Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

POINTS OF SINGLE-VALUEDNESS OF MULTIVALUED MONOTONE MAPPINGS IN FINITE DIMENSIONAL SPACES

PETAR S. KENDEROV

The multivalued mapping $T:E\to E'$ from the Banach space E to its conjugate E' is said to be monotone if $\langle x_1-x_2, y_1-y_2\rangle \geq 0$ whenever $y_i\in T(x_i)$ $x_i\in E$ $i=1, 2$. As usual $\langle x, y\rangle$ denotes the value of the continuous linear functional $y\in E'$ at the point $x\in E$.

The main aim of this note is to show that for the majority of points $x\in E$ the set $T(x)$ contains not more than one point. This was the subject of another author's paper [5] where it was proved that for a large class of Banach spaces the set $A=\{x\in E: T(x)$ has more than one point$\}$ is of first category in E. This means that the set A is "topologically small". When E is finite-dimensional $(E=R^n)$ we can ask whether the Lebesgue measure of the same set A is also small. It turned out (theorem 3) that A is a nullset. Thus every multivalued monotone mapping $T:R^n\to R^n$ is single-valued almost everywhere.

As an ilustration one special multivalued mapping is considered. The so-called metric projection $P_M:R^n\to M$ which is defined by a closed subset $M\subset R^n$ and the formula $P_M(x)=\{y\in M: \|x-y\|=\min_{z\in M}\|x-z\|\}$. This mapping is monotone (proposition 2) and therefore single-valued almost everywhere. The classical result of Reidemeister [8] that every convex function $h:R^n\to R$ is differentiable almost everywhere is also a consequence of our theorem.

We will need some definitions and results.

The monotone mapping $T:E\to E'$ is called maximal if there does not exist a monotone mapping $T_1:E\to E'$ whose graph $G(T_1)=\{(x, y)\in E\times E': y\in T_1(x)\}$ properly contains the graph of T. According to the Zorn's lemma, for any monotone mapping $T:E\to E'$ there exists a maximal monotone one $\tilde{T}:E\to E'$ such that $T(x)\subset \tilde{T}(x)$ whenever $x\in E$. For our purposes it is therefore sufficient to consider only maximal monotone mappings.

By dom T we will denote the set $\{x\in E: T(x)\neq\emptyset\}$.

Theorem 1. (T. Kato [4], R. Rockafellar [9]). *Every maximal monotone mapping $T:R^n\to R^n$ with* dom $T=R^n$ *is locally bounded. That is, for $x_0\in R^n$ a neighbourhood V of x_0 exists such that the set $T(V)=\bigcup_{x\in V} T(x)$ is bounded.*

In the sequel we will always assume that dom $T=R^n$.

Theorem 2 (F. Browder [3]). *Let $T:R^n\to R^n$ be a maximal monotone mapping. Then the graph $G(T)=\{(x, y)\in R^n\times R^n: y\in T(x)\}$ is closed and the set $T(x)$ is convex and compact for every $x\in R^n$.*

Corollary 1. *Every maximal monotone mapping $T:R^n\to R^n$ is upper semi-continuous (that is, for any open $U\supset T(x_0)$ a neighbourhood V of x_0 exists such that $T(x)\subset U$ whenever $x\in V$).*

Proof. Assuming that T is not upper semi-continuous at x_0 we find an open U and two sequences $\{x_i\}_{i=1}^{\infty}$, $\{y_i\}_{i=1}^{\infty}$ for which $U\supset T(x_0)$, $x_i\to x_0$ and $y_i\in T(x_i)\cap(R^n\setminus U)$. By Theorem 1 $\{y_i\}_{i=1}^{\infty}$ is a bounded sequence and we can choose a convergent subsequence. Suppose that $y_i \to y_0$. Then the closed set $G(T)$ contains $(x_i, y_i) \to (x_0, y_0)$. Thus (x_0, y_0) belongs to $G(T)$. Therefore $y_0 \in T(x_0)$. On the other hand, y_0 is in $R^n \setminus U$ since the last set is closed and contains the sequence $y_i \to y_0$. Then $y_0 \in T(x_0) \cap (R^n \setminus U) = \emptyset$ which is impossible. The proof is completed.

We need several notions from the convex analysis. Let K be a convex compact subset of R^n. Its support function $h_K(e)$ is defined by the formula [2]: $h_K(e) = \max_{z \in K} \langle e, z \rangle$. Here $\langle e, z \rangle$ is the usual scalar product of the vectors $z \in K$ and $e \in S = \{e \in R^n : \| e \| = 1\}$. It can be easily shown that
1) $h(e)$ is continuous on S,
2) $K_1 \subset K_2$ (both compact and convex) if and only if $h_{K_1}(e) \leq h_{K_2}(e)$ for every $e \in S$, and
3) $h_K(e) + \varepsilon$, where $\varepsilon > 0$ is the support function of the set $B_\varepsilon(K) = \{x \in R^n : \min_{y \in K} \| x - y \| \leq \varepsilon\}$.

There is one simple criterion for the set K to be a singleton. Denote by ν the usual measure on S.

P r o p o s i t i o n 1. *Let $K \subset R^n$ be compact and convex. Then the following three conditions are equivalent*:

a) *K is a one-point set,*

b) $$\int_S h_K(e)\, \nu(de) = 0,$$

c) $$\int_S (h_K(e) + h_K(-e))\, \nu(de) = 0.$$

P r o o f. First of all we note that for every summable function $f(e)$ $\int_S f(e)\, \nu(de) = \int_S f(-e)\, \nu(de)$. In particular

$2 \int_S h_K(e)\, \nu(de) = \int_S (h_K(e) + h_K(-e))\, \nu(de)$. Hence b) and c) are equivalent. Having in view that the function $h_K(e) + h_K(-e)$ is continuous and non-negative $(h_K(e) + h_K(-e) = \max_{z \in K} \langle e, z \rangle - \min_{z \in K} \langle e, z \rangle \geq 0)$ we get that c) is equivalent to $h_K(e) + h_K(-e) \equiv 0$, i. e. $\max_{z \in K} \langle e, z \rangle = \min_{z \in K} \langle e, z \rangle$ for all e from S. This is just the case when K is a single point set. So a) is equivalent to c). The proof is finished.

R e m a r k. The meaning of b) is quite clear in case of $n = 2$ because the Euclidean perimeter of the set K is just $\int_S h_K(e)\, \nu(de)$.

Returning to the maximal monotone mapping $T : R^n \to R^n$ we define a function on $R^n \times S$ putting $u(x, e) = h_{T(x)}(e)$. (According to Theorem 2 the set $T(x)$ is compact and convex.)

L e m m a 1. *For any real number r the set $\{(x, e) \in R^n \times S : u(x, e) < r\}$ is open.*

P r o o f. Let $u(x_0, e_0) = h_{T(x_0)}(e_0) < r$. Since $h_{T(x_0)}$ is a continuous function at the point e_0 we have $h_{T(x_0)}(e) + \varepsilon < r$ for some $\varepsilon > 0$ and for all e from a certain S-neighbourhood $V_1 \ni e_0$. By the upper semicontinuity of T at x_0 there exists an R^n-neighbourhood V_2 of x_0 such that $T(x) \subset B_\varepsilon(T(x_0))$ for all $x \in V_2$. Equivalently, by means of support functions, the same can be expressed as follows: $h_{T(x)}(e) \leqq h_{T(x_0)}(e) + \varepsilon$ for all e from S and x from V_2. In particular, for any $(x, e) \in V_2 \times V_1$, $u(x, e) = h_{T(x)}(e) \leqq h_{T(x_0)}(e) + \varepsilon < r$.

L e m m a 2. *For any fixed* e *from* S *the set* $N = \{x \in R^n : u(x, e) + u(x, -e) \neq 0\}$ *is a nullset with respect to the usual Lebesgue measure* μ *on* R^n.

P r o o f. It is enough to prove that $\mu(N \cap \{x \in R^n : \|x\| \leqq k\}) = 0$ for any integer k. Denote the last set by N_k. Due to Lemma 1 it is measurable. Then the function $\psi(x)$ which is 1 on N_k and 0 outside is measurable and bounded. Evidently $\mu(N_k) = \int\limits_{R^n} \psi(x)\, \mu(dx)$. Let I be the one-dimensional subspace of R^n with e as unit vector and let L be the orthocomplement of I. Then the n-dimensional measure μ of R^n is a product of the 1-dimensional measure μ_1 of I and the $n-1$-dimensional measure μ_{n-1} of L. By Fubini's theorem $\mu(N_k) = \int\limits_L \Big(\int\limits_I \psi(x)\, d\mu_1 \Big) d\mu_{n-1}$. So our aim will be reached if we show that the intersection of N_k with any line parallel to I is a one-dimensional nullset. We shall prove that such an intersection is a countable set. This will complete the proof of Lemma 2. To do this we consider a line $I_x\{x + te : -\infty < t < \infty\}$ (x being fixed) and define two functions on it: $f(t) = \max\limits_{y \in T(x+te)} \langle e, y \rangle$ and $g(t) = \min\limits_{y \in T(x+te)} \langle e, y \rangle$. Obviously, $f(t) \geqq g(t)$ for all t. For y_i from $T(x_i)$ where $x_i = x + t_i e$ $i = 1, 2$ we have $\langle e, y_1 \rangle - \langle e, y_2 \rangle = \langle e, y_1 - y_2 \rangle = (t_1 - t_2)^{-1} \langle x_1 - x_2, y_1 - y_2 \rangle$. Hence $\langle e, y_1 \rangle \leqq \langle e, y_2 \rangle$ provided $t_1 < t_2$. In this case we have the inequalities $g(t_1) \leqq f(t_1) \leqq g(t_2) \leqq f(t_2)$. Therefore $f(t)$ and $g(t)$ are non-decreasing functions of $t \in (-\infty, \infty)$ and have the same set C of points of continuity. For $t \in C$ evidently $f(t) = g(t)$. Hence $u(x + te, e) + u(x + te, -e) = \max\limits_{y \in T(x+te)} \langle e, y \rangle + \max\limits_{y \in T(x+te)} \langle -e, y \rangle = f(t) - g(t) = 0$. Thus, on the line I_x, the function $u(\cdot, e) + u(\cdot, -e)$ may differ from 0 only outside C. Since the set $(-\infty, \infty) \setminus C$ is countable the lemma is proved.

T h e o r e m 3. *Let* $T : R^n \to R^n$ *be a multivalued monotone mapping with non-empty images* (dom $T = R^n$). *Then* T *is single-valued almost everywhere.*

P r o o f. In view of Proposition 1 it is enough to show that $\int\limits_S (u(x, e) + u(x, -e))\, \nu(de) = \int\limits_S (h_{T(x)}(e) + h_{T(x)}(-e))\, \nu(de) = 0$ for almost all x from R^n. Let B be a compact subset of R^n. Consider the set $D = B \times S$ with measure $\gamma = \mu \times \nu$. By lemma 1 the non-negative function $u(x, e) + u(x, -e)$ is γ-measurable. It is also bounded from above because the open sets $W_i = \{(x, e) \in R^n \times S : u(x, e) + u(x, -e) < i\}$ $i = 1, 2, 3, \dots$ form an increasing sequence that covers the compact set D. Hence $D \subset W_{i_0}$ for some i_0. Thus our function is γ-summable and we can apply Fubini's theorem: $\int\limits_B \Big(\int\limits_S (u(x, e)$

$+(u,x,-e))\nu(de)\Big)\,\mu(dx)=\int\limits_S\Big(\int\limits_B(u(x,e)+u(x,-e))\,\mu(dx)\Big)\,\nu(de)=0$. The last equality is due to lemma 2. On the other hand, the function $u(x,e)+u(x,-e)$ is non-negative and so is $\int\limits_S u(x,e)+u(x,-e)\,\nu(de)$. All these facts imply that $\int\limits_S(u(x,e)+u(x,-e))\,\nu(de)=0$ for almost all $x\in B$. As B was an arbitrary compact subset of R^n our theorem is proved.

Our point further is to give some applications to the theorem just proved.

Let M be a closed subset of R^n. Put $P_M(x)=\{y\in R^n: \|x-y\|=\min\limits_{z\in M}\|x-z\|\}$. In this way we get a multivalued mapping $P_M: R^n\to M$ which is called a metric projection of R^n onto M. It assigns to each point $x\in R^n$ the set $P_M(x)$.

Proposition 2. *Every metric projection $P_M: R^n\to M$ is a monotone mapping.*

Proof. For $y_i\in P_M(x_i)$ $i=1,2$ $\|x_1-y_1\|\leq\|x_1-y_2\|$ and $\|x_2-y_2\|\leq\|x_2-y_1\|$. The first inequality is equivalent to $0\leq\|x_1-y_2\|^2-\|x_1-y_1\|^2=2\langle -x_1, y_2-y_1\rangle-\langle y_1,y_1\rangle+\langle y_2,y_2\rangle$. The second one is equivalent to $0\leq 2\langle -x_2, y_1-y_2\rangle-\langle y_2,y_2\rangle+\langle y_1,y_1\rangle$. Adding these two inequalities we get $0\leq\langle x_1-x_2,\ y_1-y_2\rangle$.

Corollary 2. *Every metric projection $P_M: R^n\to M$ is single-valued almost everywhere.*

As second application we shall give another proof of the well-known theorem of Reidemeister [8] that every convex function $h: R^n\to R$ is differentiable almost everywhere. Indeed, for any $x_0\in R^n$ there exists [10] a compact and convex set $\partial(x_0)\subset R^n$ (which is called subdifferential of h at the point x_0) such that $h(x)-h(x_0)\geq\langle x-x_0, y\rangle$ whenever $x\in R^n$ and $y\in\partial(x_0)$. The multivalued mapping $\partial: R^n\to R^n$ that takes every x from R^n to the set $\partial(x)$ is easily seen to be monotone. According to theorem 3 the set $A=\{x\in R^n:\partial(x)$ has more than one element$\}$ is a nullset. On the other hand for any fixed $x\in R^n$ and $e\in S$ $\lim\limits_{t\to+0}\frac{h(x+te)-h(x)}{t}=\max\limits_{y\in\partial(x)}\langle e,y\rangle$ (B. Pshenicnii [7] and J. Moreau [6]). Hence $h: R^n\to R$ is differentiable at x if and only if $\partial(x)$ is a singletone. It is therefore shown that

Theorem 4 (K. Reidemeister [8]). *Every convex function $h: R^n\to R$ is differentiable almost everywhere.*

The above mentioned result of Pshenicnii and Moreau together with Proposition 1 enables us to state

Proposition 3. *The convex function $h: R^n\to R$ is differentiable at the point $x\in R^n$ if and only if* $\lim\limits_{t\to+0}\int\limits_S\frac{h(x+te)-h(x)}{t}\,\nu(de)=0.$

It should be pointed out that Corollary 2 can be deduced from Theorem 4. Improving slightly the proof of Proposition 2 one can see that $P_M: R^n\to M$ is cyclically monotone (for the definitions and results see the paper by Rockafellar [11]). According to [11] the cyclical monotonicity of P_M is a necessary and sufficient condition for the existence of a convex function $h: R^n \to R$ having $P_M(x)$ as a subset of $\partial(x)$ whenever $x \in R^n$. It remains to apply Theorem 4.

That P_M satisfies the inequality $P_M(x) \subset \partial(x)$ for all x from R^n where ∂ is a subdifferential of a certain convex function one could also derive from E. Asplund's paper [1].

REFERENCES

1. E. Asplund. Chebyšhev sets in Hilbert space. *Trans. Amer. Math. Soc.*, **144**, 1969, 235—240.
2. W. Blaschke. Kreis und Kugel. Berlin, 1956.
3. F. E. Browder. Multivalued monotone nonlinear mappings and mappings in Banach spaces. *Trans. Amer. Math. Soc.*, **118**, 1965, 338—351.
4. T. Kato. Demicontinuity, hemicontinuity and monotonicity. *Bull. Amer. Math. Soc.*, **73**, 1967, 886—889.
5. P. Kenderov. The multivalued monotone mappings are almost everywhere single-valued. *C. R. Acad. Bulgare Sci.*, **27**, 1974, 1173—1175.
6. J. -J. Moreau. Fonctionnels sous différentiable. *C. R. Acad. Sci. Paris*, **257**, 1963, 4117.
7. Б. Н. Пшеничный. Необходимые условия экстремума. Москва, 1969.
8. K. Reidemeister. Über die singulären Randpunkte eines konvexen Körpers. *Math. Ann.*, **83**, 1921, 116—118.
9. R. T. Rockafellar. Local boundedness of nonlinear monotone operators. *Mich. Math. J.*, **16**, 1969, 397—407.
10. R. T. Rockafellar. Convex analysis. Princeton, 1970.
11. R. T. Rockafellar. Characterization of the subdifferentials of convex functions. *Pacific J. Math*, **17**, 1966, 497—510.

Centre for Research and Education in Mathematics and Mechanics
1000 Sofia P. O. Box 373

Received 10. 2. 1975