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ASYMPTOTIC DISTRIBUTION OF THE ZEROS OF POWER SERIES

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In this paper we consider power series of the type $f(z)=\Sigma A(n^{\alpha})z^n$, $\alpha>0$, where A is an entire function at most of the zero type of order $1/\alpha$. Explicit representations of the analytic extension of f onto the cut plane $\mathbb{C}^*=\{z \mid \text{if Re } z\geqq 1, \text{ then Im } z\neq 0\}$ are derived. The main results consist in the asymptotic distribution of the zeros of f near $z=1$ if A are generalized hypergeometric functions. In particular it is shown that the zeros are distributed on certain curves and their asymptotic number is established.

1. In this paper we investigate the behaviour of the zeros near $z=1$ of power series

$$f(z)=\sum_0^{\infty}B(n)z^n,\ \limsup_{n\to\infty}|B(n)|^{1/n}=1, \tag{1}$$

which admit unique analytic continuation into the set

$$\mathbb{C}^*=\{z \mid \text{Re } z<1 \text{ if Im } z=0\}. \tag{2}$$

The functions

$$f_{\varkappa}(z)=\sum_{n=0}^{\infty}(n+1)^{\varkappa}z^n,\quad \varkappa>0, \tag{3}$$

are of the from (1) and possess exactly k simple zeros ($k<\varkappa\leqq k+1$), which are negative (see [8]). In continuation of this work we ask for the zeros of such power series (1) whose functions of coefficients $B(z)$ increase more rapidly than $z^{\varkappa}$ (for example $B(z)=e^{-\sqrt{z}}$ or $B(z)=\cosh\sqrt{z}$). On the one hand, if the assumptions for $B(z)$ are not too special, we only may expect asymptotic statements about the distribution of the zeros. On the other hand, in order to get the analytic extension of $f(z)$, we need certain conditions on growth and "smoothness" of $B(z)$. Hence we consider power series of the form

$$f(z)=\sum_0^{\infty}A(n^{\alpha})z^n,\quad \alpha>0, \tag{4}$$

where $A(z)$ is an entire function at most of the zero type of order $1/\alpha$. If $A(z)=\Sigma_0^{\infty}b_kz^k$, $f_k(z)=\Sigma_1^{\infty}n^{\varkappa}z^n$ for $\varkappa>0$ and $f_0(z)=1/(1-z)$, then we have

$$\lim_{k\to\infty}k^{\alpha}|b_k|^{1/k}=0 \tag{5}$$

and

$$f(z)=\sum_0^{\infty}b_kf_{\alpha k}(z) \tag{6}$$

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for $|z|<1$. Moreover (6) is the analytic extension of $f(z)$ onto $\mathbb{C}^*$, since the series converges uniformly on every compact subset K of $\mathbb{C}^*$. For such a K put $\delta=\inf\{|1-zt|\,|\,z\in K,\ 0\leq t\leq 1\}$, which is positive by the assumption on K.

We have for $z\neq 1$ ([10]) that

(7) $$f_k(z)=P_k(z)/(1-z)^{k+1},\quad P_0(z)\equiv 1\,;\ k=0,1,2,\dots,$$

where $P_k(z)$ denotes a polynomial of degree k whose coefficients are positive and $P_k(1)=k!$. The equation (7) is an immediate consequence of the recurrence relation $f_{k+1}(z)=zf'_k(z)$. Since

$$n^{-\Theta}=\frac{1}{\Gamma(\Theta)}\int_0^1\left(\log\frac{1}{t}\right)^{\Theta-1}t^{n-1}dt,\quad n\in\mathbb{N},\ \Theta>0,$$

we have for $k<\varkappa<k+1$ ([8]) that

$$f_\varkappa(z)=\frac{1}{\Gamma(k+1-\varkappa)}\int_0^1\left(\log\frac{1}{t}\right)^{k-\varkappa}f_{k+1}(zt)\frac{dt}{t}$$

and so $|f_{ak}(z)|\leq\Gamma(ak+2)(R^{ak+1}+1)/\delta^{ak+2}$, $z\in K$. Now condition (5) implies the analyticity of $f(z)$ in K. The only singular point other than infinity is $z=1$, which is a possible accumulation point of zeros. Henceforth the following representation (Lindelöf [6], Wirtinger [12]), valid in $\mathbb{C}^*-\{0\}$, is important

(8) $$f_a(z)=\Gamma(a+1)\sum_{m=-\infty}^{\infty}\frac{1}{(2\pi im+\log(1/z))^{a+1}},\quad a>0$$

and $f_0(z)=1/(1-z)=2^{-1}+\sum\limits_{m=-\infty}^{\infty}1/(2\pi im-\log z)$, where Σ means symmetrical summation. These identities follow by application of Poisson's summation formula (e. g. Miesner, Wirsing [7]). Hence we have in $\mathbb{C}^*-\{0\}$ (see (6))

(9) $$f(z)=\frac{b_0}{2}+\sum_{k=0}^{\infty}b_k\Gamma(ak+1)\sum_{-\infty}^{\infty}\frac{1}{(2\pi im+\log(1/z))^{ak+1}},$$

from which follows that $f(z)$ can be continued analytically across the slit along the real axis for $\operatorname{Re}z>1$, where for the new branches besides 1 and ∞ $z=0$ is a singular point too [12]. In (8) the power $\exp(-(a+1)\log\log(1/z))$ is defined by the principle branch of $\log\log(1/z)$ (i. e. $\log\log(1/z)$ is real for $0<z<1$). Then we get the values of the terms of the sum (8) by the value of $(\log(1/z))^{-a-1}$, defined above, if we surround the boundary of $\mathbb{C}^*$ $|m|$ times in the positive or negative sense, respectively [12]. In view of (9) we have

(10) $$f(z)=\sum_{k=0}^{\infty}b_k\Gamma(ak+1)\frac{1}{(\log(1/z))^{ak+1}}+H(z),$$

where $H(z)$ denotes a holomorphic function in the neighbourhood of $z=1$. Now formula (10) yields the behaviour of zeros in this region. For $z\to 1$ the growth of $f(z)$ is determined by the asymptotic behaviour of the series. Therefore we next derive asymptotic developements of $f(z)\,(z\to 1)$ for a certain class of functions of coefficients $A(z)$ (special cases have been treated by

M. A. Evgrafov [2]). From these expansions we can conclude necessary conditions for the location of an infinite number of zeros; more precisely, these zeros lie in certain regions and on curves asymptotically. Furthermore the asymptotic series of $f(z)$ yields sufficient conditions for the existence of an infinite number of zeros in the neighbourhood of $z=1$ and asymptotic expansions of the densities of the zeros.

Remark 1. In the case $\alpha=1$ a theorem of Wigert [1] implies $f(z) = G(z/(1-z))$ in $\mathbb{C}-\{1\}$, where G is an entire function. If $A(z)$ is of type σ of order ϱ, $0<\varrho<1$, formula (10) gives a simple proof of the well known connection between ϱ, σ, the order ϱ_G and type σ_G of G (e. g. [1; 5]); because of the asymptotic equivalence $1/\log(1/z) \sim z/(1-z)$, $z \to 1$, and the relation between order, type and Taylor coefficients of entire functions we have

$$\varrho_G = \limsup_{k\to\infty} \frac{k \log k}{\log(1/|b_k|k!)} = \frac{\varrho}{1-\varrho}, \tag{11}$$

$$\sigma_G = \frac{1}{e\varrho_G} (\limsup_{k\to\infty} k^{1/\varrho_G} (|b_k|k!)^{1/k})^{\varrho_G} = (1-\varrho)\varrho^{\varrho/(1-\varrho)}\sigma^{1/(1-\varrho)}. \tag{12}$$

2. In order to get asymptotic expansions for $f(z)$, additionally we have to specialize the functions of coefficients $A(z)$. If we choose generalized hypergeometric functions for $A(z)$, whose asymptotic behaviour has been investigated in detail by E. M. Wright [13; 14], we cover an extended class of power series of the form (4). Put

$$f(z) = \sum_{n=0}^{\infty} {}_pG_q(\sigma n^{\alpha}) z^n, \tag{13}$$

where

$${}_pG_q(w) = \sum_{k=0}^{\infty} \frac{\varphi(k)}{k!} w^k \tag{14}$$

and $\varphi(t) = \prod_{r=1}^{p} \Gamma(a_r t+\beta_r) / \prod_{r=1}^{q} \Gamma(\varrho_r t+\mu_r)$ with $\beta_r, \mu_r, \sigma \in \mathbb{C}$, $\sigma \neq 0$ $(-\pi < \arg\sigma \leq \pi)$, $\alpha, a_r, \varrho_r > 0$ and

$$1+\sum_{r=1}^{q} \varrho_r - \sum_{r=1}^{p} a_r - \alpha > 0. \tag{15}$$

Suppose, further, that the poles of $\varphi(t)$ are all different from $t=0, 1, 2, \ldots$e Thus, $f(z)$ is of the form (4) with $b_k = \varphi(k)\sigma^k/k!$. Hence (see (9), (10)) we hav in $\mathbb{C}^*-\{0\}$

$$f(z) = \frac{\varphi(0)}{2} + \sum_{k=0}^{\infty} \frac{\varphi(k)\sigma^k}{k!} \Gamma(\alpha k+1) \sum_{-\infty}^{\infty} \frac{1}{(2\pi i m+\log(1/z))^{\alpha k+1}}$$

or

$$f(z) = \sum_{k=0}^{\infty} \frac{\varphi(k)\Gamma(\alpha k+1)}{k!} \frac{\sigma^k}{(\log(1/z))^{\alpha k+1}} + H(z), \tag{16}$$

where $H(z)$ is holomorphic in $z=1$ with

(17) $$H(z)=\frac{\varphi(0)}{2}+\sum_{k=0}^{\infty}\frac{\varphi(k)\sigma^k}{k!}\Gamma(ak+1)\sum_{m\neq 0}^{0}\frac{1}{(2\pi i m+\log(1/z))^{ak+1}}$$

and

(18) $$H(z)\to c=\varphi(0)+\sum_{k=0}^{\infty}\frac{\varphi(k)\sigma^k}{k!}\zeta(-ak),\quad z\to 1.$$

(ζ denotes the ζ-function of Riemann.)

E. g. (13) is of the following forms for the following values of parameters: for $\sigma=1$, $\sigma=1/2$, $\varphi(t)\equiv 1$ we have $f(z)=\Sigma_0^\infty e^{\sqrt{n}}z^n$; for $\sigma=a=1$, $\varphi(t)=\Gamma(t+1)/\Gamma(2t+1)$ we have $f(z)=\Sigma_0^\infty \cosh\sqrt{n}z^n$; for $\sigma=-1$, $a=2/5$, $\varphi(t)=\Gamma(t+1)/\Gamma(2t+1)$ we have $f(z)=\Sigma_0^\infty \cos n^{1/5}z^n$.

Putting $s=\log(1/z)$ (according to the choice of the principle branch of $\log\log(1/z)$ we have to choose for $\log(1/z)$ the principle branch too, i. e. $\log(1/z)$ is real for real positive z), the asymptotic behaviour of $f(z)$ for $z\to 1$ is determined by the asymptotic properties of

(19) $$g(s)=\frac{1}{s}\sum_{k=0}^{\infty}\frac{\varphi(k)\Gamma(ak+1)}{k!}(\sigma/s^a)^k$$

near $s=0$. Let be the following notations

$$a_{p+1}=a,\ \beta_{p+1}=1,\ h=\left(\prod_{r=1}^{p+1}a_r^{a_r}\right)\left(\prod_{r=1}^{q}\varrho_r^{-\varrho_r}\right),\quad \vartheta=\sum_{r=1}^{p+1}\beta_r-\sum_{r=1}^{q}\mu_r+\frac{1}{2}(q-p-1),$$

$$\gamma=1+\sum_{r=1}^{q}\varrho_r-\sum_{r=1}^{p+1}a_r,$$

(20) $$A_0=(2\pi)^{(p+1-q)/2}\gamma^{1/2-\vartheta}\left(\prod_{r=1}^{q}\varrho_r^{1/2-\mu_r}\right)\left(\prod_{r=1}^{p+1}a_r^{\beta_r-1/2}\right),$$

$$w=\sigma/s^a,\ Z=\gamma(h|w|)^{1/\gamma}\exp(i(\arg w)/\gamma),\ Z_\pm=|Z|\exp(i(\arg(-w)\pm\pi)/\gamma)$$

and for $|\arg y|\leq\pi$:

(21) $$I(y)=y^{\vartheta}e^{y}(A_0+O(y^{-1})),\qquad y\to\infty,$$

(22) $$J(y)=\sum_{r=1}^{p+1}\ \sum_{\nu=0}^{-[\operatorname{Re}\beta_r-Na_r]-1}P_{r,\nu}\,y^{-(\nu+\beta_r)/a_r}+O(y^{-N+\delta}),\quad y\to\infty,$$

(23) $$P_{r,\nu}=(y^{(\nu+\beta_r)/a_r}/m)\ \operatorname*{res}_{t=-(\nu+\beta_r)/a_r}\Gamma(-t)\Gamma(at+1)\varphi(t)y^t.$$

($m=$ order of the pole $t=-(\nu+\beta_r)/a_r$).

If the point $t=-(\nu+\beta_r)/a_r$ is not a pole of $\varphi(t)\Gamma(at+1)$, we put $P_{r,\nu}=0$. If $m>1$, $P_{r,\nu}$ is a polynomial in $\log y$ of degree $m-1$. If $m=1$, then $P_{r,\nu}$ is independent of $\log y$.

In the sequel let ϑ be real because of clearness of the formulas. Results of E. M. Wright [13] on asymptotic expansions of ${}_pG_q(w)$ yield the following

Lemma. *Let be $N\in\mathbb{N}$, ε, $\delta>0$. If condition* (15) *is satisfied, then*

$$ {}_{p+1}G_q(w)=\sum_0^\infty \frac{\varphi(k)\Gamma(ak+1)}{k!}\,w^k $$

has the following properties $(w\to\infty)$:

i) *If* $\gamma\geqq 2$, *then* ${}_{p+1}G_q(w)=I(Z_+I)+I(Z_-)+J(-w)$, $|\arg(-w)|\leqq\pi$.

ii) *If* $0<\gamma<2$,

$$ {}_{p+1}G_q(w)=\begin{cases} J(-w) & , \ |\arg(-w)|\leqq\frac{\pi}{2}(2-\gamma)-\varepsilon,\\ I(Z)+J(-w), & |\arg w|\leqq\min\left(\pi,\ \frac{3}{2}\pi\gamma-\varepsilon\right),\ |\arg(-w)|\leqq\pi. \end{cases} $$

Remark 2. (see [13, 14]). We observe that $\mathrm{Re}\,Z_- - \mathrm{Re}\,Z_+ = 2|Z|\times \sin(\arg(-w)/\gamma)\sin(\pi/\gamma)$ and so when $\gamma>1$.

$$ Z=Z_+,\ \mathrm{Re}\,Z_-<\mathrm{Re}\,Z_+ \ \text{for}\ -\pi<\arg(-w)<0, $$
$$ Z=Z_-,\ \mathrm{Re}\,Z_+<\mathrm{Re}\,Z_- \ \text{for}\ 0<\arg(-w)\leqq\pi, $$
$$ \text{and}\quad \mathrm{Re}\,Z_+=\mathrm{Re}\,Z_- \ \text{for}\ \arg(-w)=0. $$

Part i) of the lemma implies the exponential "expansion" ${}_{p+1}G_q(w)=I(Z)$ for $|\arg w|\leqq\pi-\varepsilon$ and $\gamma>2$, since $I(Z_\pm)=o(I(Z_\mp))$ for $\mathrm{Re}\,Z_\pm<\mathrm{Re}\,Z_\mp$. In the case $\gamma=2$ the exponential terms $I(Z_\pm)$ are bounded on the "critical" line $\arg(-w)=0$, and so the growth on $\arg(-w)=0$ is determined by the "algebraic expansion" J too.

Since $s=\log(1/z)$ maps the neighbourhood of $z=1$ ($|z-1|<\delta<1$) one to one onto a neighbourhood of $s=0$, in what follows it is sufficient to state the results in terms of s. Wright's lemma yields asymptotic expansions of $g(s)={}_{p+1}G_q(\sigma/s^a)/s$ inside sectors with an angle $2\pi/a$ at the origin (in the s-plane). Hence, since $s_1g(s_1)=sg(s)$, where $s_1=s\exp(2\pi i/a)$, we can obtain asymptotic expansions for $g(s)$ at $s=0$ in every branch on its Riemann surface.

3. This section is devoted to our main results. The developments given by Wright's lemma permit to decide in principle whether the number of zeros is infinite and to determine their location and density asymptotically. Obviously zeros can accumulate in such regions only, where a superposition of exponential and algebraic expansions happens. For the sake of clearness and simplicity in stating the results we choose restrictions concerning some parameters. So we only deal with the case

$$ J(-\sigma/s^a)/s=o(1)\quad (s\to 0), \tag{24} $$

which is characteristic for applications. (24) is equivalent to the condition that $\varphi(t)$ has no pole with real part greater than or equal to $-1/a$. In the cases, where $J(-\sigma/s^a)\sim K s^\xi$ (ξ arbitrarily complex), we can get similar results. For the location of the zeros we have the following

Theorem 1. *Define* $\lambda=a\vartheta/\gamma+1$; *suppose that* $f(e^{-s})$ *has infinitely many zeros in the neighbourhood of* $s=0$ *and that* (24) *is satisfied.*

i) *If* $\gamma>2$, *then all except a finite number of zeros lie in the region*

$$ |\arg(-\sigma/s^a)|\leq K_1\,|s^a/\sigma|^{2/\gamma}\quad (s\to 0) \tag{25} $$

where K_1 *is some positive constant.*

ii) *If* $\gamma=2$, *then all except a finite number of zeros lie in the region*

$$\text{(26)}\qquad |\arg(-\sigma/s^{\alpha})|\leq\begin{cases}K_2\,|s^{\alpha}/\sigma|^{1/2}\log|\sigma/s^{\alpha}| & \textit{for } \xi<0,\\ K_3\,|s^{\alpha}/\sigma|^{1/2} & \textit{for } \xi=0,\\ K_4\,|s^{\alpha}/\sigma|^{\xi+(1/2)} & \textit{for } \xi>0,\end{cases}$$

as $s\to 0$, *where* K_i *denote positive constants and* $\xi=\min(1/2,\,1/\alpha+\vartheta/2)$.

iii) *If* $0<\gamma<2$ *and* $c\neq 0$, *then all except a finite number of zeros lie on the curves*

$$\text{(27)}\qquad s^{\pm}=\tau^{\pm}(r\mp i\,(\operatorname{sign}\lambda)\,C_1\,r^{\alpha/\gamma+1}\log r\mp iC_2r^{\alpha/\gamma+1})+o\,(r^{\alpha/\gamma+1})\quad (r\to 0),$$

$$\tau^{\pm}=\exp\,(i\,(\arg\sigma\pm\pi\gamma/2)/\alpha)\quad C_j\in\mathbb{R}\quad C_1>0$$

$$\operatorname{sign}\lambda=\begin{cases}+1 & \textit{for }\ \lambda>0,\\ \ \ 0 & \textit{for }\ \lambda=0,\\ -1 & \textit{for }\ \lambda<0.\end{cases}$$

Remark 3. The assumption $c\neq 0$ is necessary, because the location of the zeros (27) follows from a comparison of the exponential expansion with the limit of the holomorphic part $H(z)$ for $z\to 1$ (see (16)—(18)). The curves (27) are similar to parabolas (see the first example of part 4).

Proof of theorem 1. i) Part i) of the lemma implies that

$$\text{(28)}\qquad f(e^{-s})=\frac{1}{s}\,\gamma^{\vartheta}\,(h\varrho)^{\vartheta/\gamma}\,(A_+(w)(A_0+O(\varrho^{-1/\gamma}))+A_-(w)\,(A_0+O(\varrho^{-1/\gamma}))),$$

where we write $w=\sigma/s^{\alpha}$, $w=-\varrho e^{i\varphi}$ and

$$A_{\pm}(w)=\exp\,(\,i(\varphi\pm\pi)\vartheta/\gamma+\gamma(h\varrho)^{1/\gamma}\,e^{i(\varphi\pm\pi)/\gamma}\,).$$

Note that in this case the term $H(z)$ can be absorbed in the "O" terms and hence may be omitted. Since there are at most a finite number of zeros in $|\arg w|\leq\pi-\varepsilon$, we may suppose that $|\varphi|$ does not exceed some suitable constant, $\pi/$ say. Moreover, it is enough, by symmetry, to consider the case in which $\varphi\geq 0$ Thus, a necessary condition for a zero is that $|A_+(w)/A_-(w)|=1+O(\varrho^{-1/\gamma})$. But

$$|A_+(w)/A_-(w)|=\exp\,(-2\gamma(h\varrho)^{1/\gamma}\sin(\pi/\gamma)\sin(\varphi/\gamma))\leq 1.$$

Thus, there is some constant K_1 such that all zeros for which $|w|$ is sufficiently large satisfy

$$\exp\,(-2\gamma(h\varrho)^{1/\gamma}\sin(\pi/\gamma)\sin(\varphi/\gamma))\geq 1-K_1\varrho^{-1/\gamma}.$$

Hence $K_2\,\varrho^{1/\gamma}\sin(\varphi/\gamma)\leq-\log(1-K_1\varrho^{-1/\gamma})$, where K_2 is some positive constant. Since $\varphi/\gamma\leq\pi/2\gamma$ we get with some positive constant K that $\varphi\leq K\varrho^{-2/\gamma}$. Omitting now the assumption $\varphi\geq 0$ we have finished the proof of part i).

ii) Again part i) of the lemma implies that (use the same notations as above)

$$f(e^{-s})=\frac{1}{s}\,2^{\vartheta}\,(h\varrho)^{\vartheta/2}\,(A_+(w)\,(A_0+O(\varrho^{-1/2}))+A_-(w)\,(A_0+O(\varrho^{-1/2})))+c+o(1).$$

Since now $A_+(w)$ and $A_-(w)$ are both bounded on the critical line, we have to take account of the contribution of the holomorphic part $H(z)$. For the same reason as in the proof in the preceeding part we may restrict our attention to the case in which $0\leq\varphi\leq\pi/2$. Then $A_+(w)$ is bounded by 1. In what follows, we use K to denote positive constants which may be different at each occurrence. A necessary condition for any zero is that

(29) $$|A_-(w)|\,(A_0+O(\varrho^{-1/2}))\leqq|A_+(w)|\,(A_0+O(\varrho^{-1/2}))+O(\varrho^{-1/\alpha-\vartheta/2}).$$

Further we have from the definition of $A_\pm(w)$ that

(30) $$|A_\pm(w)|=\exp(\mp 2(h\varrho)^{1/2}\sin(\varphi/2)).$$

First suppose that $\xi\leqq 0$. Since $|A_+(w)|\leqq 1$, the first term on the right of (29) may be absorbed in the second one. Hence we obtain, by (30),

$$\exp(2(h\varrho)^{1/2}\sin(\varphi/2))\leqq K\varrho^{-\xi},$$

and thus $2(h\varrho)^{1/2}\sin(\varphi/2)\leqq K-\xi\log\varrho$, which implies that

$$\varphi\leqq\begin{cases} K\varrho^{-1/2}\log\varrho & \text{for } \xi<0,\\ K\varrho^{-1/2} & \text{for } \xi=0.\end{cases}$$

Now suppose that $\xi>0$, that is $1/\alpha+\vartheta/2>0$. If

(31) $$2(h\varrho)^{1/2}\sin(\varphi/2)\leqq K,$$

then, since for bounded x, $e^x=1+\Theta_1 x$, $e^x=1-\Theta_2 x$, where Θ_i lie between two positive constants, it follows from (29) that

$$2(h\varrho)^{1/2}\sin(\varphi/2)=O(\varrho^{-1\ k-\gamma\ /2})+O(\varrho^{-1/2})=O(\varrho^{-\xi})$$

and thus

(32) $$\varphi\leqq K\varrho^{-\xi-(1/2)}.$$

If (31) is false, then for sufficiently large ϱ

$$|A_-(w)|\,(A_0+O(\varrho^{-1/2}))-|A_+(w)|\,(A_0+O(\varrho^{-1/2}))$$

is at least equal to a positive constant which implies that (29) cannot hold if ϱ is sufficiently large. Hence there are at most finitely many zeros for which (31) is false so that all but at most a finite number of zeros satisfy (32). Omitting now the assumption $\varphi\geqq 0$ the proof is complete.

iii) Now suppose $0<\gamma<2$. Then part ii) of the lemma implies that all except a finite number of zeros lie in such regions in which

$$f(e^{-s})=(I(Z)+J(-\sigma/s^\alpha))/s+H(e^{-s})$$

and (see (24))

(33) $$f(e^{-s})=B_0/s^{-\lambda}\exp(B_1(\sigma/s^\alpha)^{1/\gamma})(1+o(1))+c+o(1)$$

hold, where $B_0=A_0\gamma^\vartheta(h\sigma)^{\vartheta/\gamma}$, $B_1=\gamma h^{1/\gamma}$, and

(34) $$|\arg(\sigma/s^\alpha)|\leqq\min\left(\pi,\ \frac{3}{2}\pi\gamma-\varepsilon\right).$$

In view of the remark at the end of section 2 we choose $\arg w=\arg\sigma-\alpha\arg s$. Writing $s=re^{i\varphi}$ it follows that

$$\frac{|B_0|}{r^\lambda}\exp\left(B_1\frac{|\sigma|^{1/\gamma}}{r^{\alpha/\gamma}}\cos\left(\frac{\arg\sigma-\alpha\,\varphi(r)}{\gamma}\right)\right)\to|c|\quad(r\to 0)$$

and so

$$\begin{aligned}\cos((\arg\sigma-\alpha\,\varphi(r))/\gamma)&\to +0,\quad \lambda<0\\ \cos((\arg\sigma-\alpha\,\varphi(r))/\gamma)&\to -0,\quad \lambda>0\\ \cos((\arg\sigma-\alpha\,\varphi(r))/\gamma)&\to\ \ 0,\quad \lambda=0.\end{aligned}$$

Thus it follows, by (34), if $\lambda \neq 0$, that

$$(\arg \sigma - \alpha \varphi(r))/\gamma = \pm(\pi/2 + \varepsilon(r) \operatorname{sign} \lambda), \tag{35}$$

where $\varepsilon(r) \to +0$ as $r \to 0$. Now we conclude (note that $\cos((\arg \sigma - \alpha\varphi(r))/\gamma) = -(\operatorname{sign}\lambda)\sin\varepsilon(r)$, B_0, $B_1 \neq 0$)

$$-B_1 \frac{|\sigma|^{1/\gamma}}{r^{\alpha/\gamma}} (\operatorname{sign}\lambda)\sin\varepsilon(r) = \lambda \log r + \log \frac{|c|}{|B_0|} + o(1), \tag{36}$$

$$\sin \varepsilon(r) = -\frac{|\lambda|}{B_1 |\sigma|^{1/\gamma}} r^{\alpha/\gamma} \log r - (\operatorname{sign}\lambda) \frac{\log |c/B_0|}{B_1 |\sigma|^{1/\gamma}} r^{\alpha/\gamma} + o(r^{\alpha/\gamma}),$$

$$\varepsilon(r) = -C_1' r^{\alpha/\gamma} \log r - C_2' r^{\alpha/\gamma} + o(r^{\alpha/\gamma})$$

and

$$s = r e^{i\varphi(r)} = r \exp\left(i(\arg\sigma \mp \gamma \left(\frac{\pi}{2} + \varepsilon(r) \operatorname{sign}\lambda)\right)/\alpha\right)$$

$$= \tau^{\mp} r \exp\left(\mp i \frac{\gamma}{\alpha} \varepsilon(r) \operatorname{sign}\lambda\right) = \tau^{\mp} r \left(1 \mp \varepsilon(r) i \frac{\gamma}{\alpha} \operatorname{sign}\lambda + O(\varepsilon^2(r))\right)$$

$$= \tau^{\mp}(r \pm i(\operatorname{sign}\lambda) C_1 r^{\alpha/\gamma+1} \log r \pm i C_2 r^{\alpha/\gamma+1}) + o(r^{\alpha/\gamma+1}).$$

In the case $\lambda = 0$ (35) has to be replaced by $(\arg\sigma - \alpha\varphi(r))/\gamma = \pm(\pi/2 + \varepsilon(r))$, where $\varepsilon(r) \to 0$, as $r \to 0$, and then the following computations are similar.

Obviously sufficient conditions for the existence of an infinite number of zeros in the ranges (25) and (26) depend on special constellations of the parameters involved. We restrict our attention to the curves (27). Denoting by $N^{\pm}(r)$ the number of zeros on (27), whose modulus is greater than or equal to r we prove the following

Theorem 2. *Suppose that* (24) *is satisfied. If* $0 < \gamma < 2$ *and* $c \neq 0$, *then there is an infinite number of zeros on the curves* (27) *and*

$$N^{\pm}(r) = \frac{\gamma(h|\sigma|)^{1/\gamma}}{2\pi r^{\alpha/\gamma}} + O(|\log r|) \quad (r \to 0).$$

Proof. With the notations above we have from (33) for some $B_0' \neq 0$ and real l' that $f(e^{-s}) = B_0' w^{l'} \exp[B_1 w^{1/\gamma}](1 + o(1)) + c + o(1)$. Putting $\zeta = w^{1/\gamma}$ we get ($l = l'\gamma$)

$$f(e^{-s}) = F(\zeta) = B_0' \zeta^l e^{B_1\zeta}(1 + o(1)) + c + o(1), \quad (\zeta \to \infty), \tag{37}$$

where we may assume that the branches of the logarithms are chosen such that ζ^l and the functions involved in the o-terms are holomorphic in the uppe and lower half-plane respectively. (Actually we can get from (17), (18) and (21) that the o-terms can be replaced by $O(\zeta^{-1})$ and $O(\zeta^{-\gamma/\alpha})$.) Now we have to investigate the zeros of $F(\zeta)$ in the neighbourhood of the rays $\arg \zeta = \pm\pi/2$ (see (35)). More precisely, if we take account of (35), (36) and the following lines of the proof of theorem 1, it follows that the corresponding zeros of $F(\zeta)$ are distributed on the curves ($\zeta = Re^{i\psi}$)

$$\zeta^{\pm} = \pm i(R \pm i(\operatorname{sign} l) K_1 \log R \pm i K_2) + o(1), \quad (R \to \infty), \tag{38}$$

where K_j are constants and $K_1 > 0$. Without loss of generality we are dealing with the positive sign in (38). We denote by $n(\varrho)$ the number of zeros of $F(\zeta)$ on the line ζ^+ whose modulus is less than or equal to ϱ. Following Polya ([9], p. 287) we consider the region G_ϱ bounded by the following curves

$$C_1=\{\zeta=\xi+i\eta \mid \xi=k\log\eta,\quad 1\leq\eta\leq\varrho\},$$
$$C_2=\{\zeta=\xi+i\varrho \mid k\log\varrho\geq\xi\geq-k\log\varrho\},$$
$$C_3=\{\zeta=\xi+i\eta \mid \xi=-k\log\eta,\ \varrho\geq\eta\geq 1\},$$

where k is a sufficiently large positive constant. Further suppose that $F(\zeta)\neq 0$ for $\zeta\in C_j$. Now we shall see that an application of the argument principle to G_ϱ implies that

$$n(\varrho)=\frac{B_1}{2\pi}\varrho+O(\log\varrho),\quad (\varrho\to\infty). \tag{39}$$

Since, by definition, $n(\varrho)$ is non-decreasing, (39) will follow if we prove it for a sequence ϱ_N of values of ϱ increasing to infinity such that $\varrho_{N+1}-\varrho_N=O(1)$ as $N\to\infty$. We may therefore suppose that ϱ satisfies

$$\arg B_0'+\frac{1}{2}\pi l+B_1\varrho=\arg c+2N\pi, \tag{40}$$

where N is an integer. Since $\arg\zeta$ tends to $\pi/2$ uniformly on C_2 as ϱ tends to infinity, it then follows from (40) and (37) that $\arg F(\zeta)\to\arg c$ uniformly on C_2 as $\varrho\to\infty$. Hence, since $c\neq 0$, the contribution of C_2 given by the argument principle is bounded. Clearly, by (37), the contributions of C_1 ard C_3 give $B_1\varrho+O(\log\varrho)$, which implies (39). By construction, there are no zeros on C_1 and C_3 for $\eta\geq\eta_0$ and so we can distort the parts of C_1 and C_3 for $\eta<\eta_0$ such that the finitely many zeros of $F(\zeta)$ with $1\leq\operatorname{Im}\zeta<\eta_0$ are not located on C_1 and C_3. Since these manipulations do not affect (39), the proof is complete, if we remind the definitions of w and ζ.

4. This section is devoted to applications. Consider the function $f(z)=\sum_0^\infty\cosh\sqrt{n}\,z^n$ ($\sigma=\alpha=\gamma=1$, $\varphi(t)=\Gamma(t+1)/\Gamma(2t+1)$). We have from (18) that (for formulas concerning the Riemann ζ-function see for example [11, pp. 19, 20])

$$c=1+\sum_0^\infty\frac{\zeta(-k)}{(2k)!}=\frac{1}{2}+\sum_1^\infty\frac{\zeta(1-2k)}{(4k-2)!}=\frac{1}{2}+\sum_1^\infty\frac{(-1)^k b_k}{2k(4k-2)!}=\frac{1}{2}+\sum_1^\infty(-1)^k\frac{\zeta(2k)}{k(2\pi)^{2k}}\frac{(2k)!}{(4k-2)!},$$

where the absolute values of the members of the latter series form a strictly monotone decreasing sequence. Hence $c>1/2-b_1/4=11/24$. Since $\varphi(t)=\Gamma(t+1)/\Gamma(2t+1)$ is an entire function, (24) is satisfied. Now we get from the theorems 1 and 2 that the infinitely many zeros lie on the lines

$$z^{\pm}=1\pm ir-6r^2\log r+Lr^2+o(r^2),\quad r\to 0,\ L \text{ real},$$

with $n(r)=N^+(r)+N^-(r)=1/4\pi r+O(|\log r|)$, where $n(r)$ denotes the number of zeros in $\{z \mid |z-1|\geq r\}$. The existence of an infinite number of zeros of this function can also be proved by the following argument.) Since $\cosh\sqrt{z}$ is an entire function of type 1 of order 1/2, remark 1 yields $l(z=G(z/(1-z))$, where G is an entire function of order 1. If we assume that $f(z)$ only has a finite number of zeros, Hadamard's factorization theorem implies $f(z)=P(z/(1-z))\exp[Kz/(1-z)]$, where P denotes a polynomial. But this is a contradiction to the expansion $f(z)\sim\{2^{-1}\sqrt{\pi}\exp[1/4\log(1/z)]\}/(\log(1/z))^{3/2}$, $(0<z\to 1-0)$, which follows from the lemma and (33). In the neighbourhood of $z=\infty$ the zeros cannot accumulate, for $w=-1$ is a regular point of $G(w)$.

The latter example shows that for some power series we may conclude sufficient conditions for the existence of infinitely many zeros from the lemma more simply than from theorem 2. If $\alpha=1$, then ${}_pG_q(\sigma z)$ is an entire function of mean type of order $1/(1+\gamma)$; remark 1 implies $f(z)=\Sigma_0^\infty\, {}_pG_q(\sigma n)\, z^n$ possesses an analytic continuation into $\mathbb{C}-\{1\}$ and $f(z)=G(z/(1-z))$, where G denotes an entire function of mean type of order $1/\gamma$. In view of (33) we have for $\arg\sigma=\arg\log(1/z): f(z)\sim\frac{B_0}{(\log(1/z))^\lambda}\exp\left(B_1\left(\frac{\sigma}{\log(1/z)}\right)^{1/\gamma}\right)(z\to 1)$. Considerations, analogous to the example above, yield the following general result:

Theorem 3. *Let be* $1/\gamma\notin\mathbb{N}$ *or* $\lambda\notin\mathbb{N}\cup\{0\}$, *then*

$$f(z)=\sum_0^\infty {}_pG_q(\sigma n)\, z^n$$

has infinitely many zeros.

Another method for investigating the behaviour of the zeros of functions of the type (13) is illustrated by the following example. Using the representation [4]

$$e^{-a\sqrt{n}}=\frac{a}{2\sqrt{\pi}}\int_0^1 t^n\exp\left(\frac{a^2}{4\log t}\right)\frac{dt}{t(\log(1/t))^{3/2}} \qquad (a>0)$$

as Hausdorff-moment sequence,

$$f(z)=\sum_0^\infty e^{-a\sqrt{n}}z^n \quad (\sigma=-a, \alpha=1/2,\ \varphi(t)\equiv 1)$$

admits unique analytic continuation into $\mathbb{C}^*$, given by

$$f(z)=\frac{a}{2\sqrt{\pi}}\int_0^1\frac{1}{1-zt}\exp\left(\frac{a^2}{4\log t}\right)\frac{dt}{t(\log(1/t))^{3/2}}.$$

Because

$$g(t)=\frac{a}{2\sqrt{\pi}}\int_0^t\exp\left(\frac{a^2}{4\log\tau}\right)\frac{d\tau}{\tau(\log(1/\tau))^{3/2}}$$

is monotone ($t\geq 0$) and $g(1)>g(0)$, $f(z)\neq 0$ in $\mathbb{C}^*$(see Peyerimhoff [8]).

In order to prove the existence of infinitely many zeros of power series $f(z)=\Sigma_0^\infty B(n)\,z^n$ ($B(z)$ is an entire function of mean type of order α, $0<\alpha<1$), there is another method, which is based on certain arithmetical properties of $B(n)$ (see [3], for example $f(z)=\Sigma_0^\infty\cosh\sqrt{n}\; z^n$).

Remark 4. i) In the case $f(z)=\Sigma_0^\infty e^{\sigma n^\alpha}z^n$ $(0<\alpha<1,\ \varphi(t)\equiv 1)$ (33) yields the exponential expansion

$$f(z)\sim\frac{B_0}{(\log(1/z))^\lambda}\exp((1-\alpha)\alpha^{\alpha/(1-\alpha)}\ \sigma^{1/(1-\alpha)}/(\log(1/z))^{\alpha/(1-\alpha)}),$$

$$(\arg\sigma-\frac{\pi}{2}(1-\alpha)+\varepsilon)/\alpha\leq\arg\log(1/z)\leq(\arg\sigma+\frac{\pi}{2}(1-\alpha)-\varepsilon)/\alpha,$$

which agrees with (11), (12).

ii) Our statements may be extented to power series of the form

$$\sum_1^\infty A(n^\alpha)n^\varrho z^n \quad (\varrho>0)\ \text{and}\ \sum_2^\infty A(n^\alpha)n^\varrho(\log n)^\nu z^n \quad (\nu\in\mathbb{N})$$

(differentiation with respect to ϱ).

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