Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

APPROXIMATION BY POLYNOMIALS WHICH KEEP THE MONOTONY OF THE APPROXIMATED FUNCTION

GEORGI L. ILIEV

Let $f\in C_\Delta$, $\Delta=[-1, 1]$, $-1=x_0<x_1<\cdots<x_s=1$; H_n be the set of algebraic polynomials of degree not greater than n, $\widehat{H}_n^s(f;\ x_0;\ x_1;\ldots;\ x_s)=\{P:P\in H_n;\ P'(x)\geqq 0$ if f is monotonely increasing for $x\in[x_{i-1}, x_i]$ and $P'(x)\leqq 0$ if f is monotonely decreasing for $x\in[x_{i-1}, x_i]$, $i=1,\ldots, s\}$; $\widehat{E}_n^s(f; x_0\ldots; x_s)=\inf\{\|f-P\|:P\in\widehat{H}_n^s(f;\ x_0;\ldots;x_s)\}$.

In the present paper the following estimation has been proved

$$\widehat{E}_n^s(f;x_0;\ldots;x_s)\leqq C_1(s)\omega(f;n^{-1}),$$

where $\omega(f;\delta)=\sup\{|f(x')-f(x'')|:|x'-x''|\leqq\delta\}$.

1. Let $f\in C_\Delta$, $\Delta=[-1, 1]$ be a function monotonely increasing in Δ, H_n be the set of algebraic polynomials of degree not greater than n, $H_n^1=\{P:P\in H_n,\ P'(x)\geqq 0,\ x\in\Delta\}$,

$$\widehat{E}_n^1(f)=\inf\{\|f-P\|_\Delta : P\in H_n^1\}=\inf\{\max\{|f(x)-P(x)|:x\in\Delta\}:P\in H_n^1\}.$$

In [1] Lorentz and Zeller prove that

$$E_n^1(f)=O(\omega(f;n^{-1})), \tag{1}$$

where $\omega(f;\ \delta)$ is the modulus of continuity of the function f:

$$\omega(f;\ \delta)=\sup\{|f(x')-f(x'')|:\ |x'-x''|\leqq\delta\}.$$

This result is remarkable with the fact that, in spite of the imposed additional condition on the approximating apparatus, the exact order for the uniform polynomial approximation, obtained by Jackson theorem, remains the same

Let $-1=x_0<x_1<\cdots<x_s=1$, $H_n^s(x_0;\ x_1;\cdots;\ x_s)=\{P:$

$$P\in H_n;\varepsilon(-1)^iP'(x)\geqq 0 \text{ for } x\in[x_{i-1}, x_i], i=1,\ldots, s;\ \varepsilon=\pm 1\}.$$

For $f\in C_\Delta$ define the number:

$$E_n^s(f;\ x_0;\ \cdots;\ x_s)=\inf\{\|f-P\|_\Delta:\ P\in H_n^s(x_0;\ \cdots;\ x_s)\}.$$

In [2] the following theorem is proved giving an exact answer to the problem for the order of approximation of partially monotone functions by partially monotone polynomials.

Theorem 1. *If $f\in C_\Delta$ is monotone in any of the subintervals $[x_i, x_{i+1}]$ of Δ, changing its monotony at the points x_i, $-1=x_0<x_1<\cdots<x_s=1$; $i=1, 2,\ldots, s-1$, then for every positive integer n:*

SERDICA Bugaricae mathematicae publicationes. Vol. 5, 1979, p. 283—286.

$$E_n^s(f;\ x_0;\ \cdots;\ x_s)\le c(s)\omega(f;\ n^{-1}).$$

The problem for partially monotone approximations has also been considered in the papers [3; 4; 5; 6].

Let $f\in C_\Delta,\ -1=x_0<x_1<\cdots<x_s=1,$

$$\widehat{H}_n^s(f;\ x_0;\ x_1;\ \cdots;\ x_s)=\{P:\ P\in H_n;\ P'(x)\ge 0$$

if f is monotonely increasing for $x\in[x_{i-1}, x_i]$ and $P'(x)\le 0$
if f is monotonely decreasing for $x\in[x_{i-1}, x_i]$, $i=1,\ldots,s\}$.

Remark. When defining $\widehat{H}_n^s(f;\ x_0;\ \cdots;\ x_s)$ no conditions are imposed on $P\in\widehat{H}_n^s(f;\ x_0;\ \cdots;\ x_s)$ if in the subinterval $[x_{j-1}, x_j]$ the function f is not monotone.

Let

$$\widehat{E}_n^s(f;\ x_0;\ \cdots;\ x_s)=\inf\{\|f-P\|_\Delta:\ P\in\widehat{H}_n^s(f;\ x_0;\ \cdots;\ x_s)\}.$$

A basic result of the present paper is the following

Theorem 2. *Under the above notations and assumptions*

$$\widehat{E}_n^s(f;\ x_0;\ \cdots;\ x_s)\le c_1(s)\omega(f;\ n^{-1}).$$

2. We will prove Theorem 2 in the particular case, when $x_0=-1$, $x_1=0$, $x_2=1$; f is monotonely decreasing in the subinterval $[x_0, x_1]$ and arbitrary in $[x_1, x_2]$. In the general case the theorem is proved by induction from this particular case and Theorem 1, by using the methods in [2].

Lemma 1. *If $f\in C_\Delta$, $f\in \mathrm{Lip}_M 1$, $f(x)=0$ for $x\in[-1, 0]$, then for every positive integer n there exists a polynomial $P\in H_n$, such that $P'(x)\le 0$ for $x\in[-1, 0]$ and $\|f-P\|_\Delta\le c_2Mn^{-1}$, where c_2 is an absolute constant.*

Proof. It is clear that f is a function with a bounded variation and can be represented in the following manner $f(x)=f_1(x)-f_2(x)$, where $f_1(x)=f_2(x)=0$ for $x\in[-1, 0]$, $f_1(x)$ and $f_2(x)$ are monotonely increasing for $x\in[0, 1]$, $f_1\in\mathrm{Lip}_M 1$, $f_2\in\mathrm{Lip}_M 1$.

From (1) and [1] it follows that for every n there exists an absolute constant c_3 and a polynomial $P_2\in H_n$, such that

$$\|f_2-P_2\|_\Delta\le c_3Mn^{-1},\quad P_2'(x)\ge 0 \text{ for } x\in[-1, 1]. \tag{2}$$

Theorem 1 and [2] yield that for every n there exists an absolute constant c_4 and a polynomial $P_1\in H_n$, such that

$$\|f_1-P_1\|_\Delta\le c_4Mn^{-1},\quad P_1'(x)\le 0 \text{ for } x\in[-1, 0] \tag{3}$$

and $P_1'(x)\ge 0$ for $x\in[0,1]$.

Let $P=P_1-P_2\in H_n$. From (2) and (3) it follows that

$$\|f-P\|_\Delta=\|f_1-f_2-P_1+P_2\|_\Delta\le\|f_1-P_1\|_\Delta+\|f_2-P_2\|_\Delta\le c_2Mn^{-1}$$

and, besides, $P'(x)=P_1'(x)-P_2'(x)\le 0$ for $x\in[-1,0]$.

Thus, the Lemma is proved.

Theorem 3. *Let $f\in C_\Delta$, $\Delta=[-1,1]$, $f\in\mathrm{Lip}_M 1$ and let f be monotonely decreasing for $x\in[-1, 0]$. Then, for any positive integer n there exists an* *absolute constant c_5 and a polynomial $P\in H_n$ such that for $x\in\Delta$: $\|f-P\|_\Delta \leq c_5Mn^{-1}$ and $P'(x)\leqq 0$ for $x\in[-1,0]$.*

Proof. This theorem follows from (1) and Lemma 1. Assume first that $f(0)=0$. Then f can be represented in the following way

$$f(x)=\varphi_1(x)+\varphi_2(x),\quad x\in[-1,1]:$$

$$\varphi_1(x)=\begin{cases}0 & , x\in[-1,0],\\ f(x), & x\in[0,1],\end{cases}\qquad \varphi_2(x)=\begin{cases}f(x), & x\in[-1,0],\\ 0 & , x\in[0,1].\end{cases}$$

From (1) follows that there exists a polynomial $P_2\in H_n$ for which

$$\|\varphi_2-P_2\|_\Delta\leq c_6Mn^{-1},\ P_2'(x)\geq 0\ \text{for}\ x\in[-1,1]. \tag{4}$$

From Lemma 1 it follows that there exists a polynomial $P_1\in H_n$ for which

$$\|\varphi_1-P_2\|_\Delta\leq c_7Mn^{-1},\quad P_1'(x)\leq 0\ \text{for}\ x\in[-1,0]. \tag{5}$$

If $P=P_1+P_2$, then from (4) and (5) we obtain: $\|f-P\|_\Delta\leq c_5Mn^{-1}$, $P'(x)\geq 0$ for $x\in[-1,0]$.

If $f(0)\neq 0$ we study the function $\psi(x)=f(x)-f(0)$. From the above considerations it follows that there exists a polynomial $Q\in H_n$, for which

$$\|\psi-Q\|_\Delta\leq c_7Mn^{-1},\ Q'(x)\leq 0\ \text{for}\ x\in[-1,0].$$

Then for the polynomial $P=Q+f(0)$ we will have $\|f-P\|_\Delta\leq c_7Mn^{-1}$, $P'(x)\leq 0$ for $x\in[-1,0]$.

Theorem 4. *Let $f\in C_\Delta$, $\Delta=[-1,1]$ and let f decreases monotonely in $[-1,0]$. For any positive integer n there exists an absolute constant c_8 and a polynomial $P\in H_n$ for which $\|f-P\|_\Delta\leq c_8\omega(f;n^{-1})$ and $P'(x)\leq 0$ for $x\in[-1,0]$.*

Proof. Without loss of generality we might consider that $f(0)=0$. Then it is easily deducted that there exists a function f_1 for which

$$f_1(x)\ \text{is monotonely decreasing in}\ [-1,0]; \tag{6}$$

$$f_1(x)=0\ \text{for}\ x\in[-\delta/2,\ \delta/2],\ 0<\delta<1; \tag{7}$$

$$f_1(x)=f_1(-1)\ \text{for}\ x\in(-\infty,-1],\ f_1(x)=f_1(1)\ \text{for}\ x\in[1,\infty); \tag{8}$$

$$\|f_1-f\|_\Delta\leq\omega(f;\ \delta); \tag{9}$$

$$\omega(f_1;\ \tau)\leq\omega(f;\ \tau)\ \text{for any}\ \tau>0. \tag{10}$$

Let

$$\varphi(x)=\frac{1}{\delta}\int\limits_{x-\delta/2}^{x+\delta/2}f_1(t)dt. \tag{11}$$

From (10) and (11) we obtain:

$$|\varphi(x)-f_1(x)|=\frac{1}{\delta}\int\limits_{x-\delta/2}^{x+\delta/2}|f_1(t)-f_1(x)|\,dt \tag{12}$$

$$\leq\frac{1}{\delta}\int\limits_{x-\delta/2}^{x+\delta/2}\omega(f_1;\ |t-x|)\,dt\leq\omega(f_1;\ \delta/2)\leq\omega(f;\ \delta/2)$$

The function φ is the first function of Steklov with a step δ for the function f_1. The function φ is differentiable in Δ and

$$\varphi'(x)=\{f_1(x+\delta/2)-f_1(x-\delta/2)\}/\delta \tag{13}$$

(see [7, p. 188]).

From (6), (7) and (13) follows that φ is monotonely decreasing for $x\in[-1,0]$ and $\|\varphi'\|_\Delta\leqq\delta^{-1}\omega(f_1;\ \delta)\leqq\delta^{-1}\omega(f;\ \delta)$.

Then $\varphi\in \mathrm{Lip}_M 1$, where $M=\delta^{-1}\omega(f;\ \delta)$.

Theorem 3 yields that there exists $P\in H_n$ for which

$$\|\varphi-P\|_\Delta\leqq c_5n^{-1}\delta^{-1}\omega(f;\ \delta) \tag{14}$$

and $P'(x)\leqq 0$ for $x\in[-1,0]$.

From (9), (12) and (14) it follows that

$$\|f-P\|_\Delta\leqq\omega(f;\ \delta)+\omega(f;\ \delta/2)+c_5n^{-1}\delta^{-1}\omega(f;\ \delta). \tag{15}$$

If $\delta=n^{-1}$, from (15) we obtain the proof of the theorem since $P'(x)\leqq 0$ for $x\in[-1,0]$.

REFERENCES

1. G. Lorentz, K. Zeller. Degree of approximation by monotone polynomials, I. *J. Approxim. Theory*, **1**, 1968, 501—504.
2. G. Iliev. Exact estimates for partially monotone approximation. *Anal. Math.*, **4**, 1978, 181—197.
3. J. Roulier. Nearly monotone approximation. *Proc. Amer. Math. Soc.*, **47**, 1978, 84—88.
4. D. Newman, E. Passov, L. Raymon. Piecewise monotone polynomial approximation. *Trans. Amer. Math. Soc.*, **172**, 1972, 465—472.
5. E. Passov, L. Raymon. Monotone and comonotone approximation. *Proc. Amer. Math. Soc.*, **42**, 1976.
6. G. Iliev. Exact estimations under the partially monotone approximation and interpolation. *C. R. Acad. Bulg. Sci.*, **30**, 1977, 491—494.
7. Н. Ахиезер. Лекции по теории аппроксимации. Москва, 1947.

Centre for Mathematics and Mechanics
1090 Sofia P. O. Box 373

Received 13. 7. 1978