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BETTI NUMBERS OF DIFFERENTIABLE MANIFOLDS AND MULTIPLICITY OF EIGENVALUES*

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Betti numbers of a compact differentiable manifold are viewed as eigenvalues of positive compact operators, through the classical Hodge theorem. A theory for the computation of the geometric multiplicity of an eigenvalue is outlined and a formula for the calculus of Betti numbers is derived.

Everybody knows that a compact orientable surface is topologically determined by its genus g. The problem which I shall consider in this paper is the following.

Suppose that the compact and orientable surface is given through one of its atlas, i. e. suppose that a (finite) set of maps covering the surface is given with the relevant "connecting" homeomorphisms. Can we compute the genus g of the surface?

We shall assume that the differential structure introduced in the manifold by the given atlas is $\mathcal{C}^\infty$. The problem is a particular case of the following one, which, actually, is the one we shall consider in this paper.

Given an atlas (finite set of maps and connecting homeomorphisms) of a $\mathcal{C}^\infty$ differentiable, orientable and compact manifold V^r of dimension $r \geq 1$, compute the Betti numbers of V^r.

In dealing with this rather unusual problem in Analysis, in order to avoid any misunderstanding, it seems to me necessary to say that the word "compute" must be understood in the sense of Numerical Analysis, i. e. to give a mathematical procedure which, no matter how analitically sophisticated, is such that, using only the "data" of the problem (i. e. the functions which give the connecting homeomorphisms) it can be programmed on an automatic computer. On the other hand, we shall not assume any "convenience hypothesis" like the one which consists in supposing that the maps of the atlas constitute a simple covering of V^r. In fact in this case the homology of V^r is the same as the homology of the nerve of the covering. This is a classical result due to Leray. The circumstance that such kind of coverings exist [1] is of no help for computational purposes. In fact, we have to consider that not only simple coverings exist, but also triangulations [2, pp. 125—135] of the variety exist: to use a triangulation would make the problem trivial. However, the mere existence of some mathematical object is, generally, something very different from the actual computations connected with this object. This point of view, although commonly accepted by people working in Analysis, in particular in Partial Differential Equations, could not be so familiar to scientists, even outstanding, working in other fields.

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Let $M_1, M_2, \ldots, M_q$ be (open) intervals of the cartesian space X^r. Let $M_{i1}, M_{i2}, \ldots, M_{iq}$ be open sets (some eventually empty) contained in M_i and such that $M_{ii} = M_i$. Denote by τ_{ik} a $\mathcal{C}^\infty$-homeomorphism of the closure $\bar{M}_{ik}$ of M_{ik} into $\bar{M}_{ki}$. Suppose that

1) τ_{ii} = identity;

2) $\tau_{ik} = \tau_{ki}^{-1}$;

3) $x \in M_{ik},\ \tau_{ih}(x_i) \in M_{hk} \Rightarrow \tau_{hk}[\tau_{ih}(x_i)] = \tau_{ik}(x_i)$;

4) $\tau_{ik}(\partial M_{ik} \cap M_i) \subset \partial M_k$;

5) for every $x \in \partial M_i$, there exists h such that $x_i \in \bar{M}_{ih},\ \tau_{ih}(x_i) \in M_h$;

6) if $x_i = (x_i^1, \ldots, x_i^r)$ is the point of M_i and if we consider the jacobian matrix $\partial\tau_{ik}/\partial x_i$ we have det $[\partial\tau_{ik}/\partial x_i] > 0$.

The collection $[M_i; \tau_{ik}]$ which is the "datum" of our problem is an atlas.

There exists a $\mathcal{C}^\infty$ differentiable compact oriented manifold V^r (determined up to a $\mathcal{C}^\infty$ homeomorphism), which has $[M_i, \tau_{ik}]$ as its atlas (see 3 p. 545]).

If we consider the equivalence relation: $x_i \in M_i \cong x_k \in M_k$, whenever $x_i = \tau_{ki}(x_k)$ [i. e. $x_k = \tau_{ik}(x_i)$], the point x of V^r is the equivalence class determined by x_i; x_i is the image of x in the map M_i. Let $\varphi_k(x^1, \ldots, x^r)$ be a real valued function belonging to $\mathcal{C}^\infty(X^r)$ and such that

$$\varphi_k(x^1, \ldots, x^r) \begin{cases} > 0 \text{ in } M_k, \\ = 0 \text{ in } X^r - M_k \end{cases} \quad (k = 1, \ldots, q).$$

Define on V^r the function

$$\tilde{\varphi}_k(x) \begin{cases} = \varphi_k(x_k) \text{ if } x_k \text{ is the image of } x \text{ in } M_k, \\ = 0 \text{ elsewhere.} \end{cases}$$

We have $\tilde{\varphi}_k(x) \in \mathcal{C}^\infty(V^r)$. Set $\varphi(x) = \Sigma_{k=1}^q \tilde{\varphi}_k(x)$. Since $\varphi(x) > 0$ on V^r, we can consider $\psi_k(x) = \tilde{\varphi}_k(x)/\varphi(x)$ and we have $\Sigma_{k=1}^q \psi_k(x) = 1$.

Thus, we have a special partition of unity of V^r such that supp $\psi_k(x)$ is contained in (the domain of V^r having like image in the atlas) M_k.

Set $a_{ij}^{(h)}(x) = \psi_h(x)\,\delta_{ij}$ and consider $a_{ij}^{(h)}(x)$ like the components of a covariant symmetric tensor. Set $a_{ij}(x) = \Sigma_{h=1}^q a_{ij}^{(h)}(x)$ and $ds^2 = a_{ij}(x)dx^i dx^j$.

Since the $a_{ij}(x)$ are the components of a covariant symmetric positive tensor, we have introduced in V^r a Riemannian metric.

Let v be a $\mathcal{C}^\infty$ real k-form (i. e. differential form of degree k), which in a local coordinate system is represented by

$$v = \frac{1}{k!} v_{s_1 \ldots s_k} dx^{s_1} \ldots dx^{s_k}.$$

It is well known that the differential dv of v is the $(k+1)$-form which is represented by

$$dv = \frac{1}{k!} dv_{s_1 \ldots s_k} dx^{s_1} \ldots dx^{s_k}$$

$$= \frac{1}{(k+1)!} \left(\sum_{h=1}^{k+1} (-1)^{h-1} \frac{\partial v_{j_1 \ldots j_{h-1} j_{h+1} \ldots j_{k+1}}}{\partial x^{j_h}} dx^{j_1} \ldots dx^{j_{k+1}} \right).$$

Moreover $ddv=0$.

The adjoint form of the k-form v is the $(r-k)$-form

$$*v=\frac{1}{(r-k)!}*v_{s_{k+1}\dots s_r}\,dx^{s_{k+1}}\dots dx^{s_r},$$

where

$$*v_{s_{k+1}\dots s_r}=\frac{1}{k!}\,\delta^{1\dots r}_{s_1\dots s_r}\,[\det(a_{ij})]^{1/2}a^{s_1 i_1}\dots a^{s_k i_k}v_{i_1\dots i_k}.$$

$\{a^{hi}\}$ is the symmetric contravariant positive tensor associated to a_{ij}, i. e. $a^{hi}a_{ij}=\delta^h_j$.

The co-differentiation operator δ is defined by $\delta=(-1)^{r(k+1)+1}*d*$ and maps the k-form v into its co-differential δv, which is a $(k-1)$-form. We have $\delta\delta v=0$.

The Laplace-Beltrami differential operator for k-forms is the following: $\Delta=d\delta+\delta d$. (Actually one gets the classical Laplace-Beltrami operator for o-forms (i. e. scalar functions), replacing Δ by $-\Delta$).

Let I be the identity operator (for k-forms) and set $G=(\Delta+I)^{-1}$. If we consider the space $\mathfrak{L}^2_k(V^r)$, i. e. the space of real k-forms, which have locally $\mathfrak{L}^2$ coefficients, we may introduce a Hilbert structure in this space by means of the following scalar product: $(u,v)=\int u\wedge *v$, where $\int$ stands for integration extended over the oriented manifold V^r. We can perform this integration in terms of our "data". In fact

$$\int u\wedge *v=\sum_{h=1}^{q}\int_{M_h}\psi_h\frac{1}{k!(r-k)!}\delta^{s_1\dots s_r}_{1\dots r}u_{s_1\dots s_r}*v_{s_{k+1}\dots s_r}\,dx^1\dots dx^r.$$

The operator G is a positive compact operator (briefly PCO) in the Hilbert space $\mathfrak{L}^2_k(V^r)$ (see [4, p. 154]). Because of the Hodge theorem [(5, p. 159], [6, p. 225]) we have that the Betti number of the dimension k: b_k equals the geometric multiplicity of the largest "eigenvalue" $\mu=1$ of the PCO G. It is understood that if $b_k=0$ the "eigenvalue" $\mu=1$ has geometric multiplicity zero, in other words $\mu=1$ is not an eigenvalue for G.

As a consequence we may assert that the problem of the computation of b_k is a particular case of the more general problem concerning the computation of the geometric multiplicity p of the largest eigenvalue of a PCO. (Actually our concern is a little more general. In fact we are interested in the following problem. Let μ be a positive number not less than the maximum eigenvalue of a given PCO; compute the geometric multiplicity p of μ, where $p=0$, if μ is not an eigenvalue for G.)

For the solution of this problem we shall use the Hilbert space group theoretic approach which has been used in eigenvalue theory in the papers [7; 8]. Following this approach we are led to consider a sequence of rotohomothetic invariants providing the computation of p.

Let $\mu_1\geq\mu_2\geq\dots\geq\mu_k\geq\dots$ be the sequence of the eigenvalues of the PCO G of the Hilbert space S each repeated according to its (geometric) multiplicity. Suppose that for some integer $n>0$ G^n has a finite Hilbert-Schmidt trace. Set for $s=1,2,\dots$

$$\mathfrak{J}^n_s(G)=\frac{(-1)^s}{2\pi i}\int_{+C}\frac{\prod_{k=1}^{\infty}(1-\lambda\mu_k^n)}{\lambda^{s+1}}\,d\lambda,$$

where C is any rectifiable contour of the complex plane enclosing the origin

Set

$$\psi_n^{(s)}(G) = \frac{1}{\mu_1^{ns}} \mathfrak{J}_s^n(G).$$

$\psi_n^{(s)}(G)$ is a roto-homothetic invariant (i. e. an orthogonal invariant of degree zero) such that

i) $$\psi_n^{(s)}(G) > \psi_{n+1}^{(s)}(G);$$

(1) ii) $$\lim_{n\to\infty} \psi_n^{(s)}(G) \begin{cases} = 0 & \text{if } p < s, \\ = \binom{p}{s} & \text{if } p \geq s \end{cases}$$

(see [7, p. 259]).

To the actual computation of the orthogonal invariant $\mathfrak{J}_s^n(G)$ provides the following representation theorem (see [6 p. 333])

(2) $$\mathfrak{J}_s^n(G) = \frac{1}{s\,!} \sum_{h_1, \ldots, h_s} \det\{(G^n u_{h_i}, u_{h_j})\} \quad (i, j = 1, \ldots, s),$$

where $u_1, \ldots, u_k, \ldots$ is an arbitrary complete orthonormal system in the space S.

Returning to the operator $G = (\Delta + I)^{-1}$, we know, from the theory of elliptic operators, that, for $n > r/2$, G^n has a finite Hilbert-Schmidt trace, hence (2) applies. For simplicity we assume $n = 2m$, $s = 1$. From (1) we deduce

$$b_k = \lim_{m\to\infty} \mathfrak{J}_1^{2m}(G).$$

Consider the following k-form on V^r

$$\omega^{(l;\, i_1, \ldots, i_r;\, s_1, \ldots, s_k)} \begin{cases} = \psi_h(x^1, \ldots, x^r)\,(x^1)^{i_1} \ldots (x^r)^{i_r} dx^{s_k} \ldots dx^{s_k} \\ \quad \text{if } x \in \text{image of } M_l \text{ on } V^r; \\ = 0 \text{ elsewhere}; \end{cases}$$

$$(l = 1, \ldots, q,\ i_1, \ldots, i_r = 0, 1, 2, \ldots;\ s_1, \ldots, s_k = 1, \ldots, r).$$

Fixed the integer $\nu > 0$, let us orthonormalize by the Gram-Schmidt procedure the sequence $\{(\Delta + I)^\nu \omega^{(l;\, i_1, \ldots, i_r;\, s_1, \ldots, s_k)}\}$, $l = 1, \ldots, q$; $i_1, \ldots, i_r = 0, 1, 2, \ldots$; $s_1, \ldots, s_k = 1, \ldots, r$).

Denote by $\{(\Delta + I)^\nu v_h^\nu\}$ the sequence which has been obtained by the orthonormalization process. From (2) we deduce

$$\mathfrak{J}_1^{2m}(G) = \sum_{h=1}^{\infty} \|(\Delta + I)^\lambda v_h^{m+\lambda}\|^2,$$

where λ is any arbitrarily chosen non-negative integer. From (1), assuming $\lambda = 0$, we deduce the unexpectedly simple limit relation

$$b_k = \lim_{m\to\infty} \sum_{h=1}^{\infty} \| v_h^m \|^2$$

and moreover

$$\sum_{h=1}^{\infty} \| v_h^m \|^2 > \sum_{h=1}^{\infty} \| v_h^{m+1} \|^2 > b_k.$$

An analogous result has been obtained by M. P. Colautti [3], who proves a less elegant formula, which, however, is more suitable for numerical computations.

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