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LINEAR OPTIMAL CONTROL SYSTEM WITH SINGULAR PERTURBATION AND CONVEX PERFORMANCE INDEX

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In this paper a linear control system with a small parameter in the derivatives of a part of the state is considered. The behaviour of the solutions of two classical optimal control problems with convex costs for the small parameter tending toward zero is investigated. The presented approach uses some properties of the state equation and the geometrical theory of the attainable set. It gives a method for a qualitative analysis of a broad class of optimal control problems.

1. Consider for $t\in[t_0, T]$ an optimal control system described by the following state equations

$$\dot{x}=A_{11}(t)x+A_{12}(t)y+B_1(t)u,\quad \lambda\dot{y}=A_{21}(t)x+A_{22}(t)y+B_2(t)u \tag{1.1}$$

with the initial conditions $x(t_0)=x^0$, $y(t_0)=y^0$, where $x\in R^n$, $y\in R^m$, $(x(t), y(t))$ is a state variable, $u(t)\in R^r$ is a control, t_0 and T are finite constants, $t_0<T$. We assume that the real matrices $A_{ij}(t)$, $B_i(t)$ are continuous in $[t_0, T]$.

The small parameter $\lambda\in[0, 1]$ provides a singular perturbation. Letting $\lambda=0$ one gets a "low-order system"

$$\dot{x}=A_{11}(t)x+A_{12}(t)y+B_1(t)u, \tag{1.2a}$$

$$0=A_{21}(t)x+A_{22}(t)y+B_2(t)u \tag{1.2b}$$

with the same initial conditions, where the equation (1.2b) is defined in $(t_0, T]$.

The two following performance functionals are considered:

$$J(u,\lambda)=\int_{t_0}^{T}[f(x,t,\lambda)+h(u,t)]dt, \tag{1.3}$$

$$I(u,\lambda)=c(x(T),\lambda)+J(u,\lambda), \tag{1.4}$$

where the functions $f(x,t,\lambda)$, $h(u,t)$, $c(v,\lambda)$ are continuous; the function $f(x,t,\lambda)$ is convex and differentiable with respect to x for (t,λ) fixed and its derivative is continuous; the function $h(u,t)$ is strictly convex with respect to u for t fixed. It is assumed that $f(x,t,\lambda)\geq 0$ and there exist positive constants a, $p>1$, so that $h(u,t)\geq a\|u\|^p$ for every $t\in[t_0, T]$. For notational convinience we use the same symbol $\|\cdot\|$ for the norms of all spaces, leaving to the context to fix the respective meaning.

The following basic assumptions are made:

H1. *The eigenvalues of the matrix* $A_{22}(t)$ *have negative real parts for* $t\in[t_0, T]$.

H2. *The matrix*

$$M_0=\int_{t_0}^{T}\Phi_0(T,t)B_0(t)B_0'(t)\Phi_0'(T,t)dt \tag{1.5}$$

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is nonsingular, where $B_0(t)=B_1(t)-A_{12}(t)A_{22}^{-1}(t)B_2(t)$; $\Phi_0(t,\tau)$ *is the fundamental solution of the equation* $\dot{x}=A_0(t)x$, *where* $A_0(t)=A_{11}(t)-A_{12}(t)A_{22}^{-1}(t)A_{21}(t)$.

Here A' denotes the transpose of the matrix A.

This paper deals with qualitative effects due to reducing the order of the system. Part 2 contains auxiliar results, which help to determine the behaviour of the solution of optimal control problems when λ tends to zero.

Let $x^T\in R^n$ be given. We call the control $u\in L_p^{(r)}(t_0, T)$ feasible for $\lambda\in(0,\Lambda]$ if the corresponding state, determined by (1.1) with fixed initial condition satisfies $x(T)=x^T$. Let us recall that the system (1.1) is controllable with respect to x if for every $x^T\in R^n$ and every initial condition there exists a feasible control for $\lambda\in(0,\Lambda]$. The system (1.2ab) is controllable if for every x^T and every initial condition there exists a feasible control for $\lambda=0$. In part 3 it is shown that if the hypothesis H1 holds and the system (1.2ab) is controllable, then for sufficiently small λ the system (1.1) is controllable with respect to x.

The optimal control problems to determine the feasible control $u(t,\lambda)$ for $\lambda\in(0,\Lambda]$ and the feasible control $u_0(t)$ for $\lambda=0$, which minimize the performance index (1.3) are considered in part 4. It is proved that under the basic assumptions the optimal control at $\lambda=0$ is an uniformly continuous function of the parameter λ.

In part 5 a different optimal control problem is analysed, that is to minimize (1.4) over $u\in L_p^{(r)}(t_0, T)$, where the state satisfies (1.1) or (1.2ab). An existence theorem for the optimal control based on the properties of the function $c(x(T),\lambda)$ is presented. Similar results for the continuity of the optimal solution with respect to λ are obtained. In part 6 the method is applied to investigate an optimal control system which is uncontrollable at $\lambda=0$.

The behaviour of the solution of a differential equation with a small parameter in the derivative was broadly investigated by A. Tichonov [1]. A general study of singular perturbation techniques can be found in [2;3]. Probably, P. Kokotovič and P. Sannuti were the first to formulate an optimal control problem with singular perturbations. Recently, an extensive bibliography on such problems can be obtained — see the overview [5]. The research work has been confined mostly to linear dynamics with quadratic costs and has made use of explicit formulae for the optimal control applying the technique of the Riccati equation [6]. Different approaches are given in [7;8], where a constrained control problem is studied and the performance index is a linear form of the final state. In [9;10] singularly perturbated infinite-dimensional optimal control problems are analysed. In [11;12;13] problems similar to the ones presented here are discussed, but with regular perturbations.

2. Everywhere in this section it is assumed that hypothesis H1 holds. We shall denote by $\{\lambda_k\}_1^\infty$ the sequence of numbers $\lambda_k\in(0,\Lambda]$.

Lemma 1. *Suppose that the sequence* $\{\lambda_k\}_1^\infty$, $\lim\lambda_k=0$ *is given and the sequence* $\{g_k(t)\}_1^\infty$ *of continuous functions* $g_k(t)$ *is uniformly converging in* $[t_0, T]$ *to the function* $g_0(t)$. *Let* $y_k(t)$ *be the solution of the equation*

$$\lambda_k\dot{y}=A_{22}(t)y+g_k(t),\quad y(t_0)=y^0.$$

Then, for every $t^*\in(t_0, T]$

$$\lim_{k\to\infty}\ \max_{t^*\leq t\leq T}\|y_k(t)+A_{22}^{-1}(t)g_0(t)\|=0.$$

L e m m a 2. *Suppose that the sequence* $\{\lambda_k\}_1^\infty$, $\lim \lambda_k = 0$ *is given. Let us denote by* $(v_k(t), w_k(t))$ *the solution of the equation*

$$\dot{v} = A_{11}(t)v + \lambda_k^{-1} A_{12}(t)w, \quad v(t_0) = 0,$$

$$\dot{w} = A_{21}(t)v + \lambda_k^{-1} A_{22}(t)w, \quad w(t_0) = w^0;$$

and by $v_0(t)$ *the solution of the equation*

$$\dot{v} = A_0(t)v, \quad v(t_0) = -A_{12}(t_0)A_{22}^{-1}(t_0)w^0,$$

where $A_0(t) = A_{11}(t) - A_{12}(t)A_{22}^{-1}(t)A_{21}(t)$. *Then for every* $t^* \in (t_0, T]$

$$\lim_{k\to\infty} \left(\max_{t^* \leqq t \leqq T} \| v_k(t) - v_0(t) \| + \max_{t^* \leqq t \leqq T} \| w_k(t) \| \right) = 0.$$

The sequences $\{v_k(t)\}_1^\infty$, $\{w_k(t)\}_1^\infty$ *are uniformly bounded in* $[t_0, T]$.

L e m m a 3. *Suppose that the sequence* $\{\lambda_k\}_1^\infty, \lim \lambda_k = 0$ *is given. Let us denote by* $\Phi(t, \tau, \lambda_k)$ *the fundamental solution of the equation*

$$\dot{z} = A_k(t)z,$$

where

$$A_k(t) = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ \lambda_k^{-1} A_{21}(t) & \lambda_k^{-1} A_{22}(t) \end{bmatrix}, \quad \Phi(t, \tau, \lambda_k) = \begin{bmatrix} \Phi_{11}(t, \tau, \lambda_k) & \Phi_{12}(t, \tau, \lambda_k) \\ \Phi_{21}(t, \tau, \lambda_k) & \Phi_{22}(t, \tau, \lambda_k) \end{bmatrix}.$$

Suppose that t^* *is a fixed point in* (t_0, T). *Then:*

(i) *The sequences* $\{\Phi(t, t^*, \lambda_k)\}_1^\infty$, $\{\Phi(t^*, \tau, \lambda_k)\}_1^\infty$, $\{\lambda_k^{-1}\Phi_{12}(t^*, \tau, \lambda_k)\}_1^\infty$ *are uniformly bounded for* $\tau \in [t_0, t^*]$ *and* $t \in [t^*, T]$;

(ii) *The following relations hold uniformly for* $\tau \in [t_0, t^*]$ *and* $t \in [t^*, T]$:

$$\lim_{k\to\infty} \Phi_{11}(t^*, \tau, \lambda_k) = \Phi_0(t^*, \tau), \quad \lim_{k\to\infty} \Phi_{11}(t, t^*, \lambda_k) = \Phi_0(t, t^*);$$

$$\lim_{k\to\infty} \Phi_{12}(t^*, \tau, \lambda_k) = 0, \quad \lim_{k\to\infty} \Phi_{12}(t, t^*, \lambda_k) = 0;$$

(iii) *Let* $\tau_1 \in [t_0, t^*)$, $t_1 \in (t^*, T]$ *be arbitrarily chosen. Then, uniformly* for $\tau \in [t_0, \tau_1]$ *and* $t \in [t_1, T]$

$$\lim_{k\to\infty} \lambda_k^{-1}\Phi_{12}(t^*, \tau, \lambda_k) = -\Phi_0(t^*, \tau)\, A_{12}(\tau)A_{22}^{-1}(\tau),$$

$$\lim_{k\to\infty} \lambda_k^{-1}\Phi_{12}(t, t^*, \lambda_k) = -\Phi_0(t, t^*)\, A_{12}(t^*)\, A_{22}^{-1}(t^*),$$

$$\lim_{k\to\infty} \Phi_{21}(t^*, \tau, \lambda_k) = -A_{22}^{-1}(t^*)A_{21}(t^*)\Phi_0(t^*, \tau),$$

$$\lim_{k\to\infty} \Phi_{21}(t, t^*, \lambda_k) = -A_{22}^{-1}(t)A_{21}(t)\Phi_0(t, t^*),$$

$$\lim_{k\to\infty} \Phi_{22}(t^*, \tau, \lambda_k) = 0, \quad \lim_{k\to\infty} \Phi_{22}(t, t^*, \lambda_k) = 0.$$

L e m m a 4. *Suppose that the sequence* $\{\lambda_k\}_1^\infty$, $\lim \lambda_k = 0$ *and the sequences* $\{f_k(t)\}_1^\infty$, $\{g_k(t)\}_1^\infty$ *of functions* $f_k \in L_p^{(n)}(t_0, T)$, $g_k \in L_p^{(m)}(t_0, T)$, *which are weakly converging to the functions* $f_0(t)$, $g_0(t)$ *are given. Let us denote by* $x_k(t)$, $y_k(t)$ *the solutions of the equations*

$$\begin{aligned} \dot{x} &= A_{11}(t)x + A_{12}(t)y + f_k(t), \quad x(t_0) = x^0, \\ \lambda_k \dot{y} &= A_{21}(t)x + A_{22}(t)y + g_k(t), \quad y(t_0) = y^0. \end{aligned} \tag{2.1}$$

Then the sequence $\{x_k(t)\}_1^\infty$ *is uniformly bounded in* $[t_0, T]$ *and for every* $t \in [t_0, T]$

$$\lim_{k\to\infty} x_k(t) = x_0(t),$$

where $x_0(t)$ *is the solution of the equation*

$$\dot{x} = A_0(t)x + f_0(t) - A_{12}(t)A_{22}^{-1}(t)g_0(t), \quad x(t_0) = x^0. \tag{2.2}$$

Corollary 1. *If the functions* g_k *in lemma 4 are continuous and the sequence* $\{g_k\}_1^\infty$ *is uniformly bounded in* $[t_0, T]$ *then the sequence* $\{y_k\}_1^\infty$ *is also uniformly bounded in* $[t_0, T]$.

Lemma 5. *Suppose that the sequences* $\{\lambda_k\}_1^\infty$, $\lim \lambda_k = 0$, $\{f_k(t)\}_1^\infty$ *and* $\{g_k(t)\}_1^\infty$ *of continuous functions are given. Suppose that the sequence* $\{f_k(t)\}_1^\infty$ *is uniformly bounded in* $[t_0, T]$ *and converging to the continous function* $f_0(t)$ *for every* $t \in [t_0, T]$. *The sequence* $\{g_k(t)\}_1^\infty$ *is uniformly converging to the function* $g_0(t)$ *in* $[t_0, T]$. *Let us denote by* $x_k(t)$, $y_k(t)$ *the solution of the equations* (2.1) *and by* $x_0(t)$ *the solution of the equation* (2.2); *for* $t \in [t_0, T]$ *we denote* $y_0(t) = -A_{22}^{-1}(t)\ (A_{21}(t)x_0(t) + g_0(t))$.

Then for every $t^* \in (t_0, T]$

$$\lim_{k\to\infty} (\max_{t_0 \le t \le T} \| x_k(t) - x_0(t) \| + \max_{t^* \le t \le T} \| y_k(t) - y_0(t) \|) = 0.$$

We omit the somewhat lengthly proofs of the presented lemmas, which will be published elsewhere.

3. Controllability. According to assumption H1 the system (1.2) can be written in the form of

$$\dot{x} = A_0(t)x + B_0(t)u, \quad x(t_0) = x^0, \tag{3.1}$$

where $A_0(t)$, $B_0(t)$ are defined in the assumption H2. Actually H2 is the well-known controllability condition [14] for the system (3.1).

Theorem 1. *Let us assume that hypothesis H1 holds. Then if the system* (3.1) *is controllable* (*hypothesis H2 holds*) *then there exists* $\Lambda^* \in (0, \Lambda]$ *so that for every* $\lambda \in (0, \Lambda^*]$ *the system* (1.1) *is conlrollable for* x.

Proof. Let us denote $U(t, \lambda) = \Phi_{11}(T, t, \lambda)B_1(t) + \lambda^{-1}\Phi_{12}(T, t, \lambda)B_2(t)$. According to lemma 3 we obtain that for every $T^* \in [t_0, T)$ and for every sequence $\{\lambda_k\}_1^\infty$, $\lim \lambda_k = 0$

$$\lim_{k\to\infty} U(t, \lambda_k) = \Phi_0(T, t)B_0(t) \stackrel{\text{def}}{=} U_0(t) \tag{3.2}$$

uniformly in $[t_0, T^*]$ and $\{U(t, \lambda_k)\}_1^\infty$ is uniformly bounded in $[t_0, T]$. Let us introduce the matrix $M(\lambda) = \int_{t_0}^{T} U(t, \lambda)U'(t, \lambda)dt$. From (3.2) $\lim M(\lambda_k) = M_0$, where M_0 is defined in (1.5). We show that for sufficiently small λ the matrix $M(\lambda)$ is invertible. On the contrary, let us assume that for every $\lambda \in (0, \Lambda]$ there exists $\xi \in R^n$ so that $\xi' M(\lambda)\xi = 0$. We can choose sequences $\{\lambda_k\}_1^\infty$, $\lim \lambda_k = 0$ and $\{\xi_k\}_1^\infty$, $\| \xi_k \| = 1$, $k = 1, 2 \dots$ so that $\xi_k' M(\lambda_k)\xi_k = 0$ for $k = 1, 2, \dots$. Thus, there exists a converging subsequence $\{\xi_{k_i}\}_1^\infty$, $\lim \xi_{k_i} = \xi_0$, $\| \xi_0 \| = 1$ and $\lim_{k_i\to\infty} \xi'_{k_i} M(\lambda_{k_i})\xi_{k_i} = \xi_0' M_0 \xi_0$, i. e. M_0 is singular. Hence, for sufficiently large k the matrix $M(\lambda_k)$ is nonsingular.

Now we prove that the system (1.1) is controllable for x, when $\lambda=\lambda_k$. Suppose $x^0, x^T \in R^n$; $y^0 \in R^m$ are arbitrarily chosen and $u_0(t)$ is a feasible control for $\lambda=0$. Let us choose $u_k(t)=u_0(t)+\Delta u_k(t)$, where

$$\Delta u_k(t)=-U'(t,\lambda_k)M^{-1}(\lambda_k)[(\Phi_{11}(T,t_0,\lambda_k)-\Phi_0(T,t_0))x^0$$

$$+\Phi_{12}(T,t_0,\lambda_k)y^0+\int_{t_0}^{T}(U(t,\lambda_k)-U_0(t))u_0(t)dt].$$

If we set $u_k(t)$ in Cauchy's formula, we get

$$x_k(T)=\Phi_{11}(T,t_0,\lambda_k)x^0+\Phi_{12}(T,t_0,\lambda_k)y^0+\int_{t_0}^{T}U(t,\lambda_k)(u_0(t)+\Delta u_k(t))dt$$

$$=\Phi_0(T,t_0)x^0+\int_{t_0}^{T}U_0(t)u_0(t)dt=x^T.$$

Hence, the system (1.1) is controllable with respect to x.

Corollary 2. *Suppose that the initial condition (x^0, y^0) and the final condition x^T are fixed. Then for every control $u_0(t)$ feasible for $\lambda=0$ and a sequence $\{\lambda_k\}_1^\infty$, $\lim\lambda_k=0$ there exists sequence of controls $\{u_k(t)\}_1^\infty$, $u_k(t)$ is feasible for $\lambda=\lambda_k$, so that $\{u_k(t)\}_1^\infty$ is uniformly bounded in $[t_0, T]$ and for every $t\in[t_0, T)$ we have $\lim_{k\to\infty}u_k(t)=u_0(t)$. The corresponding trajectory $(x_k(t), y_k(t))$ is uniformly bounded in $[t_0, T]$ and for every $t\in[t_0, T]$*

(3.3) $$\lim_{k\to\infty}x_k(t)=x_0(t),$$

where $x_0(t)$ corresponds to the control $u_0(t)$.

Proof. Let us choose $u_k(t)$ as in theorem 1. Then, according to lemma 3 the sequence $\{u_k(t)\}_1^\infty$ is uniformly bounded in $[t_0, T]$ and converging to $u_0(t)$ for every $t\in[t_0, T)$. Hence, the sequence $\{u_k\}_1^\infty$ is weakly converging to $u_0(t)$ in $L_p^{(r)}(t_0, T)$ and, applying lemma 4 we obtain (3.3).

4. Convergence. Let us fix the initial in condition (x^0, y^0) and the final state x^T. In this part we assume that hypotheses H1 and H2 hold and $\lambda\in[0,\Lambda^*]$, where Λ^* is determined in theorem 1. Consider the optimal control problems to determine a feasible control for $\lambda\in(0,\Lambda^*)$ and a feasible control for $\lambda=0$, which minimize the performance index (1.3). According to theorem 1 and the properties of the functional (1.3), there exists unique optimal control $u(t,\lambda)$ for the system (1.1) and unique optimal control $u_0(t)$ for the system (1.2ab). Let us denote by $J(\lambda)$, J_0 the optimal values of the performance index (1.3) and by $(x(t,\lambda), y(t,\lambda))$, $(x_0(t), y_0(t))$ the corresponding optimal states.

Theorem 2. *For every $\varepsilon>0$ and $t\in[t_0, T]$ there exists $\delta\in(0,\Lambda^*]$ so that for every $\lambda\in(0,\delta]$*

(4.1) $$|J(\lambda)-J_0|+\|x(t,\lambda)-x_0(t)\|<\varepsilon$$

and $x(t,\lambda)$ is bounded on $[t_0, T]\times(0,\Lambda^*]$.

Proof. Let the sequence $\{\lambda_k\}_1^\infty$, $\lim\lambda_k=0$ be arbitrarily chosen. According to corollary 2 there exists a control $\tilde{u}_k(t)$ feasible for $\lambda=\lambda_k$ so that for every $t\in[t_0, T)$ $\lim\tilde{u}_k(t)=u_0(t)$ and the corresponding optimal state $\lim\tilde{x}_k(t)=x_0(t)$ for $t\in[t_0, T]$, $\{\tilde{x}_k(t)\}_1^\infty$ is uniformly bounded. Then

(4.2) $$\limsup_{k\to\infty}J(\lambda_k)\le\lim_{k\to\infty}J(\tilde{u}_k,\lambda_k)=J_0$$

and the sequence $\{J(\lambda_k)\}_1^\infty$ is bounded. Because of the assumption for the function $h(u, t)$ the sequence $\{u(t, \lambda_k)\}_1^\infty$ is bounded in $L_p^{(r)}(t_0, T)$, i. e. a subsequence $\{u(t, \lambda_k)\}_1^\infty$ can be chosen, denoted in the same way and converged weakly to $\bar{u}(t)$. We prove that $\bar{u}(t)$ is a feasible control for $\lambda=0$. Using the notation of theorem 1 we have

$$\lim_{k\to\infty} \int_{t_0}^{T} U(t, \lambda_k)u(t, \lambda_k)dt = \int_{t_0}^{T} U_0(t)\bar{u}(t)dt$$

due to the inequality

$$\| \int_{t_0}^{T} (U(t, \lambda_k)u(t, \lambda_k) - U_0(t)\bar{u}(t))dt \|$$

$$\leq (\int_{t_0}^{T} \|U(t, \lambda_k) - U_0(t)\|^q dt)^{1/q} (\int_{t_0}^{T} \|u(t, \lambda_k)\|^p dt)^{1/p} + \| \int_{t_0}^{T} U_0(t) (u(t, \lambda_k) - \bar{u}(t))dt \|.$$

Hence, using lemma 3 we get

$$x^T = \lim_{k\to\infty} \tilde{x}_k(T) = \lim_{k\to\infty} [\Phi_{11}(T, t_0, \lambda_k)x^0 + \Phi_{12}(T, t_0, \lambda_k) y^0$$

$$+ \int_{t_0}^{T} U(t, \lambda_k)u(t, \lambda_k)dt] = \Phi_0(T, t_0)x^0 + \int_{t_0}^{T} U_0(t)\bar{u}(t)dt,$$

that is $\bar{u}(t)$ is a feasible control for $\lambda=0$. In [14, p. 227] it is proved that the functional $J(u, \lambda)$ is weakly lower semicontinous with respect to u. Then, using (4.2) and lemma 4 we have

$$J_0 \leq J(\bar{u}, 0) \leq \lim_{k\to\infty} \inf J(\lambda_k) \leq \lim_{k\to\infty} \sup J(\lambda_k) \leq J_0.$$

Thus, from the uniqueness of $u_0(t)$ there is only one weak limit point $u_0(t)$ of the sequence $\{u(t, \lambda_k)\}_1^\infty$. According to lemma 4 the sequence $\{x(t, \lambda_k)\}_1^\infty$ is converging to $x_0(t)$ for every $t \in [t_0, T]$ and it is uniformly bounded in $[t_0, T]$.

Summarizing, if we assume the contrary of (4.1) and choose a sequence $\{\lambda_k\}_1^\infty$. $\lim \lambda_k = 0$ we come to a contradiction. Thus the proof is completed.

Let us denote by

$$D(\lambda) = \{z \mid z \in R^{n+m+1}, \ z = (J(u, \lambda), \ x(T, u, \lambda), \ y(T, u, \lambda)), \ u \in L_p^{(r)}(t_0, T)\}$$

the attainable set for the problem (1.1),(1.3)(see [14]). It is known that $d(\lambda) = (J(\lambda), x^T, y(T, \lambda))$ belongs to the boundary of $D(\lambda)$.Let $N(\lambda)$ denote the set of the normed vectors $p(\lambda)$, orthogonal to the set $D(\lambda)$, at point $d(\lambda)$, $p(\lambda) = (\theta(\lambda), q(\lambda), s(\lambda))$. From the transversality conditions $s(\lambda) = s = 0$. Let D_0 be the attainable set for the problem (3.1), (1.3), where d_1 belongs to the boundary of D_0, $d_1 = (J_0, x^T)$; N_0 is the set of normed vectors p_1, orthogonal to D_0 at the point d_1, $p_1 = (\theta_0, q_\lambda)$. Let us denote $p_1(\lambda) = (\theta(\lambda), q(\lambda))$.

Lemma 6. *For every* $\varepsilon > 0$ *there exists* $\delta > 0$ *so that if* $\lambda \in (0, \delta]$ *and* $p(\lambda) \in N(\lambda)$ *we can choose* $p_1 \in N_0$ *which satisfies* $\|p_1(\lambda) - p_1\| < \varepsilon$.

Proof. Let us assume that on the contrary there exist $\varepsilon_0 > 0$ and a sequence $\{\lambda_k\}_1^\infty$, $\lim \lambda_k = 0$ so that if $p_{1k} = p_1(\lambda_k) = (\theta(\lambda_k), q(\lambda_k))$, $\lim p_{1k} = p_1^*$ then $\min\{\|p_1 - p_1^*\| : p_1 \in N_0\} \geq \varepsilon_0$. Hence there exists $z_1^* = (J(u^*, 0), x(T, u^*, 0))$ where $u^* \in L_p^{(r)}(t_0, T)$ so that $z_1^* \in D_0$ and $(z_1^* - d_1)'p_1^* = \alpha > 0$. Let us denote $d_1(\lambda_k)$

$=(J(\lambda_k), (x^T)$; $z_1^*(\lambda_k)=(J(u^*, \lambda_k), x(T, u^*, \lambda_k))$. According to theorem 1 $\lim d_1(\lambda_k)=d_1$ and from lemma 5 $\lim z_1^*(\lambda_k)=z_1^*$. Let us denote $d(\lambda_k)=(d_1 (\lambda_k), y(T, \lambda_k))$; $z^*(\lambda_k)=(z_1(\lambda_k), y(T, u^*, \lambda_k))$. From $s(\lambda_k)=0$ we obtain

$$(z_1^*(\lambda_k)-d_1(\lambda_k))'p_{1k}=(z^*(\lambda_k)-d(\lambda_k))'p(\lambda_k)\geq a/2>0$$

for a sufficiently large k. This inequality implies that $p(\lambda_k)$ is not an orthogonal vector to $D(\lambda_k)$, which contradicts the assumption.

Let us denote by $(\eta(t), \psi(t))$ the adjoint variables [14]. Setting $\sigma(t)=\psi(t)/\lambda$ we obtain the following adjoint equation for the problem (1.1), (1.3):

$$\begin{aligned} &\dot{\eta}=-\eta A_{11}(t)-\sigma A_{21}(t)-\theta(\lambda)\frac{\partial f}{\partial x}(x(t, \lambda), t, \lambda), \\ (4.3)\qquad &\lambda\dot{\sigma}=-\eta A_{12}(t)-\sigma A_{22}(t), \\ &\eta(T)=q(\lambda),\ \sigma(T)=s(\lambda)=s=0, \end{aligned}$$

where $(\theta(\lambda), q(\lambda), s(\lambda))\in N(\lambda)$.

The adjoint equation for the problem (3.1), (1.3) has the form of

$$\dot{\eta}_0=-\eta_0 A_0(t)-\theta_0\frac{\partial f}{\partial x}(x_0(t), t, 0),\ \eta_0(T)=q_0.$$

Denoting $\sigma_0(t)=-\eta_0(t)A_{12}(t)A_{22}^{-1}(t)$ the last equation can be thus written:

$$\begin{aligned} &\dot{\eta}=-\eta_0 A_{11}(t)-\sigma_0 A_{21}(t)-\theta_0\frac{\partial f}{\partial x}(x_0(t), t, 0), \\ (4.4)\qquad &0=-\eta_0 A_{12}(t)-\sigma_0 A_{22}(t), \end{aligned}$$

which can be obtained from (4.3) by setting $\lambda=0$.

L e m m a 7. *Let us suppose that the sequence* $\{\lambda_k\}_1^\infty$, $\lim \lambda_k=0$ *and the sequence* $\{p(\lambda_k)\}_1^\infty$, $p(\lambda_k)=(\theta(\lambda_k), q(\lambda_k), s)\in N(\lambda_k)$, $\lim(\theta(\lambda_k), q(\lambda_k))=p_1\in N_0$ *are arbitrarily chosen. Let us denote by* $(\eta_k(t), \sigma_k(t))$ *the solution of the equation* (4.3) *for* $\lambda=\lambda_k$; $(\eta_0(t), \sigma_0(t))$ *the solution of the equation* (4.4). *Then for every* $T^*\in[t_0, T)$

$$(4.5)\qquad \lim_{k\to\infty}(\max_{t_0\leq t\leq T}\|\eta_k(t)-\eta_0(t)\|+\max_{t_0\leq t\leq T^*}\|\sigma_k(t)-\sigma_0(t)\|)=0$$

and $\{\sigma_k(t)\}_1^\infty$ *is uniformly bounded in* $[t_0, T]$.

P r o o f. According to theorem 2 the sequence $\{x(t, \lambda_k)\}_1^\infty$ is converging to $x_0(t)$ everywhere in $[t_0, T]$ and it is uniformly bounded. Then, applying Lemma 5 in an appropriate way we get (4.5)

Let us denote by $\eta(t, \lambda)$, $\sigma(t, \lambda)$ the solution of the adjoint equation (4.3). The optimal control $u(t, \lambda)$ is defined by the maximum principle

$$\mathcal{H}(u(t, \lambda), t, \lambda)=\max_u \mathcal{H}(u, t, \lambda),$$

where $\mathcal{H}(u, t, \lambda)=\theta(\lambda)h(u, t)+(\eta(t, \lambda)B_1(t)+\sigma(t, \lambda)B_2(t))u$. Similarly, the optimal control $u_0(t)$ for $\lambda=0$ is defined by

$$\mathcal{H}(u_0(t), t, 0)=\max_u \mathcal{H}(u, t, 0),$$

where $\mathcal{H}(u, t, 0)=\theta_0 h(u, t)+(\eta_0(t)B_1(t)+\sigma_0(t)B_2(t))u$.

T h e o r e m 3. *For every* $\varepsilon>0$ *and* $T^*\in[t_0, T)$ *there exists* $\delta>0$ *so that for every* $\lambda\in(0, \delta]$

$$\max_{t_0 \leq t \leq T^*} \| u(t, \lambda) - u_0(t) \| < \varepsilon$$

and $u(t, \lambda)$ *is bounded in* $[t_0, T] \times (0, \Lambda^*]$.

Proof. On the contrary, let us assume that there exists a positive number ε_0, a sequence $\{\lambda_k\}_1^\infty$, $\lim \lambda_k = 0$ a sequence $\{p(\lambda_k)\}_1^\infty$, $p(\lambda_k) = (\theta(\lambda_k), q(\lambda_k), s) \in N(\lambda_k)$, $\lim (\theta(\lambda_k), q(\lambda_k)) = p_1^* \in N_0$, and a number $T^*[t_0, T)$ so that

$$\max_{t_0 \leq t \leq T^*} \| u(t, \lambda_k) - u_0(t) \| \geq \varepsilon_0 .$$

Let us denote

$$G_1(t, \lambda_k) = \max_u \mathcal{H}(u, t, \lambda_k), \quad G_2(t, \tau, \lambda_k) = \max_{\| u - u_0(\tau) \| \geq \varepsilon} \mathcal{H}(u, t, \lambda_k),$$

where $\varepsilon > 0$, $\tau \in [t_0, T^*]$ are arbitrarily chosen. The maximum in the last inequality is achieved for some $\tilde{u}$, $\| \tilde{u} \| < \infty$ and due to the strict convexity of the hamiltonian $\gamma(\tau) = G_1(\tau, 0) - G_2(\tau, \tau, 0) > 0$. From the uniform continuity of $(\eta(t, \lambda), \sigma(t, \lambda))$ in $[t_0, T^*]$ with respect to λ and lemma 7 it follows that for every $u \in R^r$, $\| u - u_0(\tau) \| \leq \varepsilon$,

$$| \mathcal{H}(u, \tau, 0) - \mathcal{H}(u, t, \lambda_k) | < \gamma(\tau)/4, \tag{4.6}$$

where $t \in \Delta(\tau)$, $\Delta(\tau)$ denotes a neighbourhood of τ and $\lambda_k \in O_\tau = [0, \Lambda(\tau)]$. Using (4.6) at $u = u_0(\tau)$ we have for $t \in \Delta(\tau)$, $\lambda_k \in O_\tau$

$$G_1(t, \lambda_k) = \max \mathcal{H}(u, t, \lambda_k) \geq \mathcal{H}(u_0(\tau), t, \lambda_k) \geq G_1(\tau, 0) - \gamma(\tau)/4. \tag{4.7}$$

Now we show that for $t \in \Delta(\tau)$, $\lambda \in O_\tau$

$$G_2(t, \tau, \lambda_k) < G_1(\tau, 0) - \gamma(\tau)/2. \tag{4.8}$$

Let us assume that there exists $\tilde{u}$, $\| \tilde{u} - u_0(\tau) \| \geq \varepsilon$, so that

$$\mathcal{H}(\tilde{u}, t, \lambda_k) \geq G_1(\tau, 0) - \gamma(\tau)/2. \tag{4.9}$$

Let us denote $u^* = u_0(\tau) + \mu^*(\tilde{u} - u_0(\tau))$, $\mu^* = \varepsilon / \| \tilde{u} - u_0(\tau) \|$. Then $\| u^* - u_0(\tau) \| = \varepsilon$ and from (4.8), (4.9) and the convexity of $\mathcal{H}(u, t, \lambda)$ we come to the following contradiction:

$$G_1(\tau, 0) - \gamma(\tau) = G_2(\tau, \tau, 0) \geq \mathcal{H}(u^*, \tau, 0) \geq \mathcal{H}(u^*, t, \lambda_k) - \gamma(\tau)/4$$
$$\geq \mu^* \mathcal{H}(\tilde{u}, t, \lambda_k) + (1 - \mu^*) \mathcal{H}(u_0(\tau), t, \lambda_k) - \gamma(\tau)/4 \geq \mu^*(G_1(\tau, 0) - \gamma(\tau)/2) + (1 - \mu^*)(G_1(\tau, 0)$$
$$- \gamma(\tau)/4) - \gamma(\tau)/4 = G_1(\tau, 0) - (1 + \mu^*/2)\gamma(\tau)/2 \geq G_1(\tau, 0) - 3\gamma(\tau)/4.$$

From the covering $\bigcup_\tau \Delta(\tau) \supset [t_0, T^*]$ we can choose a finite subcovering $\{\Delta(\tau_i)\}_1^s$. Let us denote $Y = \cap_1^s O_{\tau_i}$; if $t \in [t_0, T^*]$, then there exists $\Delta(\tau_{i_0})$ so that $t \in \Delta(\tau_{i_0})$ and for sufficiently large k we have $\lambda_k \in Y$. From (4.7) and (4.8) we have

$$\begin{aligned} \mathcal{H}(u(t, \lambda_k), t, \lambda_k) - \max_{\| u - u_0(\tau_{i_0}) \| \geq \varepsilon} \mathcal{H}(u, t, \lambda_k) &= G_1(t, \lambda_k) - G_2(t, \tau_{i_0}, \lambda_k) \\ &\geq G_1(\tau, 0) - \gamma(\tau_{i_0})/4 - G_1(\tau, 0) + \gamma(\tau_{i_0})/2 = \gamma(\tau_{i_0})/4 > 0, \end{aligned} \tag{4.10}$$

hence $\| u(t, \lambda_k) - u_0(\tau_{i_0}) \| < \varepsilon$. Similarly, setting $\lambda_k = 0$ in (4.10), $\| u_0(t) - u_0(\tau_{i_0}) \| < \varepsilon$. Hence, for every $t \in [t_0, T^*]$

$$\| u(t, \lambda) - u_0(t) \| \leq \| u(t, \lambda_k) - u_0(\tau_{i_0}) \| + \| u_0(\tau_{i_0}) - u_0(t) \| < 2\varepsilon,$$

which contradicts the assumption.

To prove the uniform boundedness of $u(t, \lambda)$ we assume that there exist sequences $\{t_k\}_1^\infty$, $\lim t_k = t^* \in [t_0, T]$; $\{\lambda_k\}_1^\infty$, $\lim \lambda_k = 0$ so that $\lim_{k\to\infty} \|u(t_k, \lambda_k)\| = \infty$. Due to the inequatity $\mathcal{H}(u, t, \lambda_k) \leq -a\, \theta(\lambda_k) \| u \|^p + c_1 \| u \|$, where $c_1 > \| \eta(t_k, \lambda_k) B_1(t_k) + \sigma(t_k, \lambda_k) B_2(t_k) \|$ and using the uniform boundedness of the sequence $\{(\eta(t_k, \lambda_k), \sigma(t_k, \lambda_k)\}_1^\infty$ we obtain

$$\lim_{k\to\infty} \mathcal{H}(u(t_k, \lambda_k), t_k, \lambda_k) = -\infty.$$

Choose $\bar{u}(t) = \text{const} = \bar{u}$. From the maximum principle $\mathcal{H}(\bar{u}, t_k, \lambda_k) \leq \mathcal{H}(u(t_k, \lambda_k) t_k, \lambda_k)$. Hence $\lim \mathcal{H}(\bar{u}, t_k, \lambda_k) \leq -\infty$. But

$$-\infty \geq \lim_{k\to\infty} \mathcal{H}(\bar{u}, {}_k t, \lambda_k) \geq \theta_0 h(\bar{u}, t^*) + \eta_0(t^*) B_1(t^*)\bar{u} + \liminf_{k\to\infty} \sigma(t_k, \lambda_k) B_2(t_k) u$$

and from the boundedness of $\sigma(t_k, \lambda_k)$ according to lemma 7 we come to a contradiction. This completes the proof.

Theorem 4. *For every* $\varepsilon > 0$; $t^*, T^* \in (t_0, T)$, $t^* < T^*$ *there exists* $\delta > 0$ *so that for every* $\lambda \in (0, \delta]$

$$\max_{t_0 \leq t \leq T} \| x(t, \lambda) - x_0(t) \| + \max_{t^* \leq t \leq T^*} \| y(t, \lambda) - y_0(t) \| < \varepsilon$$

and $y(t, \lambda)$ *is uniformly bounded in* $[t_0, T] \times (0, \Lambda^*]$.

Proof. In. [11] it is proved that the optimal control $u(t, \lambda)$ is a continuous function with respect to the time t for $\lambda \in (0, \Lambda^*]$. From theorem 3 and lemma 5 it follows that $y(t, \lambda)$ converges uniformly to $y_0(t)$ in $[t^*, T^*]$. From the uniform boundedness of $u(t, \lambda)$ and corollary 1 it follows that $y(t, \lambda)$ is uniformly bounded in $[t_0, T] \times (0, \Lambda^*]$. Then $x(t, \lambda)$ satisfies $\dot{x} = A_{11}(t)x + f^*(t, \lambda)$, $x(t_0) = x^0$, where $f^*(t, \lambda) = A_{12}(t) y(t, \lambda) + B_1(t) u(t, \lambda)$ converges to $f^*(t) = A_{12}(t) y_0(t) + B_1(t) u_0(t)$ almost everywhere in $[t_0, T]$ and it is uniformly bounded. Hence $x(t, \lambda)$ uniformly converges to $x_0(t)$ in $[t_0, T]$.

5. On the problem with free final state. In this part we shall show that the previous considerations can help to obtain similar results for the correctness of the problem with fixed initial and free final state. The performance index (1.4) is minimized. First we present an existence condition for the optimal control.

Theorem 5. *Let us suppose that there exists a compact set* $\Gamma \subset R^n$ *and numbers* $b \geq 0$, $\nu > p$ *so that for every* $\lambda \in [0, \Lambda]$ *and for every* $v \notin \Gamma$ *the function* $c(v, \lambda)$ *satisfies*

$$c(v, \lambda) \geq -b \, |v|^\nu . \tag{5.1}$$

Then for every $\lambda \in [0, \Lambda]$ *there exists an optimal control.*

Proof. First we show that there exists a bounded set $\Gamma_1 \subset R^n$ and a number $a_1 > 0$ so that if $\lambda \in [0, \Lambda]$, $u \in L_p^{(r)}(t_0, T)$ and $x(T, u, \lambda) \notin \Gamma_1$ then

$$J(u, \lambda) \geq a_1 \| x(T, u, \lambda) \|^p . \tag{5.2}$$

From lemma 3 there exist constant γ, γ_1 so that for every $\lambda \in (0, \Lambda]$

$$\gamma \geq \max [\, \| \Phi_0(T, t_0) x^0 \|, \| \Phi_{11}(T, t_0, \lambda) x^0 + \Phi_{12}(T, t_0, \lambda) y^0 \| \,],$$

$$\gamma_1 \geq \sup_{t_0 \leq t \leq T} \max [\, \| \Phi_0(T, t) B_0(t) \|, \| \Phi_{11}(T, t, \lambda) B_1(t) + \frac{1}{\lambda} \Phi_{12}(T, t, \lambda) B_2(t) \| \,] \times (T - t_0)^{(p-1)/p}.$$

Using Cauchy's formula and Hölder inequality we obtain

$$x(T, u, \lambda) \leq \gamma + \gamma_1 (T - t_0)^{(1-p)/p} \int_{t_0}^{T} \| u(t) \| dt \leq \gamma + \gamma_1 \left(\int_{t_0}^{T} \| u(t) \|^p dt \right)^{1/p}.$$

Let us choose $\Gamma_1 = \{v \mid v \in R^n, \| v \| < 2\gamma\}$. Then if $x(T, u, \lambda) \notin \Gamma_1$ we get

$$\| x(T, u, \lambda) \| \leq 2\gamma + 2\gamma_1 \| u \|_{L_p} - \| x(T, u, \lambda) \| \leq 2\gamma_1 \| u \|_{L_p} \leq 2\gamma_1 (J(u, \lambda)/a)^{1/p},$$

where the properties of the functions f, g are used. Let us suppose that $\lambda \in [0, \Lambda]$ is fixed. We prove now that $I(u, \lambda)$ has a lower bound. Let us assume that there exists a sequence $\{u_k(t)\}_1^\infty$, $u_k \in L_p^{(r)}(t_0, T)$ so that $\lim I(u_k, \lambda) = -\infty$. But from $J(u_k, \lambda) \geq 0$ we get $\lim \| x(T, u_k, \lambda) \| = \infty$. Then for a sufficiently large k we have

$$I(u_k, \lambda) = c(x(T, u_k, \lambda), \lambda) + J(u_k, \lambda)$$

$$\geq -b \| x(T, u_k, \lambda) \|^r + a_1 \| x(T, u_k, \lambda) \|^p = \| x(T, u_k, \lambda) \|^r (-b + a_1 \| x(T, u_k, \lambda) \|^{p-r}) > 0$$

and we come to a contradiction. Hence we can choose a minimizing sequence $\{\tilde{u}_k\}_1^\infty$. Let us choose u_0 arbitrarily in $L_p^{(r)}(t_0, T)$. Then for e sufficiently large k we have $I(\tilde{u}_k, \lambda) \leq I(u_0, \lambda)$ and from (5.3) $\{x(T, \tilde{u}_k, \lambda)\}_1^\infty$ is bounded. Hence from $I(\tilde{u}_k, \lambda) \geq a^p \| \tilde{u}_k \|_{L_p}^p - b \| x(T, \tilde{u}_k, \lambda) \|^r$ it follows that $\{\tilde{u}_k\}_1^\infty$ is bounded in $L_p^{(r)}(t_0, T)$. Now we can easily complete the proof using the continuity of c weak lower semicontinuity of $J(u, \lambda)$ and the closeness of the attainable set $D(\lambda)$.

The following result can be proved on the basis of this theorem.

C o r o l l a r y 3 [14]. *If the function $c(v, \lambda)$ is lower bounded or convex then there exists an optimal control.*

Let us assume that hypothesis H1 from the introductory section and condition (5.1) hold. Let us denote by $\mathcal{O}(\lambda) \subset L_p^{(r)}(t_0, T)$ the set of the optimal controls for $\lambda \in [0, \Lambda]$; $I(\lambda)$ is the optimal value of the performance index.

T h e o r e m 6. *For every $\varepsilon > 0$, $t \in [t_0, T]$ there exists $\delta \in (0, \Lambda]$ so that if $\lambda \in (0, \delta]$ and $u(t, \lambda) \in \mathcal{O}(\lambda)$ there exists $u_0(t) \in \mathcal{O}(0)$, thus if $x(t, \lambda)$, $x_0(t)$ are the corresponding optimal states, then $| I(\lambda) - I(0) | + \| x(t, \lambda) - x_0(t) \| < \varepsilon$ and $x(t, \lambda)$ is uniformly bounded in $[t_0, T] \times (0, \Lambda]$.*

P r o o f. Let us assume that there exist $\varepsilon_0 > 0$, $t^* \in [t_0, T]$ a sequence $\{\lambda_k\}_1^\infty$, $\lim \lambda_k = 0$ and a sequence $\{u_k\}_1^\infty$, $u_k \in \mathcal{O}(\lambda_k)$ thus for every $u_0 \in \mathcal{O}(0)$ the inequality $\| x(t^*, \lambda_k) - x_0(t^*) \| + | I(\lambda_k) - I(0) | \geq \varepsilon_0$ is satisfied. Let $u_0 \in \mathcal{O}(0)$. Then, using lemma 4

$$\limsup_{k \to \infty} I(\lambda_k) \leq \lim_{k \to \infty} I(u_0, \lambda_k) = I(0). \tag{5.4}$$

This inequality implies that the sequence $\{v_k\}_1^\infty$, $v_k = x(T, \lambda_k)$ is bounded. Apparently, on the contrary from (5.1) and (5.2) we get $I(\lambda_k) \geq -b \| v_k \|^r + a_1 \| v_k \|^p > I(0)$ for a sufficiently large k. Moreover $\| u_k \|_{L_p} \leq a^{-1} (I(\lambda_k) - b \| v_k \|^r)^{1/p}$ hence we can choose a weakly converging to u^* subsequence in $L_p^{(r)}(t_0, T)$ which we denote in the same way $\{u_k\}_1^\infty$. From lemma 4 it follows that the sequence of the corresponding states $\{x(t, \lambda_k)\}_1^\infty$ is converging to $x^*(t) = x(t, u^*, 0)$ everywhere in $[t_0, T]$ and it is uniformly bounded. But $I(u, \lambda)$ is weakly lower semicontinuous, hence

$$I(u^*, 0) \leq \liminf_{k\to\infty} I(u_k, \lambda_k)$$

and due to (5.4) $u^* \in \mathcal{D}(0)$, $\lim I(\lambda_k) = I(0)$, which contradicts the assumption. Thus the proof is completed.

Theorem 7. *Suppose that the function $c(v, \lambda)$ is convex with respect to v for fixed λ or the system* (3.1) *is controllable. Then for every $\varepsilon > 0$; $t^*, T^* \in (t_0, T)$, $t^* < T^*$ there exists $\delta \in (0, \Lambda]$ so that if $\lambda \in (0, \delta]$ and $u(t, \lambda) \in \mathcal{D}(\lambda)$ we can choose $u_0(t) \in \mathcal{D}(0)$ thus*

$$\max_{t_0\leq t\leq T} \| x(t, \lambda) - x_0(t) \| + \max_{t^*\leq t\leq T^*} \| y(t, \lambda) - y_0(t) \| + \max_{t_0\leq t\leq T^*} \| u(t, \lambda) - u_0(t) \| < \varepsilon,$$

where $(x(t, \lambda), y(t, \lambda))$, $(x_0(t), y_0(t))$ are the corresponding optimal states.

Proof. If we assume the contrary, then using lemmas 1, 4, 5, 7 and theorem 6 as in the proof of lemma 6 and theorems 3, 4 we come to a contradiction. The continuity of the optimal control $u(t, \lambda)$ with respect to t for $\lambda \in [0, \lambda]$ which is used in theorem 4 can be easily proved on the basis of [11, lemma 1].

Note. If the function $c(v, \lambda)$ is convex and continuously differentiable with respect to v then from [14] the optimal control is uniquely defined and satisfies the maximum principle, where

$$\eta(T, \lambda) = q(\lambda) = -\frac{\partial c}{\partial x}(x(T, \lambda), \lambda).$$

Hence, lemma 6 follows immediately.

6. An application. Finally we consider the control system

$$\begin{aligned} \dot{x} &= A_{11}(t)x + A_{12}(t)y + \lambda B_1(t)u, \\ \lambda \dot{y} &= A_{21}(t)x + A_{22}(t)y + \lambda B_2(t)u \end{aligned} \tag{6.1}$$

with the following performance index

$$J(u, \lambda) = \int_{t_0}^{T} (f(x, t, \lambda) + u'Q(t)u)dt. \tag{6.2}$$

Let us suppose that the initial state (x^0, y^0) and the final state x^T are fixed, the matrices $A_{ij}(t)$, $B_i(t)$ and the function $f(x, t, \lambda)$ satisfy the assumption, given in the introductory section; the matrix $A_{22}(t)$ satisfies the assumption H1; the matrix $Q(t)$ is symmetric, continuous and positively definite. The essential difference between the former and the latter case is that the behaviour of the optimal solution is investigated in a neighbourhood of $\lambda = 0$ for a system, which is uncontrollable for $\lambda = 0$.

Additionaly we introduce the control system

$$\dot{x} = A_0(t)x + B_0(t)v, \quad x(t_0) = x^0, \tag{6.3}$$

where the matrices $A_0(t)$, $B_0(t)$ are defined as before, and the following performance index

$$J(v) = \int_{t_0}^{T} v'Q(t)v dt. \tag{6.4}$$

Let us denote by $u(t, \lambda)$, $x(t, \lambda)$, $y(t, \lambda)$, $J(\lambda)$ the optimal solution of the problem (6.1), (6,2) and by $v_0(t)$, $x_0(t)$, J_0 the optimal solution of the problem (6.3), (6.4); $y_0(t) = -A_{22}^{-1}(t)(A_{21}(t)x_0(t) + B_2(t)v_0(t))$.

Theorem 8. *Let us suppose that the system* (6.3) *is controllable. Then for a sufficiently small* λ *the system* (6.1) *is controllable with respect to* x *and for every* $\varepsilon>0$; $t^*, T^*\in(t_0, T)$, $t^*<T^*$, *there exists* $\delta>0$ *so that*

$$\max_{t_0\le t\le T}\|x(t,\lambda)-x_0(t)\|+\max_{t^*\le t\le T^*}\|y(t,\lambda)-y_0(t)\|$$
$$+\max_{t_0\le t\le T^*}\|\lambda u(t,\lambda)-v_0(t)\|+|\lambda^2 J(\lambda)-J_0|<\varepsilon$$

if $\lambda\in(0,\delta)$.

Proof. If we set $v=\lambda u$ then the first part of the theorem follows from theorem 1. Let us consider the problem to optimize the system (6.1) with a control $v=\lambda u$ and the following performance index

$$I(v,\lambda)=\lambda^2 J(v,\lambda)+\int_{t_0}^{T}(\lambda^2 f(x,t,\lambda)+v'Q(t)v)dt.$$

Obviously, for $\lambda>0$ the solution $(x(t,\lambda), y(t,\lambda))$ of this problem is also optimal for the problem (6.1), (6.2) and the optimal control $u(t,\lambda)=v(t,\lambda)/\lambda$ Therefore, we can use the results given in part 4 to complete the proof.

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