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PARTIALLY ORDERED B^*-EQUIVALENT BANACH ALGEBRAS

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Let A be a complex unital Banach algebra with a continuous involution $x \to x^*$. Then A is called symmetric, if x^*x has non-negative spectrum for every $x \in A$, or what is equivalent, if every self-adjoint element $x = x^* \in A$ has real spectrum. The algebra A is called B^*-algebra, if its norm $\|.\|$ is a B^*-norm, i. e. satisfies the B^* condition: $\|x^*x\| = \|x\|^2$ for every $x \in A$. It is called B^*-equivalent, if a new norm $|.|$ can be introduced on A, which is a B^*-norm and equivalent to the original norm $\|.\|$.

If A is symmetric, the set K of all self-adjoint elements with non-negative spectrum is a wedge, which induces a natural partial order in A. This wedge has some special properties—it is closed, generating, the spectral radius is monotone increasing on it, etc. If A is B^*-equivalent, K is a normal cone. Our aim is, when given a complex unital Banach algebra A, which is a partially ordered linear space with wedge K, to give some necessary and sufficient conditions for K under which A becomes symmetric or B^*-equivalent with a suitable continuous involution, such that K coincides with the wedge of all self-adjoint elements with non-negative spectrum.

This note contains the proofs of some results announced by Boyadžiev (1977).

Throughout with A we denote a complex Banach algebra with unit e, norm $\|.\|$ and spectral radius $\varrho(.)$. Let $K \subset A$ be a wedge ($x+y \in K$ and $\lambda x \in K$ when $x, y \in K$ and $\lambda \geq 0$) inducing a partial order in A. We write $x \leq y$ for x, $y \in A$ if $y-x \in K$.

The wedge K is called α-normal, if a positive constant α exists, such that $\|x\| \leq \alpha \|x+y\|$ when $x, y \in K$. The wedge K is called α-commutatively normal, if positive constant α exists, such that $\|x\| \leq \alpha \|x+y\|$ when $x, y \in K$ and $xy = yx$. Evidently, if K is a α-commutatively normal wedge, it is a cone ($K \cap -K = \{0\}$).

Denoting $H = K - K$ — the real linear span of K, we consider the following conditions:

A) The wedge K is generating, i. e. $A = H + iH$.

B) $H \cap iH = \{0\}$.

C) If $x \in H$, then $x^2 \in K$.

D) The wedge K is closed.

E1) The spectral radius $\varrho(.)$ is monotone increasing on commuting elements of K, i. e. $\varrho(x) \leq \varrho(x+y)$ when $x, y \in K$ and $xy = yx$.

E2) The wedge K is α-commutatively normal.

F) If $x, y \in H$, then $i(xy - yx) \in H$ too.

G) If $x, y \in K$ and $xy = yx$, then $xy \in K$ too.

Lemma 1. *Let in A hold A), B), C). Then:*

a) *The set H is a real linear subspace of A and every $x \in A$ is uniquely decomposed $x = a + ib$ with $a, b \in H$. The correspondence $x = a + ib \to x^* = a - ib$, $a, b \in H$ is a linear involution on A ($(x^*)^* = x$, $(\lambda x + \mu y)^* = \bar{\lambda} x^* + \bar{\mu} y^*$) and H coincides with the set of all self-adjoint elements $x = x^* \in A$.*

b) *If $x, y \in H$ and $xy = yx$, then $xy \in H$.*

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c) The unit e belongs to K.

Proof. a) The proof is obvious.

b) We have $xy=[(x+y)^2-(x-y)^2]/4 \in K-K=H$.

c) Let $e=a+ib$ with $a, b \in H$. We have $a=ae=ea=a^2+iab=a^2+iba$. Hence $ab=ba$ and as $ab \in H$ and $a-a^2=iab$, it follows from B) that $a=a^2 \in K$ and $ab=0$. Also $b=eb=ab+ib^2=ib^2$, so $b=0$. Now $e=a \in K$.

Lemma 2. *Let in A hold A), B), C) and D). Then:*

a) For every $x \in H$ we have $-\varrho(x)e \leq x \leq \varrho(x)e$.

b) The subspace H is closed.

c) The linear involution $x \to x^*$ introduced in a) of the previous lemma is continuous.

Proof. a) Let $x \in H$ and $0<t<\varrho(x)^{-1}$ (if $\varrho(x)=0$, then $\varrho(x)^{-1}=\infty$). There exists $y \in A$ with $y^2=e-tx$ and $y=\lim_n p_n(tx)$, where $p_n(.)$ are polinoms with real coefficients [2, I. 8. 13]. So $p_n(tx) \in H$ for every n and hence $p_n^2(tx) \in K$ for every n. As K is closed, we obtain $e-tx=y^2=\lim_n p_n^2(tx) \in K$. Letting now $t \to \varrho(x)^{-1}$ we obtain $\varrho(x)e-x \in K$. In the same way $\varrho(x)e+x \in K$, so $-\varrho(x)e \leq x \leq \varrho(x)e$. As $\varrho(x) \leq \|x\|$ for every $x \in A$, we have also $-\|x\|e \leq x \leq \|x\|e$ for every $x \in H$.

b) Let $x_k \in H$ and $x_k \to x$. For every k we have $\|x_k\|e+x_k \in K$, so $\|x\|e+x \in K$ as $\|x_k\| \to \|x\|$ and K is closed. Now $x=(x+\|x\|e)-\|x\|e \in H$.

c) Follows from the closed graph theorem [2, V. 36. 1.].

Lemma 3. *Let B be a complex algebra and $V \subseteq B$ — a real linear subspace, such that*

a) $B=V+iV$,

b) $V \cap iV=\{0\}$,

c) *$i(ab-ba) \in V$ and $ab+ba \in V$, when $a, b \in V$.*

Then the mapping $x=a+ib \to x^*=a-ib$, for $a, b \in V$, is an algebraic involution (or only involution) on B ($(x^*)^*=x$, $(\lambda x+\mu y)^*=\bar\lambda x^*+\bar\mu y^*$, $\lambda, \mu \in \mathbb{C}$, $(xy)^*=y^* x^*$ when $x, y \in B$) and V coincides with the set of self-adjoint elements.

For proof see [2, I. 12. 7.].•

Theorem 1. *Let A), B), C), D) and E1) hold in A. Then:*

a) *If $x \in K$, the element $e+x$ is invertible and $(e+x)^{-1} \in K$.*

b) *If $x \in H$, then its spectrum $\mathrm{Sp}(x)$ is real, and if $x \in K$, then $\mathrm{Sp}(x)$ is non-negative ($\mathrm{Sp}(x) \geq 0$).*

c) *If $x \in H$ and $\mathrm{Sp}(x) \geq 0$, then $x \in K$.*

d) *If $x, y \in K$ and $xy=yx$, then $xy \in K$ (i. e. G) holds).*

e) *If $A_0 \subseteq A$ is a commutative complex Banach subalgebra containing the unit and self-adjoint (if $x \in A_0$, then $x^* \in A_0$) according to the continuous linear involution $x \to x^*$ on A (see Lemma 2, c)), then $x \to x^*$ is an algebraic involution on A_0, the algebra A_0 is symmetric, $H_0=H \cap A_0$ is the set of self-adjoint elements in A_0 and $K_0=\{x \mid x \in H_0,\ \mathrm{Sp}(x) \geq 0\}=K \cap A_0$.*

Proof. a) Let $x \in K$ and $0<t<\varrho(x)^{-1}$. Then $y \in H$ exists with $y^2=e-tx$ (y is a limit of real polinoms of tx and H is closed), i. e. $y^2+tx=e$. According to E1) we obtain $\varrho(e-tx)=\varrho(y^2) \leq 1<1+t$. Then $\varrho((1+t)^{-1}e-t(1+t)^{-1}x)=\varrho((1+t)^{-1}y^2)<1$ and the element $e-(1+t)^{-1}y^2=e-[(1+t)^{-1}e-t(1+t)^{-1}x]=t(1+t)^{-1}(e+x)$ is invertible. So $e+x$ is invertible too. We have $[t(1+t)^{-1}(e+x)]^{-1}=[e-(1+t)^{-1}y^2]^{-1}=\sum_{k=0}^{\infty}[(1+t)^{-1}y^2]^k=\sum_{k=0}^{\infty}[(y/\sqrt{1+t})^k]^2 \in K$, ($(y/\sqrt{1+t})^k \in H$). Hence $(e+x)^{-1} \in K$.

b) Let $x\in H$ and $0\neq\lambda\in R$ (the set of real numbers). Then $\lambda^{-2}x^2\in K$ and $e+\lambda^{-2}x^2=\lambda^{-2}(\lambda^2e+x^2)$ is invertible, so that $y=(x^2+\lambda^2e)^{-1}$ exists and $e=y(x^2+\lambda^2e)=y(x+i\lambda e)(x-i\lambda e)$. Then $x-i\lambda e$ is invertible, i. e. $i\lambda\notin \mathrm{Sp}(x)$. Let now $\lambda=\alpha+i\beta$ with $\alpha,\beta\in R$ and $\beta\neq 0$. We have that $x-\lambda e=[(x-\alpha e)-i\beta e]$ is invertible, because $x-\alpha e\in H$. So $\lambda\notin \mathrm{Sp}(x)$ and hence $\mathrm{Sp}(x)\subset R$.

Let now $x\in K$ and $\lambda<0$. Then $-\lambda^{-1}x\in K$ and it follows from a) that the element $e-\lambda^{-1}x=\lambda^{-1}(\lambda e-x)$ is invertible. So $x-\lambda e$ is invertible, hence $\lambda\notin\mathrm{Sp}(x)$. Thus $\mathrm{Sp}(x)\geq 0$.

c) Let $x\in H$ and $\mathrm{Sp}(x)\geqq 0$. For $\varepsilon>0$ and $z=x+\varepsilon e$ we have $z\in H$ and $\mathrm{Sp}(z)=\mathrm{Sp}(x)+\varepsilon>0$. Then there exists $y\in H$ with $y^2=z$ [3, 4. 7. 2]. Hence $x+\varepsilon e\in K$ and letting $\varepsilon\to 0$ we obtain $x\in K$.

d) Let $A_0\subseteq A$ be a commutative complex Banach subalgebra containing x, y and e. As x and y have real spectrum, they have in A_0 the same spectrum as in A (i. e. — non-negative), according to [3, 1. 6. 13.]. As for every $z\in A_0$ we have for it's spectrum in A_0: $\mathrm{Sp}(z, A_0)=\{f(z)\,|\,f$ — nonzero multiplicative and linear functional on $A_0\}$, we obtain that $\mathrm{Sp}(xy, A_0)\geq 0$. So $\mathrm{Sp}(xy)\geqq 0$, as $\mathrm{Sp}(xy)\subseteq \mathrm{Sp}(xy, A_0)$. According to lemma 1, b), $xy\in H$, so it follows from c) that $xy\in K$.

e) It is obvious that H_0 is the set of self-adjoint elements in A_0. As A_0 is self-adjoint, $A_0=H_0+iH_0$. As A_0 is commutative, if $x, y\in H_0$, then $i(xy-yx)=0\in H_0$ and $xy+yx=2xy\in H_0$. Fvidently $H_0\cap iH_0=\{0\}$. Then $x\to x^*$ is an (algebraic) involution on A_0, according to lemma 3.

As $\mathrm{Sp}(x)\subset R$ for every $x\in H$ (b)), then $\mathrm{Sp}(x, A_0)=\mathrm{Sp}(x)\subset R$ for every $x\in H_0$ [3, 1. 6. 13.], hence A_0 is symmetric. According to b) we have $K\cap A_0\subseteq K_0$ and according to c) $K_0\subseteq K\cap A_0$. Hence $K_0=K\cap A_0$.

Theorem 2. *Let A), B), C), D), E1) and F) hold in A_0. Then the linear involution $x\to x^*$ (see lemma 1, a) and lemma 2, c)) is algebraic on A (i. e. $(xy)^*=y^*x^*$), A is symmetric and $K=\{x\,|\,x=x^*\in A,\ \mathrm{Sp}(x)\geq 0\}$.*

Conversely, if A is a symmetric unital Banach algebra with continuous involution $x\to x^*$ and we denote $K=\{x\,|\,x=x^*\in A$ and $\mathrm{Sp}(x)\geq 0\}$ and $H=K-K$, then the conditions A), B), C), D), F), hold in A and the spectral radius is monotone increasing on K, so that E1) holds too.

Proof. For $x, y\in H$ we have $xy+yx=[(x+y)^2-(x^2+y^2)]\in H$. Then the linear involution $x\to x^*$ is algebraic, according to Lemma 3. Every $x\in H$ has real spectrum (theorem 1, b), so A is symmetric. The equality $K=\{x\,|\,x=x^*\in A,\ \mathrm{Sp}(x)\geq 0\}$ follows from b) and c) of the previous theorem.

Conversely, if A is a complex unital symmetric Banach algebra with continuons involution $x\to x^*$ and we denote $K=\{x\,|\,x=x^*\in A,\ \mathrm{Sp}(x)\geqq 0\}$ and $H=K-K$, it is known from the existing theory [3, 4. 7. 10] that K is a generating wedge and H coincides with the set of all self-adjoint elements in A. The wedge K is closed, according to a recent result of B. Aupetit [5] that the spectrum is uniformly continuous on H (in the Hausdorff metric) when A. is symmetric.

To show that $\varrho(.)$ is monotone increasing on K, we use the equality $\varrho(x)=\sup\{|f(x)|\,|\,f\in P\}$ for every $x\in H$, true for symmetric algebras, where P stands for the set of all positive linear functionals f on A with $f(e)=1$. If $f\in P$, f takes non-negative values on K [3, 4. 7. 3.], so $\varrho(x)\leq\varrho(x+y)$ when x, $y\in K$. The theorem is proved.

Now we characterize B^*-equivalent algebras, giving a conection between the norm and the partial order in A.

Theorem 3. *Let A), B), C), D) and E2) hold in A. Then every complex commutative self-adjoint and closed subalgebra $A_0 \subseteq A$ containing the unit is B^*-equivalent. Every $x \in H$ has real spectrum and* $\|x\| \leq (2a+1)\varrho(x)$.

Proof. Let $A_0 \subseteq A$ be a complex commutative Banach subalgebra containing the unit e and self-adjoint according to the linear involution $x \to x^*$ (lemma 1, a)). As in theorem 1, e) we obtain that this linear involution is algebraic on A_0 and $H_0 = H \cap A_0$ is the real linear subspace of self-adjoint elements in A_0. As H and A_0 are closed (lemma 2, b)), H_0 is closed too. As $xy \in H$ when $x, y \in H$ and $xy = yx$ (lemma 1, b)), H_0 is a real commutative Banach subalgebra of A_0 containing the unit.

In A_0 we consider the wedge $N_0 = \{x \mid x = \Sigma_{k=1}^n x_k^2,\ x_k \in H_0,\ n \geq 1\}$ and it's closure N. The wedge N_0 (and therefore N) is generating A_0 ($A_0 = H_0 + iH_0$ and if $x \in H_0$, $x = [(x+e)^2 - (x-e)^2]/4 \in N_0 - N_0 \subseteq H_0$). Also $N \subseteq H_0$. Moreover, if $x, y \in N$, then $xy \in N$, too. Let $x = \lim_k x_k$ and $y = \lim_k y_k$, $x_k, y_k \in N_0$. Evidently $x_k y_k \in N_0$, hence $xy = \lim_k x_k y_k \in N$.)

The wedge N induces in A_0 a partial order. We write $x < y$ for $x, y \in A_0$ if $y - x \in N$. Evidently, if $x, y \in A_0$ and $x < y$, then $x \leq y$, too, as $N_0 \subseteq K$ and hence $N \subseteq K$ (K is closed).

As in lemma 2, a) we obtain that $-\varrho(x)e < x < \varrho(x)e$ for every $x \in H_0$, so that e is an order unit in H_0 and every $x \in H_0$ is order bounded. For $x \in H_0$ we denote $|x| = \inf\{\lambda \mid \lambda > 0,\ -\lambda e < x < \lambda e\}$. It is easy to see, that $|.|$ is a seminorm on H_0, monotone increasing on N. Evidently $|x| \leq \varrho(x)$ and $-|x|e < x < -|x|e$ for $x \in H_0$.

We shall show now that $|x^2| \leq |x|^2$ for every $x \in H_0$. If $-e < x < e$, i. e. $e \pm x \in N$, then $0 < x^2 < e$ because $e - x^2 = (e-x)(e+x) \in N$. If now $x \in H_0$ and $|x| \neq 0$, $-x < x/|x| < e$ and therefore $0 < x^2/|x|^2 < e$. Hence $|x^2| \leq |x|^2$.

We shall see now, that $|.|$ and $\|.\|$ are equivalent on H_0. Let $x \in N$. From $0 < x < |x|e$ it follows that $0 \leq x \leq |x|e$ and according to E2) we obtain $\|x\| \leq a|x|$. Let now $x \in H_0$. We have $\|x\| = \|x + |x|e - |x|e\| \leq \|x + |x|e\| + \||x|e\| \leq a|x + |x|e| + |x| \leq (2a+1)|x|$. In particular, it follows that $|.|$ is a norm on H_0. From $|x| \leq \varrho(x)$ we also obtain $|x| \leq \|x\|$ for every $x \in H_0$. Finally we have $|x| \leq \|x\| \leq (2a+1)|x|$ for every $x \in H_0$.

It is easy to see that $\varrho(x) = |x|$ for every $x \in H_0$. Let $x \in H_0$. For every integer $k = 2^n$ we have $[(2a+1)^{-1}]^{1/k} \|x^k\|^{1/k} \leq |x^k|^{1/k} \leq \|x^k\|^{1/k}$. Therefore the limit $l = \lim_k |x^k|^{1/k}$ exists and $l = \varrho(x)$. From $|x^k| \leq |x|^k$, i. e. $|x^k|^{1/k} \leq |x|$ it follows $l \leq |x|$. So $\varrho(x) = |x|$.

For A_0, N and H_0 all the conditions of theorem 2 hold and therefore A_0 is symmetric. Hence every $x \in H_0$ has real spectrum.

As $\|x\| \leq (2a+1)\varrho(x)$ for every $x \in H_0$, A_0 is B^*-equivalent, according to a well-known argument (see for example [4], [7, 8.4]), with $|x|_0 = \varrho(x^*x)^{1/2}$ a B^*-norm on it, equivalent to $\|.\|$.

As every $x \in H$ can be included in some closed commutative and self-adjoint subalgebra containing the unit, every such element x has real spectrum and for it the inequality $\|x\| \leq (2a+1)\varrho(x)$ holds. The theorem is proved.

Theorem 4. *Let A), B), C), D), E2) and G) hold in A. Then besides the results from the previous theorem we have:*

a) *If $A_0 \subseteq A$ is a closed commutative self-adjoint subalgebra containing the unit, then $\{x \mid x \in H \cap A_0,\ \mathrm{Sp}(x) \geq 0\} = K \cap A_0$ (and A_0 is B^*-equivalent accordig to the previous theorem).*

b) *The spectral radius* $\varrho(.)$ *is monotone increasing on* K *and on* H *it is a norm equivalent to* $\|.\|$. *For every* $x \in H$, $\varrho(x)=\inf\{\lambda|\lambda>0, -\lambda e\leq x\leq\lambda e\}$ *and for every* $x\in K$, $\|x\|\leq\alpha\varrho(x)$.

In particular, all the conditions of theorem 1 hold in A.

Proof. According to lemma 2, a) for every $x \in H$ we denote $|x|=\inf\{\lambda|\lambda>0, -\lambda e\leq x\leq\lambda e\}$. It is easy to see that $|.|$ is a norm on H (as K is a cone, according to E2)), which monotone increases on K.

Let $A_0\subseteq A$ be a complex commutative self-adjoint Banach subalgebra containing the unit and $H_0=H\cap A_0$ be the set of self-adjoint elements in it. We know from the previous theorem that A_0 is B^*-equivalent and H_0 is a real Banach subalgebra of A_0. In A_0 we consider the cone $K_0=K\cap A_0$. It follows from G) that if $x, y\in K_0$, then $xy\in K_0$ too. With the same method as in the previous theorem we can see that $|x^2|\leq|x|^2$ for $x\in H_0$. In the same way we obtain $\|x\|\leq\alpha|x|$ for every $x\in K_0$, $|x|\leq\|x\|\leq(2\alpha+1)|x|$ for every $x\in H_0$ and therefore $\varrho(x)=|x|$ for every $x\in H_0$.

As K_0 contains the generating cone N_0 defined in the previous theorem, K_0 is generating for A_0 too. For A_0, K_0 and H_0 all the conditions of theorem 2 hold and so $K_0=\{x|x\in H_0, \mathrm{Sp}(x)\geq 0\}$ ($\mathrm{Sp}(x, A_0)=\mathrm{Sp}(x)$ for every $x\in H_0$ as this spectrum is real [3, 1.6.13]). Every element $x\in H$ can be included in some complex commutative and self-adjoint Banach subalgebra of A containing the unit, so $\varrho(x)=|x|$ and hence $\|x\|\leq(2\alpha+1)\varrho(x)$. For x in K we have $\|x\|\leq\alpha\varrho(x)$. The proof is completed.

Theorem 5. *Let A), B), C), D), E2), F) hold in A. Then the linear involution* $x\to x^*$ *(see lemma 1, a)) is algebraic on A and with it A is* B^**-equivalent* ($\varrho(x^*x)^{1/2}=|x|_0$ *for* $x\in A$ *is a* B^**-norm on A equivalent to* $\|.\|$).

If G) also holds in A, then $K=\{x|x=x^*\in A$ *and* $\mathrm{Sp}(x)\geq 0\}$.

Conversely, if A is a B^**-equivalent unital Banach algebra and we denote* $K=\{x|x=x^*\in A, \mathrm{Sp}(x)\geq 0\}$ *and* $H=K-K$, *then H is the set of self-adjoint, elements in A and for A, K and H the conditions A), B), C), D), F), G) hold and K is a* α*-normal cone for some* $\alpha>0$, *so that E2) holds too.*

Proof. Just like in theorem 2 we obtain that $x\to x^*$ is an algebraic involution on A. Theorem 3 implies that every $x\in H$ has real spectrum and $\|x\|\leq(2\alpha+1)\varrho(x)$. Then A is B^*-equivalent and $|x|_0=\varrho(x^*x)^{1/2}$, $x\in A$, is a B^*-norm on A equivalent to $\|.\|$ [4], [7, 8.4.].

If G) holds in A, the spectral radius is monotone increasing on K according to the previous theorem and all the conditions of theorem 2 hold in A, so $K=\{x|x=H, \mathrm{Sp}(x)\geq 0\}$.

Conversely, let A be a B^*-equivalent Banach algebra with involution $x\to x^*$. If we denote $K=\{x|x=x^*\in A$ and $\mathrm{Sp}(x)\geq 0\}$ and $H=K-K$, then $H=\{x|x=x^*\in A\}$ and A), B), C), D) and F) hold (well-known in the theory of B^*-equivalent algebras).

To prove G) let $x, y\in K$ and $xy=yx$. Let A_0 be a complex commutative Banach subalgebra of A containing x, y and the unit. Now we continue just like in d) of theorem 1 to obtain $xy\in K$.

To see that K is an α-normal cone for some $\alpha>0$, let $|.|$ be a B^*-norm on A equivalent to $\|.\|$, i. e. $\beta|x|\leq\|x\|\leq\gamma|x|$ for every $x\in A$ and some positive constants β, γ. As $|.|$ is a B^*-norm, $|x|=\varrho(x)$ for every $x\in H$ (well-known for B^*-algebras), then theorem 2 implies that $|.|$ is monotone increasing on K (every B^*-algebra is symmetric). For $x, y\in K$ we have

$$\|x\| \leq \gamma \|x\| \leq \gamma \|x+y\| \leq \gamma\beta^{-1} \|x+y\|. \text{ Take } \alpha=\gamma\beta^{-1}.$$

The proof is completed.

Some applications. With the help of theorem 5 we can obtain some results about unital Banach star algebras.

Let A be a complex Banach algebra with unit e and continuous involution $x \to x^*$. Let $H=\{x \mid x=x^* \in A\}$.

In A we consider the wedge: $Q=\{z \mid z=\sum_{k=1}^{n} x_k^* x_k,\ x_k \in A,\ k=1, 2, \ldots n$ $n \geq 1\}$. The following theorem holds:

Theorem 6. *The following conditions are equivalent:*

a) *The algebra A is B^*-equivalent.*

b) *The wedge Q is α-normal.*

c) *The wedge $[Q]$ (closure of Q) is α-commutatively normal.*

d) *The algebra A is symmetric and for some $\alpha>0$ we have $\|x^2\| \leq \alpha \|x^2+y^2\|$ for every $x, y \in H$ with $x^2y^2=y^2x^2$.*

Proof. If A is B^*-equivalent, Q coincides with the cone of all self-adjoint elements with non-negative spectrum in A. Theorem 5 implies then that Q is α-normal. So a) $\to$ b) is proved. The implication b) $\to$ c) is obvious — if Q is α-normal, $[Q]$ is α-normal too.

Now c) $\to$ a). As the involution $x \to x^*$ is continuous, for the set H of self-adjoint elements in A we have $H=[Q].-[Q]$. Now for A, $[Q]$, H we apply theorem 5.

As every B^*-equivalent algebra is symmetric, a) and b) imply d). Let now d) hold. The set K of self-adjoint elements with non-negative spectrum in A is a wedge and $Q \subseteq K$ [3, 4.7.10]. We shall prove that K is α-commutatively normal.

Let x, $y \in K$ and $xy=yx$. For $\varepsilon>0$ we have $\mathrm{Sp}(x+\varepsilon e)>0$ and $\mathrm{Sp}(y+\varepsilon e)>0$. Then there exist a, $b \in H$ with $a^2=x+\varepsilon e$, $b^2=y+\varepsilon e$ [3, 4.7.2.]. Evidently $a^2b^2=b^2a^2$. Now d) implies $\|x+\varepsilon e\| \leq \alpha \|x+y+2\varepsilon e\|$. Letting $\varepsilon \to 0$ we obtain $\|x\| \leq \alpha \|x+y\|$. So K is α-commutatively normal. It is also closed (see theorem 2). Therefore $[Q] \subseteq K$ (in fact they coincide) and hence $[Q]$ is α-commuiatively normal. The implication d) $\to$ c) is proved and with it the theorem too.

Theorem 5 can also be applied to obtain the well-known theorem of Vidav-Palmer.

Let A be a complex Banach algebra with unit e. We set: $S=\{f \mid f$ — a continuous linear functional on A with $f(e)=1=\|f\|\}$. (The elements of S are called normalized states.)

For $x \in A$ we denote $V(x)=\{f(x) \mid f \in S\}$. The set $V(x)$ is called numerical range of x.

The elements $x \in A$ for which $V(x) \subset R$ are called Hermitian (in the sense of Vidav). The set of Hermitian elements in A we denote with H.

Theorem of Vidav-Palmer. *If $A=H+iH$, then A is a B^*-algebra with continuous involution $x \to x^*$, such that H coincides with the set of all self-adjoint elements in A.*

Let $A=H+iH$. We denote with K the set of all Hermitian elements for which the numerical range is non-negative.

In [6, 2.5.6.] it is shown that K is a closed α-normal cone for some $\alpha>0$.

From the theory of Hermitian elements it is known that $i(xy-yx) \in H$, when x, $y \in H$ [6, 2.5.4] and $V(x)=\mathrm{co}\,\mathrm{Sp}(x)$ (convex hull) for every $x \in H$

(Vidav's lemma [6, 2.5.14.]). From this lemma it follows easily that $x \in H$ implies $x^2 \in K$ (if $x \in H$, then $x^2 \in H$ [6, 2.6.3.] and then use that every $x \in H$ has real spectrum and $\mathrm{Sp}(x^2) = \mathrm{Sp}^2(x)$). Evidently $e \in K$ and so K is generating — if $x \in H$, $x = [(x+e)^2 - (x-e)^2]/4 \in K - K \subseteq H$, so $H = K - K$. Also $H \cap iH = \{0\}$ is obvious. So for A, K, H we can apply theorem 5 to obtain that A is B^*-equivalent with a suitable involution, such that H coincides with the set of self-adjoint elements in A.

With a standard consideration we can see that A is a B^*-algebra (using (2.6.8 and 2.5.2 of [6]).

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