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CHARACTERISTIC OF SOME CLASSES OF ALMOST HERMITIAN MANIFOLDS

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For an arbitrary almost Hermitian manifold a generalized curvature tensor of Kähler type is found. For this tensor the associated Ricci and Bochner tensors are introduced. As a result one gets characteristics of some special classes of almost Hermitian manifolds.

H. Mori [3] and M. Sitaramayya [4] gave a decomposition of curvature tensors of Kähler type on a $2n$-dimensional Hermitian vector space and studied curvature tensors on a Kähler manifold.

In this paper we find a generalized curvature tensor of Kähler type in an almost Hermitian manifold and similarly to the case of a Kähler manifold using the associated tensor of Bochner and the Ricci tensor we obtain some special almost Hermitian manifolds. For these manifolds we give some characteristics.

Let M be an almost Hermitian manifold with $\dim M=2n$, J an almost complex structure ane denote the Hermitian inner product by $\langle , \rangle$. In what follows ∇ is the Levi-Civita connection and R its curvature tensor. For arbitrary vector fields X, Y, Z we consider the following tensors:

$$R'(X, Y)Z=3R(X, Y)Z+3R(JX, JY)Z-R(JY, JZ)X+R(JX, JZ)Y \tag{1}$$
$$+R(Y, JZ)JX-R(X, JZ)JY,$$

$$R^*(X, Y)Z=(R'(X, Y)Z-JR'(JX, JY)JZ)/16. \tag{2}$$

Using the properties of R it is easy to verify the following proposition.

Proposition 1. *The tensor* (1) *has the following properties:*

1) $R'(Y, X)Z=-R(X, Y)Z$;
2) $R'(X, Y)Z+R'(Y, Z)X+R'(Z, X)Y=0$;
3) $R'(JX, JY)Z=R'(X, Y)Z$.

The tensor (2) *has the following properties*:

(3)
1) $R^*(Y, X)Z=-R^*(X,Y)Z$;
2) $R^*(X, Y)Z+R^*(Y, Z)X+R^*(Z, X)Y=0$;
3) $\langle R^*(X, Y)Z, U\rangle=-\langle R^*(X, Y)U, Z\rangle$;
4) $\langle R^*(X, Y)JZ, JU\rangle=\langle R^*(X, Y)Z, U\rangle$.

From the conditions (3) it follows that the tensor R^* is a generalized curvature tensor of Kähler type. The Ricci tensor S^* of type $(0, 2)$ associated with R^* is a symmetric bilinear function defined by $S^*(X, Y)=\mathrm{trace}\,(Z\to R^*(Z, X)Y)$. The Ricci tensor Q^* of type $(1, 1)$ is defined by $\langle Q^*(X), Y\rangle=S^*(X, Y)$. From (3) is follows that $S^*(JX, JY)=S^*(X, Y)$, $Q^*(JX)=JQ^*(X)$. If X is a unit vector in the tangent space T_xM, then $S^*(X)=S^*(X, X)$ is called the Ricci curvature of X with respect to R^*.

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Let $H(X)=\langle R(X, JX)JX, X\rangle$ be the holomorphic sectional curvature of a 2-plane determined by the unit vectors X and JX. It follows immediately from (1) and (2) that $\langle R^*(X, JX)JX, X\rangle=\langle R(X, JX)JX, X\rangle$.

Now let M has a pointwise constant holomprphic sectional curvature, i. e. $H(X)=c(x)$ for an arbitrary unit vector $X\in T_xM$, where c does not depend on X. We denote

$$2^{-1}R_{I,I}(X, Y)Z=\langle Y, Z\rangle X-\langle X, Z\rangle Y+\langle JY, Z\rangle JX-\langle JX, Z\rangle JY-2\langle JX, Y\rangle JZ.$$

This tensor has also the properties (3) [2]. Therefore the tensor

$$T(X, Y, Z, U)=\langle R^*(X, Y)Z, U\rangle-8^{-1}c\langle R_{I,I}(X, Y)Z, U\rangle$$

has the following properties;

$$
\begin{aligned}
&1)\ T(X, Y, Z, U)=-T(Y, X, Z, U);\\
&2)\ T(X, Y, Z, U)+T(Y, Z, X, U)+T(Z, X, Y, U)=0;\\
(4)\quad &3)\ T(X, Y, Z, U)=-T(X, Y, U, Z);\\
&4)\ T(X, Y, JZ, JU)=T(X, Y, Z, U);\\
&5)\ T(X, JX, JX, X)=0.
\end{aligned}
$$

From a well-known theorem [2] it follows that $T=0$. Conversely, if $T=0$, then $H(X)=c$ for an arbitrary unit vector $X\in T_xM$. So we proved the following proposition.

P r o p o s i t i o n 2. *The manifold M has a pointwise constant holomorphic sectional curvature $c(x)$ if and only if the generalized curvature tensor R^* has the form*

$$R^*(X, Y)Z=8^{-1}cR_{I,I}(X, Y)Z. \tag{5}$$

For the generalized curvature tensor R^* which satisfies the conditions (3) we can apply the decomposition theorem of [3] and so we obtain the tensor of Bochner associated with R^* [1]:

$$B(X, Y)Z=R^*(X, Y)Z-\frac{1}{2(n+2)}R_{S^*,I}(X, Y)Z+\frac{S^*}{8(n+1)(n+2)}R_{I,I}(X, Y)Z.$$

Here S^* is the scalar curvature corresponding to R^* — the trace of the Ricci tensor, and

$$
\begin{aligned}
R_{S^*,I}(X, Y)Z=&\,S^*(Y, Z)X-S^*(X, Z)Y+S^*(JY, Z)JX-S^*(JX, Z)JY\\
&-2S^*(JX, Y)JZ+\langle Y, Z\rangle Q^*(X)-\langle X, Z\rangle Q^*(Y)\\
&+\langle JY, Z\rangle Q^*(JX)-\langle JX, Z\rangle Q^*(JY)-2\langle JX, Y\rangle Q^*(JZ).
\end{aligned}
$$

We shall call an almost Hermitian manifold for which $S^*(X)$ does not depend on the unit vector $X\in T_xM$, $x\in M$ a generalized Einstein manifold.

Consider a quadrilinear mapping $T\colon T_xM\times T_xM\times T_xM\times T_xM\to\mathbb{R}$ satisfying the conditions 1) — 4) of (4) and denote $k(X, Y)=T(X, Y, Y, X)$. A 2-plane E of T_xM is called holomorphic if $JE=E$ and antiholomorphic if $JE\perp E$.

P r o p o s i t i o n 3. *If $k(X, JX)=0$ for a basis of an arbitrary holomorphic 2-plane, then $k(X, Y)=0$ for a basis of an arbitrary antiholomorphic 2-plane. The inverse is also true by the condition $n\geq 3$.*

Proof. The first part of the proposition follows directly from the conditions (4) for T.

For the inverse let X be an arbitrary unit vector in T_xM and X, Y be a basis of an antiholomorphic 2-plane. Then $(X+Y)/\sqrt{2}, (JX-JY)/\sqrt{2}$ and $(X-Y)/\sqrt{2}, (JX+JY)/\sqrt{2}$ are bases of antiholomorphic 2-planes. Therefore

$$k((X+Y)/\sqrt{2}, (JX-JY)/\sqrt{2})=0, \; k((X-Y)/\sqrt{2}, (JX+JY)/\sqrt{2})=0. \tag{6}$$

Adding the two equalities of (6) and using the properties of T we obtain

$$k(X, JX)+k(Y, JY)=0. \tag{7}$$

If $n\geq 3$, we can take X, Y, Z as a basis of an antiholomorphic 3-plane and similarly one has the equalities:

$$k(X, JX)+k(Z, JZ)=0, \; k(Y, JY)+k(Z, JZ)=0. \tag{8}$$

From (7) and (8) we can conclude that $k(X, JX)=0$, which proves our assertion.

Let M has a pointwise constant holomorphic sectional curvature $H(X)=c(x)$, $X\in T_xM$, $\langle X, X\rangle=1$ and denote $K^*(X, Y)=\langle R^*(X, Y)Y, X\rangle$. Proposition 2 implies (5). Applying propostion 3 to the tensor $T=R^*-(c/8)R_{I,I}$ we obtain that the conditions

$$H(X)=c, \; 4K^*(X, Y)=c \tag{9}$$

are equivalent if $n\geq 3$. Here X, JX is a basis of an arbitrary holomorphic 2-plane and X, Y, is a basis of an arbitrary antiholomorphic 2-plane. In terms of an adapted basis $(u_i, Ju_i)\, i=1, \ldots, n$ the conditions (9) can be written in the form

$$H_i=c, \; 4K^*_{ij}=4K^*_{i\bar{j}}=c, \; i\neq j, \tag{10}$$

where $H_i=H(u_i)=H(Ju_i)$, $K^*_{ij}=K^*(u_i, u_j)$, $K^*_{i\bar{j}}=K^*(u_i, Ju_j)$. Using the formulae

$$S^*_i=S^*(u_i)=\sum_{j=1, j\neq i}^{n}(K^*_{ij}+K^*_{i\bar{j}})+H_i; \; i=1, \ldots, n; \; S^*=2\sum_{i=1}^{n}S^*_i \tag{11}$$

and summing (10) we obtain

$$c=S^*/n(n+1). \tag{12}$$

So we proved the next proposition 4.

Proposition 4. *If $n\geq 3$ an almost Hermitian manifold with a pointwise constant holomorphic curvature is characterized by the conditions* (9) *or* (10).

Similarly, taking $T=B$ we obtain also proposition 5.

Proposition 5. *An almost Hermitian manifold with vanishing generalized tensor of Bochner is characterized by the conditions*

$$\frac{4}{n+2}S^*_i-H_i=\frac{S^*}{(n+1)(n+2)}, \; i=1, \ldots, n; \tag{13}$$

$$\frac{4}{n+2}\frac{S^*_i+S^*_j}{2}-4K^*_{ij}=\frac{4}{n+2}\frac{S^*_i+S^*_j}{2}-4K^*_{i\bar{j}}=\frac{S^*}{(n+1)(n+2)}, \; i\neq j. \tag{14}$$

More precisely: for $n \geq 2$ (13) implies (14) and vice versa, for $n \geq 3$ (14) implies (13).

Corollary. *Every almost Hermitian manifold with a pointwise constant holomorphic sectional curvature has a vanishing generalized tensor of Bochner and it is a generalized Einstein manifold.*

In fact the first equality of (10) and (12) imply

$$S_i^* = \frac{S^*}{2n}, \quad \frac{4}{2+n} S_i^* - H_i = \frac{S^*}{(n+1)(n+2)}.$$

The inverse is also true, i. e. if an almost Hermitian manifold is a generalized Einstein manifold and has a vanishing generalized tensor of Bochner, then it has a pointwise constant holomorphic sectional curvature. This follows directly from (13).

Now let us consider an almost Hermitian manifold M and suppose that in every point $x \in M$ holds the condition

$$(15) \qquad \lambda S^*(X) + \mu H(X) = c, \quad X \in T_x M, \ \langle X, X \rangle = 1,$$

where c does not depend on X and λ, μ are real parameters. We can write (15) using an adapted basis in the form

$$(16) \qquad \lambda S_i^* + \mu H_i = c, \quad i = 1, \ldots, n.$$

From (15) in a well-known way follows (see also [5])

$$\frac{\lambda}{8} R_{S^*, I}(X, Y)Z + \mu R^*(X, Y)Z = \frac{c}{8} R_{I, I}(X, Y)Z$$

and from here

$$(17) \qquad \lambda \frac{S_i^* + S_j^*}{2} + \mu 4 K_{ij}^* = \lambda \frac{S_i + S_j^*}{2} + \mu 4 K_{\bar{i}\bar{j}}^* = c, \quad i \neq j; \ i, j = 1, \ldots, n.$$

Taking in account (11) from (17) follows

$$(18) \qquad (\lambda(n-2) + 4\mu) S_i^* + \lambda S^*/2 = 2(n+1)c, \quad i = 1, \ldots, n.$$

Adding these equalities we obtain

$$(19) \qquad c = [(n+1)\lambda + 2\mu]/2n(n+1).$$

Substituting (19) in (18) the equality (16) gets the form $((n+2)\lambda + 4\mu)(S_i^* - S^*/2n) = 0$. From the last equality applying the propositions 3, 4 and 5 we obtain the following proposition 6.

Proposition 6. *Let M be an almost Hermitian manifold satisfying in each point the condition* (15). *Then*

1. *If* $(n+2)\lambda + 4\mu = 0$, $\mu = -1$, $\lambda = 4/(n+2)$, *then M has a vanishing generalized tensor of Bochner.*

2. *If* $(n+2)\lambda + 4\mu \neq 0$ *and*

2.1) $\mu = 0$, *then M is a generalized Einstein manifold;*

2.2) $\mu \neq 0$, *then M has a pointwise constant holomorphic sectional curvature.*

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