_2-space X_2 is called an equivariant mapping if $fT = Tf$.*

Every single-valued equivariant mapping $f: X_1 \to X_2$ induces a single-valued continuous mapping $\tilde{f}: \tilde{X}_1 \to \tilde{X}_2$ such that $\pi_2 \tilde{f} = \tilde{f}\pi_1$, where $\pi_i: X_i \to \tilde{X}_i$ is the natural projection of the space X_i onto its orbit space $\tilde{X}_i$, $i=1,2$.

The following lemma gives us the naturality of Smith's exact sequence.

Lemma 1. [3, Appendix B, § 15, for the homology]. *Let X_i be free Z_2-spaces with orbit spaces $\tilde{X}_i$ and projections π_i, $i=1,2$. If $f: X_1 \to X_2$ is an equivariant continuous single-valued mapping and $\tilde{f}: \tilde{X}_1 \to \tilde{X}_2$ is induced by the mapping f, then the following diagram is commutative:*

$$
\begin{array}{ccccccccc}
\xrightarrow{\sigma^i} & H^i(\tilde{X}_2) & \xrightarrow{\partial^i(X_2)} & H^{i+1}(\tilde{X}_2) & \xrightarrow{\pi_2^{i+1}} & H^{i+1}(X_2) & \xrightarrow{\sigma^{i+1}} & H^{i+1}(\tilde{X}_2) & \to \cdots \\
(2) & \tilde{f}^i \downarrow & & \tilde{f}^{i+1} \downarrow & & f^{i+1} \downarrow & & \tilde{f}^{i+1} \downarrow & \\
\xrightarrow{\sigma^i} & H^i(X_1) & \xrightarrow{\partial^i(X_1)} & H^{i+1}(\tilde{X}_1) & \xrightarrow{\pi_1^{i+1}} & H^{i+1}(X_1) & \xrightarrow{\sigma^{i+1}} & H^{i+1}(\tilde{X}_1) & \to \cdots
\end{array}
$$

In (2) the horizontal exact sequences are Smith's sequences of the free Z_2-spaces X_i, $i=1,2$. The homomorphisms $f^* = \{f^i\}$, $\tilde{f}^* = \{\tilde{f}^i\}$ are induced by the mappings f and $\tilde{f}$ respectively.

The naturality of Smith's exact sequence is a very useful property. Another tool needed is Yang's homomorphism [9].

Suppose that X is a free Z_2-space and F a closed subset in X such that $F \cup T(F) = X$. The closed set $B = F \cap T(F)$ is a free Z_2-space, by $\tilde{B}$ we shall denote the orbit space of B. The identity inclusion $k: B \to X$ is an equivariant mapping which induces the mapping $\tilde{k}: \tilde{B} \to \tilde{X}$.

Lemma 2 ([9]). *There are homomorphisms (Yang's homomorphisms) $\Delta^i: H^i(\tilde{B}) \to H^{i+1}(\tilde{X})$ for every i, such that*

$$\partial^i(X) = \Delta^i \tilde{k}^i. \tag{3}$$

Definition 3. *Smith's homomorphism $s_{i,j}(X): H^i(\tilde{X}) \to H^{i+j}(\tilde{X})$ of the free Z_2-space X is the homomorphism*

$$s_{i,j}(X) = \partial^{j-1}(X) \ldots \partial^i(X).$$

Corollary 1. *By the assumptions of Lemma 1, it follows:*

$$\textit{if } s_{i,j}(X) \neq 0, \textit{ then } s_{i,j-1}(B) \neq 0.$$

Indeed, from Lemmas 1 and 2 we obtain $s_{i,j}(X) = \Delta^{j-1} s_{i,j-1}(B)\tilde{k}_i$.

We shall use this corollary later to prove the main theorem.

Example. $s_{i,n-1}(S^{n-1}) \neq 0$ for $0 \leq i \leq n-1$.

Remarks. The projection $\pi: X \to \tilde{X}$ is a two-sheet covering space. Every such covering space (in the case of paracompact X) is given by a pull back diagram

$$(4)\qquad \begin{array}{ccc} X & \xrightarrow{\Phi} & S^N \\ \pi \downarrow & & \downarrow p \\ \tilde{X} & \xrightarrow{\varphi} & RP^N \end{array}$$

where N is sufficiently large when X is compact or $N=\infty$, otherwise (Φ, φ) is the mapping of the covering space: $(X, \tilde{X})$ on the standard covering space (S^N, RP^N) (RP^N is the real projective space of dimension N).

Let $\alpha \in H^1(RP^N)$ be the non-trivial cohomology class. It is well known that $H^*(RP^N)=Z_2[\alpha]/\alpha^{N+1}=0$ (in the case of $N<\infty$) or $H^*(RP^N)=Z_2[\alpha]$ in the case of $N=\infty$. Here $Z_2[\alpha]$ is the polynomial ring over Z_2 with one generator α. The cohomology class $\varphi^*(\alpha)$ is called the characteristic class of the covering space $(X, \tilde{X})$. For $0\leq i<j$, $s_{i,j}(X)\varphi^*(\alpha^i)=\varphi^*(\alpha^j)$. Smith's homomorphism $s_{i,j}(X)$ is not zero if and only if $\varphi^*(\alpha^j)\neq 0$. The maximum of the integers j such that $\varphi^*(\alpha^j)\neq 0$ is called Smith's index of the free Z_2-space X.

Actually, two sheet covering spaces over the space X are classified by the set $[X, RP^\infty]$ of all homotopy classes of mappings of the space X in the infinite real projective space RP^∞ (if X is a paracompact space). Having in mind that infinite dimension real projective space RP^∞ is the space of Eilenberg-MacLane $K(Z_2, 1)$ and the theorem that $H^1(X)=[X, RP^\infty]$ ([11]), it is obvious that the characteristic class $\varphi^*(\alpha)$ of the covering space $(X, \tilde{X})$ characterise the covering space $(X, \tilde{X})$ up to an equivalence of covering spaces.

2. An involution of Stiefel's manifold of two frames in R^n.

Definition 4. *Stiefel's manifold V of two frames in $R^n\times R^n$ is the space* $V=\{(x, y)=(x_1, \ldots, x_n, y_1, \ldots, y_n)\in R^n\times R^n \mid \Sigma x_i^2=\Sigma y_i^2=1,\ \Sigma x_i y_i=0\}$.

The space V is a compact $2n-3$ dimensional manifold without boundary.

Lemma 3 ([5]). *$H^i(V)=Z_2$ for $i=0, n-2, n-1, 2n-3$, and $H^j(V)=0$ for $j\neq 0, n-2, n-1, 2n-3$.*

We shall consider the following fixed point free involution $T: V\to V$: $T(x_1, \ldots, x_n, y_1, \ldots, y_n)=(x_1, \ldots, x_n, -y_1, \ldots, -y_n)$ for $(x_1, \ldots, x_n, y_1, \ldots, y_n)$. By $\tilde{V}$ we shall denote the orbit space of the involution $T: V\to V$. The manifold $\tilde{V}$ is the manifold of all non-oriented line elements on the sphere S^{n-1}. Its cohomology is well known too [6].

Lemma 4. *In the exact sequence (1) of the free Z_2-space V the homomorphism $\partial^{n-1}(V)$ is an epimorphism and the homomorphisms $\partial^j(V)$ are isomorphisms for $n\leq j\leq 2n-4$.*

It follows from (1) and Lemma 3 that $\partial^i(V)$ are isomorphisms for $n\leq i\leq 2n-5$ and $\partial^{n-1}(V)$ is an epimorphism.

Let us consider the right-hand side of the exact sequence (1)

$$(5)\qquad 0\to H^{2n-4}(\tilde{V})\xrightarrow{\partial^{2n-4}(V)} H^{2n-3}(\tilde{V})\xrightarrow{\pi^{2n-3}} H^{2n-3}(V)\xrightarrow{\sigma^{2n-3}} H^{2n-3}(\tilde{V})\to 0.$$

The group $H^{2n-3}(V)$ is isomorphic to the group Z_2 (Lemma 3), and from (5) we obtain that $H^{2n-3}(\tilde{V})=Z_2$, hence the homomorphisms σ^{2n-3} and $\partial^{2n-4}(V)$ are isomorphisms.

For given real $0<\theta<\pi$ we shall consider the set $A_\theta=\{(x, y)=(x_1, \ldots, x_n, y_1, \ldots, y_n) \in R^n \times R^n \mid \Sigma x_i^2=\Sigma y_i^2=1, \Sigma x_i y_i=\cos \theta, i=1, \ldots, n\}$. If $\theta=\pi/2$, then $A_\theta=V$.

The space A_θ is a free Z_2-space with respect to the involution $T: A_\theta \to A_\theta$ given by $T(x_1, \ldots, x_n, y_1, \ldots, y_n)=(y_1, \ldots, y_n, x_1, \ldots, x_n)$ for every $(x_1, \ldots, x_n, y_1, \ldots, y_n) \in A_\theta$.

Lemma 5. *There is an equivariant homeomorphism* $h: A_\theta \to V$

$$h(x_1, \ldots, x_n, y_1, \ldots, y_n)=(\lambda(x_1+y_1), \ldots, \lambda(x_n+y_n),$$

$$\mu(y_1-x_1), \ldots, \mu(y_n-x_n)),$$

where $\lambda=(2[1+\cos \theta])^{-1/2}$ *and* $\mu=(2[1-\cos \theta])^{1/2}$.

Corollary 2. *Smith's exact sequence of the free Z_2-space A_θ is isomorphic to Smith's exact sequence of the free Z_2-space V.*

Corollary 3. *In the exact sequence (1) of the free Z_2-space A_θ the homomorphisms $\partial^i(A_\theta)$ are isomorphisms for $n \leq i \leq 2n-4$ and $\partial^{n-1}(A_\theta)$ is an epimorphism.*

3. Single-valued continuous mappings of S^{n-1} in R^l, $l \leq n-2$. For a given single-valued continuous mapping $f: S^{n-1} \to R^l$ we set

$$B_\theta(f)=\{(x, y) \in A_\theta \mid f(x)=f(y)\}.$$

Lemma 6. *If $l \leq n-2$, then the set $B_\theta(f)$ is not empty.*

This lemma follows from [7]. Here we shall give another proof. Suppose that the set $B_\theta(f)=0$, i. e., for every $(x, y) \in A_\theta$ it follows $f(x) \neq f(y)$. Let $\varphi: V \to S^{l-1}$ be given by

$$\varphi(u, v)=(f(x(u, v))-f(y(u, v)))\,\|f(x(u, v))-f(y(u, v))\|^{-1},$$

where

$$(6)\qquad \begin{aligned} x(u, v)&=(\varrho(\mu u_1-\lambda v_1), \ldots, \varrho(\mu u_n-\lambda v_n)),\\ y(u, v)&=(\varrho(\mu u_1+\lambda v_1), \ldots, \varrho(\mu u_n+\lambda v_n)) \end{aligned}$$

for $u=(u_1, \ldots, u_n)$, $v=(v_1, \ldots, v_n)$, $\varrho=(2 \sin \theta)^{-1}$ and λ and μ are given in 2.

Let $S^{n-2}=\{x=(x_1, \ldots, x_n) \in S^{n-1} \mid x_n=0\}$ and $\psi: S^{n-2} \to V$ be the mapping $\psi(x_1, \ldots, x_{n-1})=(1, 0, \ldots, 0, x_1, \ldots, x_{n-1})$ for $(x_1, \ldots, x_{n-1}) \in S^{n-2}$.

The mapping $\varphi\psi: S^{n-2} \to S^{l-1}$ has the following properties:

a) $\varphi\psi$ is a single-valued continuous mapping,

b) $\varphi\psi(-x_1, \ldots, -x_{n-1})=-\varphi\psi(x_1, \ldots, x_{n-1})$ for every point $(x_1, \ldots, x_{n-1}) \in S^{n-2}$,

c) $l-1<n-2$.

It follows from [8] that such mapping does not exist, i. e., we have contradiction, hence $B_\theta(f) \neq \emptyset$.

Theorem 2. *Let $f: S^{n-1} \to R^l$ be a single-valued continuous mapping. If $0<\theta<\pi$ and $l \leq n-2$, then* $\dim B_\theta(f) \geq 2n-l-3$.

Here dim is the covering dimension ([2, Ch. 3]).

The set $B_\theta(f)$ is a free Z_2-space with respect to the involution $T: B_\theta(f) \to B_\theta(f)$ given by $T(x, y)=(y, x)$ for $(x, y) \in B_\theta(f)$. By $\widetilde{B}_\theta(f)$ we denote the orbit space of $B_\theta(f)$.

Lemma 7. $H^{2n-l-3}(\widetilde{B}_\theta(f)) \neq 0$.

This lemma immediately implies Theorem 2. Indeed, suppose that $H^{2n-l-3}(\widetilde{B}_\theta(f))\neq 0$, then $\dim \widetilde{B}_\theta(f)\geq 2n-l-3$ ([2, ch. 8]). The projection $B_\theta(f)\to\widetilde{B}_\theta(f)$ is a local homeomorphism and the spaces $B_\theta(f)$ and $\widetilde{B}_\theta(f)$ are compact spaces, hence $\dim B_\theta(f)\geq 2n-l-3$.

Let us prove Lemma 7. The mapping $f\colon S^{n-1}\to R^l$ is given by $f=(f_1, \ldots, f_l)$, where $f_i\colon S^{n-1}\to R^1$ are real-valued functions, $i=1,\ldots,l$.

Let us consider the sets $C_0=A_\theta$, $D_0=C_0$, $D_i=\{(x,y)\in A_\theta \mid f_s(x)=f_s(y)$ for every $1\leq s\leq i\}$ and $C_i=\{(x,y)\in D_{i-1} \mid f_i(x)\geq f_i(y)\}$ for $1\leq i\leq l$. The sets D_i and C_i have the following properties:

a) $C_i\subset D_{i-1}$, $D_i\subset C_i$;

b) D_i and C_i are compact spaces;

c) D_i are free Z_2-spaces;

d) $C_i\cap T(C_i)=D_i$;

e) $C_i\cup T(C_i)=D_{i-1}$;

f) $D_l=B_\theta(f)$.

Now we are in a position to prove that $s_{n-1,2n-3-i}(D_i)\neq 0$. We shall prove this by induction with respect to the integer i. For $i=0$ we have $s_{n-1,2n-3}(D_0)\neq 0$ (Lemma 4, Corollary 2). Suppose that for $0\leq i<l$ we have $s_{n-1,2n-3-i}(D_i)\neq 0$. It follows from Corollary 1 that

$$s_{n-1,2n-3-i}(D_i)=A^{2n-4-i}(D_{i+1})s_{n-1,2n-4-i}(D_{i+1})\widetilde{k}^{n-1}.$$

Therefore $s_{n-1,2n-4-i}(D_{i+1})\neq 0$ and the lemma is proved.

4. The single-valued mappings of S^{2k} in R^{2k} — another proof of the theorem H. The points of s-dimensional real projective space RP^s are $(s+1)$-tuples $[u_1:u_2:\cdots:u_{s+1}]$ such that $\Sigma u_i^2=1$ $[u_1:\cdots:u_{s+1}]=[v_1:\cdots:v_{s+1}]$ if and only if $u_i=\varepsilon v_i$, where $\varepsilon=\pm 1$ for $1\leq i\leq s+1$. If $s_1<s_2$ there is an inclusion $j\colon RP^{s_1}\to RP^{s_2}$ given by $j[u_1:\cdots:u_{s_1}]=[u_1:u_2:\cdots:u_{s_1}:0:\cdots:0]$ for $[u_1:\cdots:u_{s_1}]\in RP^{s_1}$.

Let us consider the mapping $\varphi\colon RP^{2k}\to V_{2k+1}$ given by $\varphi([u_1:\cdots:u_{2k+1}])=(a_1(u), a_2(u))$, where

$$a_1(u)=(-2u_{2k+1}u_1, \ldots, -2u_{2k+1}u_{2k}, 1-2u_{2k+1}^2),$$

$$a_2(u)=(-2u_{2k}u_1, \ldots, -2u_{2k}u_{2k-1}, 1-2u_{2k}^2, -2u_{2k}u_{2k+1})$$

for every $u=[u_1:\cdots:u_{2k+1}]$.

The image of RP^{2k-2} by the mapping φ coincides with the point $(0,0,\ldots,0,1;0,\ldots,0,1,0)$. Let $P_{2k,2k-2}$ be the factor space of RP^{2k} in which RP^{2k-2} is collapsed into a point. The mapping φ induces a mapping $\hat{\varphi}$ of the space $P_{2k,2k-2}$ into the space V_{2k+1}. It is straightforward to see that $\hat{\varphi}$ is a homeomorphism of the space $P_{2k,2k-2}$ into V_{2k+1}.

The standard cell structure of RP^{2k} induces a cell structure of $P_{2k,2k-2}=\tau^0\cup\tau^{2k-1}\cup\tau^{2k}$ (τ^i is the i-dimensional cell) [13]. Let $i\colon S^{2k-1}\to P_{2k,2k-2}$ be the characteristic mapping of the cell τ^{2k-1}. The mapping $\tilde{\varphi}i\colon S^{2k-1}\to V^{2k+1}$ has the following properties:

a) $\tilde{\varphi}i$ is a homeomorphism;

b) the image of the mapping $\tilde{\varphi}i$ coincides with the set

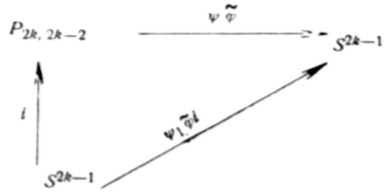
$$S_1^{2k-1}=\{(0,\ldots,0,1;v_1,\ldots,v_{2k},0)\in V_{2k+1}\}.$$

Suppose that for some single-valued continuous mapping $f: S^{2k}\to R^{2k}$ and $0<\theta<\pi$ the set $B_\theta(f)=\{(u,v)\in A_\theta \mid f(u)=f(v)\}$ is empty. The mapping $\psi: V_{2k+1}\to S^{2k-1}$ given by

$$\psi(u,v)=(f(x(u,v))-f(y(u,v)))\,|f(x(u,v))-f(y(u,v))|^{-1},$$

where $x(u,v)$ and $y(u,v)$ are given in (6), has an odd degree on the sphere S_1^{2k-1}. Indeed, the mapping $\psi_1=\psi|_{S_1^{2k-1}}$ is an odd mapping. Therefore the degree of the mapping $\psi_1\tilde{\varphi}i$ is not zero, *i.* e., the homomorphism $(\psi_1\tilde{\varphi}i)_{2k-1}$ is not zero. Now we need the following commutative diagram

(7)



It follows from diagram (7) that the homomorphism

$$(\psi\tilde{\varphi})_{2k-1}: H_{2k-1}(P_{2k,2k-2}, Z)\to H_{2k-1}(S^{2k-1}, Z)$$

is not zero. But the group $H_{2k-1}(S^{2k-1}, Z)=Z$ is isomorphic to the group of all integers Z and $H_{2k-1}(P_{2k,2k-2}, Z)$ is isomorphic to Z_2, hence $(\psi\tilde{\varphi})_{2k-1}\neq 0$ is impossible. Therefore $B_\theta(f)$ is not empty for every f.

5. The single-valued mappings $f: S^{2k}\to R^{2k}$ — a second proof of the theorem H. Let X be a free Z_2-space. The index SI (X) (Smith's index of the Z_2-space X) is defined as follows $\mathrm{SI}(X)=\max\{i \mid s_{0,i}(x)\neq 0\}$ (see also remark in 1).

On the space V_{2k+1} we shall consider a free involution $T: V_{2k+1}\to V_{2k+1}$ given by $T(u,v)=(u,-v)$ for every $(u,v)\in V_{2k+1}$.

L e m m a 8 ([10, Ch. 2, § 5]) $\mathrm{SI}(V_{2k+1})=2k$.

As soon as V_{2k+1} and A_θ are equivariant homotopic, then the following corollary is true.

C o r o l l a r y 4. $\mathrm{SI}(A_\theta^{2k+1})=2k$, if $0<\theta<\pi$.

For a single-valued continuous mapping $f: S^{2k}\to R^{2k}$ we consider the mapping $\varphi: A_\theta^{2k+1}\to R^{2k}$ given by $\varphi(u,v)=f(u)$ for $(u,v)\in A_\theta^{2k+1}$. It follows from [4] that the set $B_\theta(f)=\{(u,v)\in A_\theta^{2k+1} \mid \varphi(u,v)=\varphi(v,u)\}$ is not empty.

6. Multi-valued acyclic mappings $F: S^{n-1}\to R^l$, $0\leq l\leq n-2$.

D e f i n i t i o n 5. *The multi-valued mapping $\psi: X\to Y$ is said to be an acyclic mapping if*: a) *ψ is upper semicontinuous,* b) *the set $\psi(x)$ is an acyclic compact set for every $x\in X$.*

Let us remind that the compact set K is called acyclic if K is connected and $H_i(K)=0$ for $i\geq 1$.

D e f i n i t i o n 6. *For the multi-valued acyclic mapping $\psi: X\to Y$ by* $\dim\psi$ *we shall denote* $\max(\dim\psi(x), x\in X)$.

Theorem 3. *Let* $F: S^{n-1} \to R^l$ *be an acyclic multi-valued mapping and* $l \leq n-2$. *If* $B_\theta(F) = \{(u, v) \in A^n_\theta \mid F(u) \cap F(v) \neq \Phi\}$, *then* $\dim_{Z_2} B_\theta(F) \geq 2n - l - \dim F - 3$. *Here* $0 < \theta < \pi$.

Corollary 5. *In the assumptions of Theorem 3 it follows that*

$$\dim_{Z_2} B_\theta(F) \geq 2(n-l) - 3.$$

It is obvious that $\dim F \leq l$. Then the following assertion is true.

Corollary 6. *In the assumptions of Theorem 3*

$$\dim B_\theta(F) \geq 2(n-l) - 3.$$

To prove Theorem 3, let us consider the mapping $\Phi: A^n_\theta \to R^l \times R^l$ given by $\Phi(u, v) = F(u) \times F(v)$ for every $(u, v) \in A^n_\theta$. The mapping Φ is an acyclic one.

The graph $\Gamma(\Phi) = \{(u, v, x, y) \in A^n_\theta \times R^l \times R^l \mid x \in F(u), y \in F(v)\}$ of the mapping Φ is a compact set and the projection $p: \Gamma(\Phi) \to A^n_\theta (p(u, v, x, y) = (u, v), (u, v, x, y) \in \Gamma(\Phi))$ has the following properties: (a) p is a closed mapping; (b) $p^{-1}(u, v) = F(u) \times F(v)$ for every $(u, v) \in A^n_\theta$. It follows from Vietoris-Begles theorem ([12]) that the homomorphism $p^*: H^*(A^n_\theta) \to H^*(\Gamma(\Phi))$ is an isomorphism.

There is an involution $T: \Gamma(\Phi) \to \Gamma(\Phi): T(u, v, x, y) = (v, u, y, x)$ for $(u, v, x, y) \in \Gamma(\Phi)$. The mapping $p: \Gamma(\Phi) \to A^n_\theta$ is obviously an equivariant one. Therefore, the mapping p induces a mapping $\tilde{p}: \tilde{\Gamma}(\Phi) \to \tilde{A}^n_\theta$ of the orbit space of $\Gamma(\Phi)$ in the orbit space of A^n_θ. The mapping $\tilde{p}$ has the following properties: a) $\tilde{p}$ is a closed mapping; b) the set $\tilde{p}^{-1}(\xi_0)$ is homeomorphic to the set $F(u_0) \times F(v_0)$ for $\xi_0 \in \tilde{A}^n_\theta$ and $(u_0, v_0) \in \xi_0$.

Again from Vietoris-Begle's theorem ([12]) we obtain that the homomorphism $\tilde{p}^*: H^*(\tilde{A}^n_\theta) \to H^*(\tilde{\Gamma}(\Phi))$ is an isomorphism.

Lemma 8. *The homomorphisms* $\partial^i(\Gamma(\Phi))$ *are isomorphisms for* $n \leq i \leq 2n-5$ *and the homomorphism* $\partial^{n-1}(\Gamma(\Phi))$ *is an epimorphism.*

The equivariant mapping $p: \Gamma(\Phi) \to A^n_\theta$ induces the following commutative diagram

$$\begin{array}{ccccccccc}
\to & H^i(\tilde{A}^n_\theta) & \to & H^i(A^n_\theta) & \to & H^i(\tilde{A}^n_\theta) & \xrightarrow{\partial^i(A^n_\theta)} & H^{i+1}(\tilde{A}^n_\theta) & \to \cdots \\
(8) & \tilde{p}^i \downarrow & & p^i \downarrow & & \tilde{p}^i \downarrow & & \tilde{p}^{i+1} \downarrow & \\
\to & H^i(\tilde{\Gamma}(\Phi)) & \to & H^i(\Gamma(\Phi)) & \to & H^i(\tilde{\Gamma}(\Phi)) & \xrightarrow{\partial^i(\Gamma(\Phi))} & H^{i+1}(\tilde{\Gamma}(\Phi)) & \to \cdots
\end{array}$$

The horizontal exact sequences are Smith's sequences of $\Gamma(\Phi)$ and A^n_θ. The vertical homomorphisms are isomorphisms.

From Corollary 3 and (6) we obtain Lemma 8.

Let us consider the single-valued continuous mapping $f: \Gamma(\Phi) \to R^l$ given by $f(u, v, x, y) = x$ for every $(u, v, x, y) \in \Gamma(\Phi)$. The set

$$B(f) = \{(u, v, x, y) \in \Gamma(\Phi) \mid f(u, v, x, y) = f(v, u, y, x)\}$$

is a closed invariant set in the space $\Gamma(\Phi)$. Therefore the set $B(f)$ is a compact free Z_2-space.

Denoting by $\tilde{B}(f)$ the orbit space of $B(f)$ we have

Lemma 9. *If $l \leq n-2$, then $H^{2n-3-l}(\tilde{B}(f)) \neq 0$.*

It can be proved as Lemma 7.

Lemma 10. *If $l \leq n-2$, then* $\dim_{Z_2} B(f) \geqq 2n-l-3$.

The natural projection $q: B(f) \to \tilde{B}(f)$ is a local homeomorphism. Since $B(f)$ is a compact space, then $\dim_{Z_2} B(f) = \dim_{Z_2} \tilde{B}(f)$. From Lemma 9 we have $H^{2n-l-3}(\tilde{B}(f)) \neq 0$, hence $\dim_{Z_2} B(f) \geqq 2n-l-3$.

Lemma 11. *The mapping $r: B(f) \to B_\theta(F)$ given by $r = p \mid B(f)$ has the following properties*: a) r *is closed*; b) $\dim_{Z_2} r^{-1}(x) \leq \dim F(x) \leq \dim F$; c) $r(B(f)) = B_\theta(F)$.

It follows from Lemma 11 that we can apply to the mapping r the Hurewicz theorem for decreasing dimension mappings ([13, Ch. 4, § 7]). It follows from this theorem that

$$\dim_{Z_2} B_\theta(F) \geqq \dim_{Z_2} B(f) - \max\{\dim_{Z_2} r^{-1}(\xi), \xi \in B_\theta(F)\} \geqq 2n-l-3-\dim F.$$

7. Multi-valued mappings $F: S^{2k} \to R^{2k}$.

Theorem 4. *Let $F: S^{2k} \to R^{2k}$ be a multi-valued acyclic mapping. If $0 < \theta < \pi$, then the set $B_\theta(F) = \{(u, v) \in A_\theta^{2k+1} \mid F(u) \cap F(v) \neq \emptyset\}$ is not empty.*

Let us consider the multi-valued acyclic mapping $\Phi: A_\theta^{2k+1} \to R^{2k} \times R^{2k}$ given by $\Phi(u, v) = F(u) \times F(v)$ for $(u, v) \in A_\theta^{2k+1}$.

The compact set $\Gamma(\Phi) = \{(u, v, x, y) \in A_\theta^{2k+1} \times R^{2k} \times R^{2k} \mid (x, y) \in F(u) \times F(v)\}$ is a free Z_2-space with respect to the involution $T: \Gamma(\Phi) \to \Gamma(\Phi)$, $T(u, v, x, y) = (v, u, y, x)$, $(u, v, x, y) \in \Gamma(\Phi)$.

Using the method of 6 we can prove the following

Lemma 12. $\mathrm{SI}(\Gamma(\Phi)) = \mathrm{SI}(A_\theta^{2k+1})$.

Corollary 7. $\mathrm{SI}(\Gamma(\Phi)) = 2k$.

It follows from [9] that the set $C = \{(u, v, x, y) \in \Gamma(\Phi) \mid \varphi(u, v, x, y) = \varphi(v, u, y, x)\}$ is not empty (where $\varphi: \Gamma(\Phi) \to R^{2k}$ is the mapping given by $\varphi(u, v, x, y) = x$ for $(u, v, x, y) \in \Gamma(\Phi)$).

Let $q: C \to B_\theta(F)$ be the mapping given by $q(u, v, x, y) = (u, v)$ for $(u, v, x, y) \in C$. The set C is not empty, hence the set $B_\theta(F)$ is not empty.

REFERENCES

1. A. Borel. Seminar on transformation groups. Princeton, 1960.
2. W. Hurewicz, H. Wallman. Dimension theory. Princeton, 1944.
3. S. Lefschetz. Algebraic topology. Princeton, 1942.
4. C. Yang. On theorems of Borsuk-Ulam, Kakutani-Yamabe-Yojobo and Dyson I. *Ann. Math.*, **60**, 1954, 262—282.
5. E. Stiefel. Richtungfelder und Fernparallelismus in *n*-dimensionalen Mannifaltigkeiten. *Comment. Math. Helv.*, **8**, 1935—1936, 3—51.
6. T. Kiang. The manifolds of linear elements of *n*-sphere. *Bull. Amer. Math. Soc.*, **51**, 1941, 417—428.
7. H. Hopf. Eine Verallgemeinerung bekannter Abbildungs- und Überdeckungssätze. *Port. Math.*, **4**, 1944, 130—139.
8. K. Borsuk. Drei Sätze über die *n*-dimensionale Euklidische Sphäre. *Fund. Math.*, **20** 1933, 177—190.
9. G. Skordev. On a theorem of Bourgin-Yang. *Serdica*. 1. 1975, No. 2, 183—197.
10. Wu Wentsün. A theory of imbedding, immersion and isotopy of polytopes, in an Euclidean space. Peking, 1965.
11. P. Huber. Homotopical cohomology and Čech cohomology. *Math. Ann.*, **144**, 1961.
12. G. Bredon. Sheaf theory. New York, 1967.
13. J. Whitehead. On the groups $\pi_r(V_{n,m})$ and sphere bundles. *Proc. London Math. Soc.*, (2), **49**, 1947, 474—481.

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