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THE REPRESENTATION OF ANALYTIC FUNCTIONS BY MEANS OF SERIES IN LAGUERRE FUNCTIONS OF THE SECOND KIND

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It is proved that an analytic function f can be represented by a series in Laguerre functions of the second kind $\{M_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ iff f is a K_α-transformation of a suitable entire function. Here K_α is the modified Bessel function of the third kind with index α.

The system of Laguerre functions of the second kind with parameter $\alpha>-1$ is defined in the region $\mathbb{C}\setminus[0,+\infty]$ by the equalities

$$(1)\qquad M_n^{(\alpha)}(z)=-\int_0^\infty \frac{t^\alpha \exp(-t)L_n^{(\alpha)}(t)}{t-z}\,dt,\qquad n=0,\ 1,\ 2,\dots,$$

where $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ are the Laguerre polynomials with parameter α. Using Rodrigues formula [1, II, p. 188, (5)] from (1) we get easily that

$$(2)\qquad M_n^{(\alpha)}(z)=-\int_0^\infty \frac{t^{n+\alpha}\exp(-t)}{(t-z)^{n+1}}\,dt,\qquad n=0,\ 1,\ 2,\dots$$

If $\operatorname{Re} z<0$, we denote by $l(z)$ the ray $\{\zeta=(-z).t,\ 0\leqq t<+\infty\}$. Then, from (2) it follows that

$$(3)\qquad M_n^{(\alpha)}(z)=-\int_{l(z)} \frac{\zeta^{n+\alpha}\exp(-\zeta)}{(\zeta-z)^{n+1}}\,d\zeta=-(-z)^\alpha\int_0^\infty \frac{t^{n+\alpha}\exp(zt)}{(1+t)^{n+1}}\,dt.$$

Having in view the integral representation [1, I, p. 273, (10)] of Tricomi's confluent hypergeometrical function, from (3) we get the following representation of the Laguerre functions of second kind ($n=0,\ 1,\ 2,\ \dots,\ z\in\mathbb{C}\setminus[0,+\infty]$)

$$(4)\qquad M_n^{(\alpha)}(z)=-\frac{2(-z)^{\alpha/2}}{\Gamma(n+1)}\int_0^\infty t^{n+\alpha/2}\exp(-t)K_\alpha(2\sqrt{-zt})\,dt,$$

where $K_\alpha(z)$ is the modified Bessel function of the third kind with index α.

In this paper using the integral representation (4) we consider the problem of expanding an analytic function in series of the kind

$$(5)\qquad \sum_{n=0}^{\infty} b_n M_n^{(\alpha)}(z).$$

The region of convergence of the series (5) can be described by means of a formula of Cauchy-Hadamard type. More precisely, the following statement holds [2, p. 283].

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Lemma 1. *If* $\mu_0 = \max\{0, \limsup_{n\to+\infty}(2\sqrt{n})^{-1}\ln|b_n|\}$, *the series* (5) *is absolutely uniformly convergent on every compact subset of the region* $\overline{\Delta^*(\mu_0)}$: $\mathrm{Re}\,(-z)^{1/2} > \mu_0$ *and diverges at every point of the region* $\mathbb{C}\setminus\overline{\Delta^*(\mu_0)}\cup[0, +\infty)$ ($\Delta^*(+\infty)=\emptyset$, $\Delta^*(0)=\mathbb{C}\setminus[0, +\infty)$).

Let $0\leq\tau<+\infty$ and $B(\tau)$ denote the class of the entire functions Ψ having the property that

$$\limsup_{|w|\to+\infty}(2\sqrt{|w|})^{-1}(\ln|\Psi(w)|-|w|)\leq\tau. \tag{6}$$

Lemma 2. *The entire function*

$$\Psi(w)=\sum_{n=0}^{\infty}(n!)^{-1}b_n w^n \tag{7}$$

belongs to the class $B(\tau)$ *iff*

$$\limsup_{n\to+\infty}(2\sqrt{n})^{-1}\ln|b_n|\leq\tau. \tag{8}$$

Proof. (a) If $\Psi\in B(\tau)$, then for every $\delta>0$,

$$|b_n|=O\Big\{n!\int_{|w|=n}|w|^{-n-1}|\Phi(w)|\,ds\Big\}=O\{n!\,n^{-n}\exp[n+2(\tau+\delta)\sqrt{n}]\}$$

and therefore, $\limsup_{n\to+\infty}(2\sqrt{n})^{-1}\ln|b_n|\leq\tau+\delta$.

(b) Suppose that (8) holds. Then, having in view the asymptotic formula [3, (8.22.3)] for the Laguerre polynomials, we can conclude that for every $\delta>0$ there exists $M(\delta)>0$ such that $|b_n|\leq M(\delta)L_n^{(0)}[-(\tau+\delta)^2]$ $(n=0, 1, 2, \ldots)$ (Let us note that $L_n^{(0)}(-x)>0$ if $x>0$.) Using [1, II, p. 189, (18)] we get that

$$|\Psi(w)|=O\Big\{\sum_{n=0}^{\infty}(n!)^{-1}L_n^{(0)}[-(\tau+\delta)^2]\,|w|^n\Big\}=O\{e^{|w|}J_0[2\sqrt{-(\tau+\delta)^2|w|}]\}$$

$$=O\{e^{|w|}J_0[2i(\tau+\delta)\sqrt{|w|}]\}=O\{e^{|w|}I_0[2(\tau+\delta)\sqrt{|w|}]\},$$

where $J_0(z)$ is the Bessel function of the first kind and $I_0(z)$ is the modified Bessel function of the first kind, both with zero index.

Further, the asymptotic formula [1, II, p. 86, (5)] gives that $|\Psi(w)|=O\{|w|^{1/4}\exp[|w|+2(\tau+\delta)\sqrt{|w|}]\}$ and since $\delta>0$ is arbitrary, we get (6).

Theorem 1. *Let* $0\leq\mu_0<+\infty$, $\alpha>-1$ *and* f *be a complex function analytic in the region* $\Delta^*(\mu_0)$. *In order that* f *can be expanded in this region in a series of the kind* (5) *is necessary and sufficient that for* f *holds an integral representation*

$$f(z)=-2(-z)^{\alpha/2}\int_0^{\infty}t^{\alpha/2}\exp(-t)\Psi(t)K_\alpha(2\sqrt{-zt})\,dt, \tag{9}$$

where $\Psi\in B(\mu_0)$.

Proof. First of all we note that if $\Psi\in B(\mu_0)$, the integral in (9) is absolutely uniformly convergent on every compact subset $K\subset\Delta^*(\mu_0)$. Indeed, let $\mu_0<\mu<+\infty$ be chosen so that $K\subset\Delta^*(\mu)$. Then from the inequality $|\Psi(t)|=O\{\exp[t+2(\mu_0+\delta)\sqrt{t}]\}$, where $\delta=(\mu-\mu_0)/2$, and the asymptotic formula [1, II, p. 86, (7)] it follows that if $t\to+\infty$

$$(10)\qquad t^{\alpha/2}\exp(-t)\,|\,\Psi(t)K_\alpha(2\sqrt{-zt})\,|=O\{t^{\alpha/2}\exp(-2\delta\sqrt{t})\}$$

uniformly on K.

Let us suppose that the representation (9) holds, where $\Psi\in B(\mu_0)$. If the function Ψ is given by the expansion (7), from Lemmas 1, 2 it follows that the series (5) is convergent in the region $\Delta^*(\mu_0)$. Then, if we define $R_\nu(z) = f(z)-\sum_{n=0}^{\nu} b_n M_n^{(\alpha)}(z)$ and use the integral representation (4), we can write

$$R_\nu(z)=-2(-z)^{\alpha/2}\int_0^\infty t^{\alpha/2}\exp(-t)\left\{\sum_{n=\nu+1}^{\infty}(n!)^{-1}b_n t^n\right\}K_\alpha(2\sqrt{-zt}).$$

From Lemma 2 it follows that the function $\Psi^*(w)=\sum_{n=0}^{\infty}(n!)^{-1}\,|\,b_n\,|\,w^n$ also belongs to the class $B(\mu_0)$. Therefore, if we replace Ψ by Ψ^* in (10), we can assert that for every $\varepsilon>0$ there exists $T=T(\varepsilon)>0$ such that

$$\int_T^\infty t^{\alpha/2}\exp(-t)\Psi^*(t)\,|\,K_\alpha(2\sqrt{-zt})\,|\,dt<\varepsilon.$$

Then, for every $\nu=0, 1, 2, \dots$ we get that

$$\left|\int_T^\infty t^{\alpha/2}\exp(-t)\left\{\sum_{n=\nu+1}^{\infty}(n!)^{-1}b_n t^n\right\}K_\alpha(2\sqrt{-zt})dt\right|$$

$$\leqq\int_T^\infty t^{\alpha/2}\exp(-t)\Psi^*(t)\,|\,K_\alpha(2\sqrt{-zt})\,|\,dt<\varepsilon.$$

Further, there exists a $N=N(\varepsilon)$ such that if $\nu>N$ and $0\leqq t\leqq T$, then $|\sum_{n=\nu+1}^{\infty}(n!)^{-1}b_n t^n|<\varepsilon$. Therefore,

$$\left|\int_0^T t^{\alpha/2}\exp(-t)\left\{\sum_{n=\nu+1}^{\infty}(n!)^{-1}b_n t^n\right\}K_\alpha(2\sqrt{-zt})dt\right|$$

$$=O\left\{\varepsilon\int_0^\infty t^{\alpha/2}\exp(-t)\,|\,K_\alpha(2\sqrt{-zt})\,|\,dt\right\}=O(\varepsilon).$$

Now, $R_\nu(z)=O(\varepsilon)$ if $\nu>N$, i. e. the series (5) represents the function f in the region $\Delta^*(\mu_0)$.

Let f be analytic in the region $\Delta^*(\mu_0)(0\leqq\mu_0<+\infty)$ and have a representation by the series (5) in this region. Then, from Lemmas 1, 2 it follows that the function Ψ defined by (7) belongs to the class $B(\mu_0)$. By means of the integral transformation (9) the function Ψ defines a complex function $\tilde{f}$ analytic in the region $\Delta^*(\mu_0)$. But we have just seen that $\tilde{f}$ can be represented in this region by the series (5) and, therefore, $f=\tilde{f}$.

As an application of Theorem 1 we shall get a necessary and sufficient condition for a complex function f, analytic in the half-plane $H^+(\tau_0)$: $\operatorname{Im} z>\tau_0$ $(0\leqq\tau_0<+\infty)$, to be represented in this half-plane by a series in Hermite functions of the second kind $\{G_n(z)\}_{n=0}^{\infty}$. The last system is defined by the equalities $(z\in\mathbb{C}\setminus(-\infty, +\infty))$ $G_n(z)=-\int_{-\infty}^{\infty}(t-z)^{-1}\exp[(-t^2)H_n(t)]dt$, $n=0, 1, 2, \dots$, where $\{H_n(z)\}_{n=0}^{\infty}$ are the Hermite polynomials.

Using the relations [1, II, p. 193, (2), (3)] between Laguerre and Hermite polynomials, we can write the corresponding formulas, which express the Hermite functions of second kind in terms of the Laguerre functions of second kind, namely

$$G_{2n}(z)=(-1)^n 2^{2n} n! z M_n^{(-1/2)}(z^2), \quad G_{2n+1}(z)=(-1)^n 2^{2n+1} n! M_n^{(1/2)}(z^2).$$

The above relations, the equalities $K_{1/2}(z)=K_{-1/2}(z)=\sqrt{\pi/2z}\exp(-z)$ [1, II, p. 5, (14); p. 9, (39)] and Theorem 1 lead to the following result

Theorem 2. *The complex function f, analytic in the half-plane* $H^+(\tau_0)$ $(0\le\tau_0<+\infty)$, *can be expanded in this half-plane in a series of Hermite functions of the second kind iff for f holds the representation*

$$f(z)=\int_0^\infty \{\Psi_1(t^2)+t\Psi_2(t^2)\}\exp(-t^2+2izt)dt,$$

where $\Psi_1, \Psi_2 \in B(\tau_0)$,

Remark. Theorem 2 holds if we replace the half-plane $H^+(\tau_0)$ by $H^-(-\tau_0)$: $\operatorname{Im} z<-\tau_0$ and z by $-z$.

As a second application of Theorem 1 we shall prove that under the assumption $\operatorname{Re}(-z)^{1/2}>\operatorname{Re}(-\zeta)^{1/2}(z, \zeta\in\mathbb{C})$ holds the equality

$$2\zeta^{-\alpha/2}(-z)^{\alpha/2}\int_0^\infty J_\alpha(2\sqrt{\zeta t})K_\alpha(2\sqrt{-zt})dt=\frac{1}{\zeta-z}. \tag{11}$$

The system of Laguerre polynomials as well as the system of Laguerre functions of the second kind satisfies the linear recurrence $(n+1)y_{n+1}+(z-2n-\alpha-1)y_n+(n+\alpha)y_{n-1}=0$. Moreover, there is a formula of Cristoffel-Darboux type, namely

$$\frac{1}{z-\xi}=\sum_{n=0}^{\nu}\frac{1}{J_n^{(\alpha)}}L_n^{(\alpha)}(\zeta)M_n^{(\alpha)}(z)+\frac{\Delta_{\nu+1}^{(\alpha)}(\zeta, z)}{z-\zeta}, \tag{12}$$

where $J_n^{(\alpha)}=\Gamma(n+\alpha+1)/\Gamma(n+1)$ and $\Delta_{\nu+1}^{(\alpha)}(\zeta, z)=(\nu+1)/J_\nu^{(\alpha)}\{L_\nu^{(\alpha)}(\zeta)M_{\nu+1}^{(\alpha)}(z)-L_{\nu+1}^{(\alpha)}(\zeta)M_\nu^{(\alpha)}(z)\}$.

With the aim of (12) and the asymptotic formulas for Laguerre polynomials [3, (8.22.2), (8.22.3)] and for Laguerre functions of the second kind [2, p. 272, (11)] one can prove that $1/(z-\zeta)=\sum_{n=0}^\infty L_n^{(\alpha)}(\zeta)M_n^{(\alpha)}(z)/J_n^{(\alpha)}$ provided that $\operatorname{Re}(-z)^{1/2}>\operatorname{Re}(-\zeta)^{1/2}$.

For every $\zeta\in\mathbb{C}$ the entire function

$$\Psi_\alpha(\zeta, w)=\sum_{n=0}^\infty\frac{L_n^{(\alpha)}(\zeta)}{n!J_n^{(\alpha)}}w^n=\sum_{n=0}^\infty\frac{L_n^{(\alpha)}(\zeta)}{\Gamma(n+\alpha+1)}w^n$$

is in the class $B(\tau)$, where $\tau=\operatorname{Re}(-\zeta)^{1/2}$. Then, (11) follows immediately from Theorem 1, while $\Psi_\alpha(\zeta, w)=\exp w(\zeta w)^{-\alpha/2}J_\alpha(2\sqrt{\zeta w})$ [1, II, p. 189, (18)].

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