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PARTIALLY MONOTONE INTERPOLATION

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Theorems are proved on interpolation and approximation (in Hausdorff and uniform distance) of piece-wise monotone functions by algebraic polynomials which follow the same monotony pattern. The estimates for the degree of the interpolating polynomial and the orders of approximation are exact.

1. During the last ten years an increasing number of publications have appeared on the approximation or interpolation of functions with definite properties, where the approximating element is required to possess the same properties.

In the papers [1—14] a monotone or piece-wise monotone function f is approximated by algebraic polynomials of degree not greater than n conserving entirely the monotony of the function f. A basic result in this area is that the order of this approximation is $\omega(f; n^{-1})$, where $\omega(f; \delta)$ is the continuity modulus of the function f:

$$\omega(f; \delta)=\sup\{|f(x')-f(x'')|:|x'-x''|\leq\delta\}.$$

Let the points (x_i, y_i), $i=0, 1, \ldots, m$, $x_i=i/m$, $y_i<y_{i+1}$, $y_0=0$, $y_m=1$ be given. Wolibner [15] and Young [16] proved that there exists an algebraic polynomial P, such that $P(x_i)=y_i$, $i=0, 1, \ldots, m$ and P is monotone in [0, 1] without giving, however, an estimate for the degree of the polynomial. Nikoltcheva [17] makes this result more precise by showing that this polynomial can be of degree $cm\ln m$ when $\Delta y_i=y_{i+1}-y_i>m^{-\alpha}$, $m<\alpha<1$. This estimate for the degree of an interpolation monotone polynomial is exact.

In the present paper two theorems have been proved giving an exact answer concerning the degree of the partially monotone interpolation polynomial:

Theorem 1. *Let the points* (x_i, y_i), (x_{-i}, y_{-i}), $i=1, \ldots, m$, $(x_0=0$, $y_0=0)$; $x_i=i/m$, $x_{-i}=-i/m$, $y_m=y_{-m}=1$, $\Delta y_i=y_{i+1}-y_i>0$, $\Delta y_{-i}=y_{-(i+1)}-y_{-i}>0$, $i=0, \ldots, m-1$; $A=\max\{\Delta y_i : 1-m\leq i\leq m-1\}$, $B=\min\{\Delta y_i : 1-m\leq i\leq m-1\}$ *be given.*

Let $k\geq 1$ *be an integer. If n satisfies* $k\ln n/n\leq 1/8m, 4m(A+B)/n^{k/4}. B<1$, *then there exists an algebraic polynomial* $P_{n,k}\in H^2_{4n+1}$, *for which*:

$$P_{n,k}(x_i)=y_i, \quad P_{n,k}(x_{-i})=y_{-i}, \quad i=1, \ldots, m,$$
$$P_{n,k}(0)=0,$$

where H^2_s *is the class of algebraic polynomials P of degree not greater than s, for which*:

$$P'(x)\leq 0 \quad for \quad x\in[-1, 0],\ P'(x)\geq 0 \quad for \quad x\in[0, 1].$$

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Theorem 2. *For the notations of Theorem* 1, *let* $B\geq cm^{-\beta}$, $m>\beta>1$, $n>m>\max\{8/c, e^5\}$. *If* $n>45\beta m\ln m$, *then there exists an algebraic polynomial* $P_n\in H^2_{4n+1}$, *for which* $P_n(x_i)=y_i$, $P_n(x_{-i})=y_{-i}$, $i=1,\ldots,m$, $P_n(0)=0$.

In the last part of this paper some corollaries have been obtained from Theorem 1 and Theorem 2 for the partially monotone approximation of partially monotone functions with respect to the Hausdorff distance and the partially monotone local approximations.

2. To prove Theorem 1 we need the following three lemmas that have been proved in [17] and [18].

Lemma 1. *For every positive integer* k, $1\leq k\leq n/2\ln n$, *there exists an algebraic polynomial* $A_{n,k}\in H_{2n}$, $(H_n=\{P:P(x)=a_nx^n+\ldots+a_1x+a_0\}$ *with the properties*

$$A_{n,k}\geq 0,\quad x\in[-1,1] \tag{1}$$

$$A_{n,k}(x)\leq 2e^4/n^{2k-1},\ \lambda_{k,n}\leq|x|\leq 1,\ \text{where } \lambda_{k,n}=k\ln n/n(=(k\ln n)/n), \tag{2}$$

$$\int_{-\lambda_{k,n}}^{\lambda_{k,n}} A_{n,k}(x)dx\geq 1. \tag{3}$$

Lemma 2. *If* $|\varepsilon_{i,j}|<\varepsilon$; $j, i=1,\ldots,m$ *and* $M>m\varepsilon(A+B)/B$, $A=\max\{b_i$ $1\leq i\leq m\}$. $B=\min\{b_i:1\leq i\leq m\}$, $b_i>0$, $i=1,\ldots,m$ *then the system*

$$\begin{bmatrix} M+\varepsilon_{1,1} & \varepsilon_{1,2} & \cdots & \varepsilon_{1,m}\\ \varepsilon_{2,1} & M+\varepsilon_{2,2} & \cdots & \varepsilon_{2,m}\\ \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots\\ \varepsilon_{m,1} & \varepsilon_{m,2} & \cdots & M+\varepsilon_{m,m}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\ \cdot\\ \cdot\\ x_m\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\ \cdot\\ \cdot\\ b_m\end{bmatrix}$$

has a unique positive solution.

Lemma 3. *For every* $n>1$ *and* δ, $8\ln n/n\leq\delta\leq 1$, *there exists an algebraic polynomial* $\sigma_{n,\delta}(x)\in H_{2n}$, *such that*:

$$|1+\sigma_{n,\delta}(x)|\leq e^{-n\delta/4}\quad \text{for}\quad x\in[-1,-\delta], \tag{4}$$

$$|1-\sigma_{n,\delta}(x)|\leq e^{-n\delta/4}\quad \text{for}\quad x\in[\delta,1], \tag{5}$$

$$-1\leq\sigma_{n,\delta}(x)\leq 1\quad \text{for}\quad |x|\leq\delta, \tag{6}$$

$$\sigma_{n,\delta}(x)\leq 0\quad \text{for}\quad x\in[-1,0];\quad \sigma_{n,\delta}(x)\geq 0\quad \text{for}\quad x\in[0,1]. \tag{7}$$

We will prove the following

Lemma 4. *If* $3k\ln n/n\leq a\leq 1-(2k\ln n/n)$, *then for any* k, $8\leq k\leq n/2\ln n$, *there exists an algebraic polynomial* $A_{n,k}(x,a)\in H_{4n}$ *with the properties*;

$$A_{n,k}(x,a)\leq 0\quad \text{for}\quad x\in[-1,0],\ A_{n,k}(x,a)\geq 0\quad \text{for}\quad x\in[0,1], \tag{8}$$

$$|A_{n,k}(x,a)|\leq 4e^4/n^{2k-1}\quad \text{for}\quad -1\leq x\leq a-2\lambda_{k,n}, \tag{9}$$

$$|A_{n,k}(x,a)|\leq 4e^4/n^{2k-1}\quad \text{for}\quad a+2\lambda_{k,n}\leq x\leq 1;\ \lambda_{k,n}=k\ln n/n \tag{10}$$

$$\int_{a-2\lambda_{k,n}}^{a+2\lambda_{k,n}} A_{n,k}(x,a)dx\geq 1-1/n^{k/4}. \tag{11}$$

Proof. Take the polynomial $A_{n,k}((x-\alpha)/2)\sigma_{n,\delta}(x)$, where $A_{n,k}$ is the polynomial from Lemma 1, $\sigma_{n,\delta}$ is the polynomial from Lemma 3, $\delta = k \ln n/n$.

By Lemma 1 and Lemma 3 it follows that

1. $A_{n,k}(x, \alpha) \leq 0$ for $x \in [-1, 0]$, $A_{n,k}(x, \alpha) \geq 0$ for $x \in [0,1]$ since $A_{n,k}((x-\alpha)/2) \geq 0$ for $x \in [-1, 1]$, $\sigma_{n,\delta}(x) \leq 0$ for $x \in [-1, 0]$, $\sigma_{n,\delta}(x) \geq 0$ for $x \in [0, 1]$.

2. Let $x \in [-1, \alpha - 2\lambda_{k,n}]$. Then (by Lemma 1):

$$(12) \qquad A_{n,k}\left(\frac{x-\alpha}{2}\right) \leq 2e^4 n^{-2k+1}.$$

Since $\delta = k \ln n/n$ and $\alpha > 3k \ln n/n$, Lemma 3 yields

$$(13) \qquad |\sigma_{n,\delta}(x)| \leq 1 + e^{-n\delta/4} = 1 + 1/n^{k/4}.$$

From (12) and (13) one gets

$$(14) \qquad |A_{n,k}(x, \alpha)| = |A_{n,k}((x-\alpha)/2)|\,|\sigma_{n,\delta}(x)| \leq 2e^4 n^{-2k+1}(1+n^{-k/4}) \leq 4e^4 n^{-2k+1}.$$

3. For $\alpha + 2\lambda_{k,n} \leq x \leq 1$, again by Lemma 1 and Lemma 3, one obtains

$$|A_{n,k}((x-\alpha)/2)| \leq 2e^4 n^{-2k+1}, \quad |\sigma_{n,\delta}(x)| \leq 1 + n^{-k/4}.$$

or $|A_{n,k}(x, \alpha)| \leq 4e^4 n^{-2k+1}$.

4. Lemma 1 and Lemma 3 and the mean values theorem yield:

$$\int_{\alpha-2\lambda_{k,n}}^{\alpha+2\lambda_{k,n}} A_{n,k}(x, \alpha)dx = \int_{\alpha-2\lambda_{k,n}}^{\alpha+2\lambda_{k,n}} A_{n,k}\left(\frac{x-\alpha}{2}\right)\sigma_{n,\delta}(x)dx = \sigma_{n,\delta}(\xi_\alpha)\int_{\alpha-2\lambda_{k,n}}^{\alpha+2\lambda_{k,n}} A_{n,k}\left(\frac{x-\alpha}{2}\right)dx,$$

where $\alpha - 2\lambda_{k,n} \leq \xi_\alpha \leq \alpha + 2\lambda_{k,n}$. Or

$$\int_{\alpha-2\lambda_{k,n}}^{\alpha+2\lambda_{k,n}} A_{n,k}(x, \alpha)dx = 2\sigma_{n,\delta}(\xi_\alpha)\int_{-\lambda_{k,n}}^{\lambda_{k,n}} A_{n,k}(y)dy \geq (1+\tau_\alpha)\int_{-\lambda_{k,n}}^{\lambda_{k,n}} A_{n,k}(y)dy,$$

where $|\tau_\alpha| \leq 1/n^{k/4}$, $\int_{-\lambda_{k,n}}^{\lambda_{k,n}} A_{n,k}(y)dy \geq 1$.

Thus, the Lemma is proved.

Lemma 5. *If* $3k \ln n/n \leq \alpha \leq 1 - 2k \ln n/n$, *then for any* k, $8 \leq k \leq n/2 \ln n$, *there exist an algebraic polynomial* $B_{n,k}(x, \alpha) \in H_{4n+1}$, *with the properties*

$$(15) \qquad B_{n,k}(x, \alpha) \leq 0 \quad \textit{for} \quad x \in [-1, 0],$$

$$B_{n,k}(x, \alpha) \geq 0 \quad \textit{for} \quad x \in [0, 1],$$

$$(16) \qquad 0 \leq B_{n,k}(x, \alpha) \leq 2/n^{k/4} \quad \textit{for} \quad -1 \leq x \leq \alpha - 2\lambda_{k,n},$$

$$(17) \qquad |B_{n,k}(x, \alpha) - N| \leq 2/n^{k/4} \quad \textit{for} \quad \alpha + 2\lambda_{k,n} \leq x \leq 1,\ \lambda_{k,n} = k \ln n/n,$$

$$(18) \qquad 0 \leq |B_{n,k}(x, \alpha)| \leq N + 2/n^{k/4} \quad \textit{for} \quad \alpha - 2\lambda_{k,n} \leq x \leq \alpha + 2\lambda_{k,n},$$

where $N = \int_{-\lambda_{k,n}}^{\lambda_{k,n}} A_{n,k}(y)dy \geq 1$,

$$(19) \qquad B_{n,k}(0, \alpha) = 0.$$

Proof. This Lemma follows from Lemma 4 for

$$B_{n,k}(x, a)=\int_{-1}^{x} A_{n,k}(y, a)dy-\int_{0}^{1} A_{n,k}(y, a)dy.$$

Proof of Theorem 1. Let $\varepsilon>0$ be arbitrary and $c=m\varepsilon(A+B)/B$. Consider the polynomial

$$P_{n,k}(x)=\sum_{i=1}^{m} a_{m-i+1}cB_{n,k}(-x, (2i-1)/2m)+\sum_{i=1}^{m} a_{m+i}\, cB_{n,k}(x, (2i-1)/2m).$$

Obviously, $P_{n,k}(x)\in H_{4n+1}$. We will show that the system

$$P_{n,k}(x_{-i})=y_{-i},\quad P_{n,k}(x_i)=y_i;\quad i=1, \dots, m, \tag{20}$$

has a positive solution with respect to $\{a_i\}_{i=1}^{2m}$.

The condition of the Theorem $k\ln n/n\leqq 1/8m$ and (16) and (17) imply

$$B_{n,k}(-j/m, (2i-1)/2m)\leqq 2/n^{k/4},\quad 1\leqq j\leqq m,$$

$$B_{n,k}(j/m, (2i-1)/2m)\leqq 2/n^{k/4},\quad 1\leqq j\leqq i-1,$$

$$B_{n,k}(j/m, (2i-1)/2m)=N+\delta_{k,n},\ i\leqq j\leqq m,$$

where $N\geqq 1$ does not depend on i, j; $|\delta_{k,n}|\leqq 2/n^{k/4}$.

But then the system (20) might be written as follows:

$$\begin{bmatrix} L_{11} & L_{12} \\ & \\ L_{21} & L_{22} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_{2n} \end{bmatrix}=\begin{bmatrix} y_{-m} \\ y_{-(m-1)} \\ \cdot \\ \cdot \\ \cdot \\ y_m \end{bmatrix},$$

$$L_{11}=\begin{bmatrix} M+\delta_{11} & M+\delta_{1,2} & M+\delta_{1,3} & \dots & M+\delta_{1,m-1} & M+\delta_{1,m} \\ \delta_{21} & M+\delta_{2,2} & M+\delta_{2,3} & \dots & M+\delta_{2,m-1} & M+\delta_{2,m} \\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ \delta_{m-1,1} & \delta_{m-1,2} & \delta_{m-1,3} & \dots & M+\delta_{m-1,m-1} & M+\delta_{m-1,m} \\ \delta_{m,1} & \delta_{m,2} & \delta_{m,3} & \dots & \delta_{m,m-1} & M+\delta_{m,m} \end{bmatrix},$$

$$L_{12}=\begin{bmatrix} \delta_{1,m+1} & \delta_{1,m+2} & \delta_{1,m+3} & \dots & \delta_{1,2m-1} & \delta_{1,2m} \\ \delta_{2,m+1} & \delta_{2,m+2} & \delta_{2,m+3} & \dots & \delta_{2,2m-1} & \delta_{2,2m} \\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot \\ \delta_{m-1,m+1} & \delta_{m-1,m+2} & \delta_{m-1,m+3} & \dots & \delta_{m-1,2m-1} & \delta_{m-1,2m} \\ \delta_{m,m+1} & \delta_{m,m+2} & \delta_{m,m+3} & \dots & \delta_{m,2m-1} & \delta_{m,2m} \end{bmatrix},$$

$$L_{21}=\begin{bmatrix}\delta_{m+1,1} & \delta_{m+1,2} & \delta_{m+1,3} & \dots & \delta_{m+1,m-1} & \delta_{m+1,m}\\ \delta_{m+2,1} & \delta_{m+2,2} & \delta_{m+2,3} & \dots & \delta_{m+2,m-1} & \delta_{m+2,m}\\ \dots & \dots & \dots & \dots & \dots & \dots\\ \delta_{2m,1} & \delta_{2m,2} & \delta_{2m,3} & \dots & \delta_{2m,m-1} & \delta_{2m,m}\end{bmatrix},$$

$$L_{22}=\begin{bmatrix}M+\delta_{m+1,m+1} & \delta_{m+1,m+2} & \delta_{m+1,m+3} & \dots & \delta_{m+1,2m-1} & \delta_{m+1,2m}\\ M+\delta_{m+2,m+1} & \delta_{m+2,m+2} & \delta_{m+2,m+3} & \dots & \delta_{m+2,2m-1} & \delta_{m+2,2m}\\ \dots & \dots & \dots & \dots & \dots & \dots\\ M+\delta_{2m,m+1} & M+\delta_{2m,m+2} & M+\delta_{2m,m+3} & \dots & M+\delta_{2m,2m-1} & M+\delta_{2m,2m}\end{bmatrix},$$

where $|\delta_{i,j}|\leqq 2c/n^{k/4}$, $M=cN$.

From this system, by subtracting j-th row from the $j+1$-th, $j=1,\dots,m-1$, and by subtracting the $m+j$-th row from the $m+j-1$-th one, $j=1,\dots,m$, we obtain the system:

$$\begin{bmatrix}M+\varepsilon_{1,1} & \varepsilon_{1,2} & \dots & \varepsilon_{1,2m}\\ \varepsilon_{2,1} & M+\varepsilon_{2,2} & \dots & \varepsilon_{2,2m}\\ \dots & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots\\ \varepsilon_{2m,1} & \varepsilon_{2m,2} & \dots & M+\varepsilon_{2m,2m}\end{bmatrix}\begin{bmatrix}a_1\\ a_2\\ \cdot\\ \cdot\\ a_{2m}\end{bmatrix}=\begin{bmatrix}\Delta y_{-(m-1)}\\ \Delta y_{-(m-2)}\\ \cdot\\ \cdot\\ \Delta y_{m-1}\end{bmatrix}, \tag{21}$$

where $|\varepsilon_{i,j}|\leqq 4c/n^{k/4}$, $\Delta y_i>0$.

But $c=m\varepsilon(A+B)/B$. Therefore, $M>m\varepsilon(A+B)/B$, since $M=cN>C$. Lemma 2 yields that if $|\varepsilon_{i,j}|<\varepsilon$, then the system (21) has a positive solution, i. e. the following condition must be satisfied: $4c/n^{k/4}<\varepsilon=cB/m(A+B)$, or

$$4m(A+B)/n^{k/4}B<1. \tag{22}$$

Therefore $\ln n>4k^{-1}\ln((A+B)4m/B)$ must be fulfilled.

Since the condition (22) is formulated in the conditions of the theorem, Lemma 2 implies that the system (21), i. e. the system (20) as well, have a positive solution. But then the polynomial $P_{n,k}(x)$ will satisfy the conditions of the theorem, since $B_{n,k}(x,(2i-1)/2m)$ and $B_{n,k}(-x,(2i-1)/2m)$ are monotonely decreasing in $[-1,0]$ and monotonely increasing in $[0,1]$. The condition $P_{n,k}(0)=0$ is obtained from the fact that $B_{n,k}(0,(2i-1)/2m)=0$.

Thus, the theorem is proved.

Proof of Theorem 2. If $B\geqq cm^{-\beta}$, then $(A+B)/B\leqq 2m^{\beta}/c$. Then the conditions of Theorem 1 take the form: $k\ln n/n<1/8m$, $8m^{\beta+1}/cn^{k/4}<1$.

Let $k=(\beta+2)/4$. The second condition becomes $8m^{\beta+1}/cn^{\beta+2}<1$.

Since $n>m>8/c$, then this condition is satisfied.

For $k=(\beta+2)/4$, the first condition assumes the form:

$$\left(\frac{\beta+2}{4}\right)\frac{\ln n}{n}<\frac{1}{8m}. \tag{23}$$

But by definition $\beta>1$. Therefore, $(\beta+2)/4<3\beta/4$.

Then (23) takes the form

(24) $$6\beta \ln n/n < 1/m.$$

Let $n = \tau\beta m \ln m$, where τ is an arbitrary positive number for the time being. Then (24) might be written in the form:

(25) $$\frac{6 \ln (\tau\beta m \ln m)}{\tau \ln n} < 1.$$

But

$$\frac{6 \ln (\tau\beta m \ln m)}{\tau \ln m} < 6 \frac{\ln \tau\beta + 2 \ln m}{\tau \ln m} = 6\left(\frac{\ln \tau\beta}{\tau \ln m} + \frac{2}{\tau}\right) \leqq 6\left(\frac{\ln \tau}{\tau \ln m} + \frac{3}{\tau}\right) \leqq 6\left(\frac{\ln \tau}{5\tau} + \frac{3}{\tau}\right).$$

The above inequality follows from the condition of the theorem, according to which $m > e^5$. If $\tau = 45$, then

$$6\left(\frac{\ln \tau}{5\tau} + \frac{3}{\tau}\right) < 1,$$

and therefore (25) is fulfilled. The theorem is proved.

Using Lemma 1 from [19] it can be proved that the order $O(m \ln m)$ from Theorem 2 cannot be improved. This will become clear from the considerations below.

3. We will state some corollaries from Theorem 2 giving exact estimates for the partially monotone approximations with respect to the Hausdorff distance and the partially monotone local approximations.

The Hausdorff distance between functions was introduced by Bl. Sendov and B. Penkov [20] and further developed in the theory of approximations by Bl. Sendov and his school [21].

Let the set F_Δ consist of all bounded along the axis y and closed point sets in the plane which are convex with respect to the x axis and whose projection on the x axis coincides with the interval Δ. The Hausdorff distance in the set F_Δ is defined by

$$r(F, G) = \max\{\sup_{A \in F} \inf_{B \in G} d(A, B), \sup_{A \in G} \inf_{B \in G} d(A, B)\},$$

where $d(A, B) = \max\{|a_1 - b_1|, |a_2 - b_2|\}$,

$$A = (a_1, a_2), \quad B = (b_1, b_2); \quad F \in F_\Delta, \quad G \in F_\Delta.$$

Let f be a function bounded in Δ. By $\bar{f}$ denote the complemented graph of f:

$$\bar{f} = \cap \{F : F \in F_\Delta, \quad f \subset F\},$$

where $\bar{f}$ denotes the graph of the function f.

Obviously, if f is a continuous function, then $f = \bar{f}$.

The Hausdorff distance $r(f, g)$ between two functions f and g, bounded in Δ, is determined as a Hausdorff distance between their complemented graphs: $r(f, g) = r(\bar{f}, \bar{g})$.

Let us denote by

$$E_n(f)_r = \inf_{P \in H_n} r(f, P)$$

the best approximation of the function f, bounded in the set $\Delta=[a, b]$, by means of algebraic polynomials of degree not greater than n. A fundamental result in the theory of Hausdorff approximations is the universal estimate obtained by Bl. Sendov [22], as follows:

$$E_n(f)_r = O(\ln n/n).$$

This estimate is kept when a monotone bounded function is approximated by a monotone polynomial of H_n with respect to the Hausdorff distance (see [10]).

Theorem 3. *If f is a continuous function in $[-1, 1]$, monotonely decreasing in $[-1, 0]$ and monotonely increasing in $[0, 1]$, then for any positive integer n, there exists a polynomial $P \in H_n^2$, for which, if $x \in [-1, 1]$:*

$$|f(x) \div P(x)| \leq a \ln n/n, \tag{26}$$

where a is an absolute positive constant, and

$$|f(x) \div P(x)| = \max\{\inf_{t\in[-1,1]} \max\{|x-t|, |f(x)-P(t)|\},$$

$$\inf_{t\in[-1,1]} \max\{|x-t|, |f(t)-P(x)|\}\}.$$

Proof. Let m be a positive integer. Consider the system of points (x_{-i}, y_{-i}), (x_i, y_i), where $x_i = i/m$, $x_{-i} = -i/m$, $y_i = f(i/m)+(i-1)/m^3$, $y_{-i} = f(-i/m)+(i-1)/m^3$, $i=1, \ldots, m$. From Theorem 2, for $m>e^5$, it follows that there exists a polynomial $P_1(x)$ of degree $135\, m \ln m$, which is of the class $H^2_{135\, m \ln m}$, for which: $y_i = P_1(x_i)$, $y_{-i} = P_1(x_{-i})$, $i=1, \ldots, m$.

Let $x \in [x_{i-1}, x_i]$ or $[x_{-i}, x_{-i+1}]$. It is easily seen that

$$\inf_{t\in[-1,1]} \max\{|x-t|, |f(x)-P_1(t)|\} \leq 1/m + 1/m^3 < 2/m,$$

$$\inf_{t\in[-1,1]} \max\{|x-t|, |f(t)-P_1(x)|\} \leq 1/m + 1/m^3 < 2/m.$$

Therefore, if $m = n/\ln n$,

$$|f(x) \div P_1(x)| \leq 2 \ln n/n.$$

But, if $m = n/\ln n$, then $135 m \ln m \leq bn$.
Then there exists a polynomial $P \in H_n^2$, for which

$$|f(x) \div P(x)| \leq a \ln n/n.$$

From the definition of a Hausdorff distance it is clear that, if f is continuous in $[-1, 1]$, $P \in H_n$:

$$\max_{-1\leq x\leq 1} |f(x) \div P(x)| = r(f, P).$$

Then the estimate (26) from Theorem 3 can be written, as follows:

$$r(f, P) \leq a \ln n/n. \tag{27}$$

As seen, the estimate (27) does not depend on the structural properties (for example, the continuity modulus) of the continuous function f. Then, since the complemented graph of every function, bounded in $[-1, 1]$, can be approximated arbitrarily well by means of continuous functions with respect to the Hausdorff distance, then the estimate (27) holds also for the Hausdorff approximation of arbitrary, bounded partially monotone functions in $[-1, 1]$, by means of partially monotone polynomials.

Before going further on, we have to define the concept of a local continuity modulus for a given function f, continuous in $[-1, 1]$. This definition can be found in [19].

Let f be continuous in $[-1, 1]$, $x\in[-1, 1]$. The number

$$\omega(f, x; \delta)=\sup_{|h|\leqq\delta}|f(x+h)-f(x)|, \quad \delta>0$$

is called a local continuity modulus for the function f at the point $x\in[-1, 1]$.

Obviously,

$$\omega(f, x; \delta)\leqq\omega(f; \delta), \quad x\in[-1, 1],$$

$$\sup_{x\in[-1, 1]}\omega(f, x; \delta)=\omega(f; \delta).$$

From the Jackson Theorem it is known that for any positive integer n and continuous f, there exists a polynomial $P\in H_n$, for which

(28) $$|f(x)-P(x)|\leqq 12\omega(f; n^{-1}), \quad x\in[-1, 1].$$

It is known, however, that in (28) $\omega(f; n^{-1})$ cannot be replaced by $\omega(f, x; n^{-1})$. In this connection, V. Popov in [19] set the problem for the local approximations and solved it in the following way:

For every positive integer n and continuous f, there exists an algebraic polynomial $P\in H_n$, for which

(29) $$|f(x)-P(x)|\leqq\omega(f, x; \ln n/n)+O(n^{-1}).$$

The estimate (29) is exact.

We will prove the following theorem giving an exact (with respect to the order) estimate for the partially monotone local approximations:

Theorem 4. *If f is continuous in $[-1, 1]$, monotone decreasing in $[-1, 0]$ and monotone increasing in $[0, 1]$, then, for any positive integer n, there exists a polynomial $P\in H_n^2$, for which, if $x\in[-1, 1]$*

(30) $$|f(x)-P(x)|\leqq c\omega(f, x; \ln n/n)+O(n^{-1}),$$

where c is an absolute constant.

Proof. An estimate of the type

$$|f(x)-P(x)|\leqq c_1\omega(f, x; \ln n/n)+O(\ln n/n)$$

can be easily proved by the well-known inequality:

$$|f(x)-P(x)|\leqq\omega(f, x; r(f, P))+r(f, P)$$

and by Theorem 3.

We will prove the estimate (30) by using directly the problem for the partially monotone interpolation.

Let m be a positive integer. Consider the system of points (see Theorem 3) (x_i, y_i), (x_{-i}, y_{-i}), $i=-m, \ldots, m$. Without loss of generality and in connection with our further considerations, we might assume that $y_m=y_{-m}=1$, $y_1=y_{-1}=0$. Then, by Theorem 2 it follows that there exists a polynomial $P \in H^2_{135\, m \ln m}$, $m>e^5$, for which:

$$y_i=P(x_i), \quad y_{-i}=P(x_{-i}), \quad i=1, \ldots, m.$$

Let $x \in [x_{i-1}, x_i]$ and let $f(x) \geq P(x)$ first. Then:

$$(31) \qquad |f(x)-P(x)|=f(x)-P(x) \leq f(x_i)-P(x_{i-1})$$

$$=f(x_i)-f(x_{i-1})+f(x_{i-1})-P(x_{i-1})=f(x_i)-f(x_{i-1})+f(x_{i-1})-f(x_{i-1})-(i-2)/m_8$$

$$\leq \omega(f, x;\ x_i-x_{i-1})+1/m^2 \leq \omega(f, x;\ m^{-1})+m^{-2}.$$

If $f(x) \leq P(x)$, then

$$|f(x)-P(x)|=P(x)-f(x) \leq P(x_i)-f(x_{i-1})=P(x_i)-f(x_i)+f(x_i)-f(x_{i-1})$$

$$\leq \frac{i-1}{m^8}+\omega(f, x;\ m^{-1}) \leq \omega(f, x;\ m^{-1})+m^{-2}.$$

In the same way the inequality (31) is proved when $x \in [x_{-i}, x_{-i+1}]$. But $P(x)$ is of degree not greater than $135\, m \ln m$. If we set $m=n/\ln n$, $P(x)$ becomes of degree not greater than $135\, n$, i. e. $P \in H^2_{135n}$ and

$$|f(x)-P(x)| \leq \omega(f, x;\ c_1 \ln n/n)+O(n^{-1}).$$

The above inequality, which holds for $P \in H_{135n}$, implies the validity of the theorem.

From the fact that the result (20) is exact it follows that the estimate in the above theorem is also exact with respect to the order. On the other hand Theorem 4 implies that the estimate of Theorem 2 is also exact with respect to the order of n, since, if we assume that in Theorem 2 might become smaller in order, depending on m, then the order in Theorem 4 can be immediately improved, which, as already mentioned, is impossible.

From Theorem 4, as a particular case, follows the same estimate for the monotone local approximations as well.

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