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PERFECTION OF IDEALS GENERATED BY THE PFAFFIANS OF AN ALTERNATING MATRIX, 2

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Let R be a noetherian commutative ring, X be an alternating $s\times s$ matrix whose above-diagonal entries are algebraically independent over R, and let $Pf_{2t}(X)$ denote the ideal in $R[X_{ij}]$ generated by the pfaffians of all the alternating $2t\times 2t$ submatrices of X. It is proved that $Pf_{2j}(X)$ is perfect of depth $(s-2t+1)(s-2t+2)/2$, and that it is prime when R is a domain. The theorem is a consequence of a more general result, involving ideals generated by both pfaffians and determinants of submatrices of X.

The second part of the paper contains the end of the proof of the main results.

6. Construction of a generic matrix for $A_{H,n}$. To compute depth $A_{H,n}$ we shall construct an alternating matrix with special properties U depending on H and n. The technical details of the construction take up the greater part of the present paragraph (and of the paper). The construction of U will be made by an induction on m (Induction 1) accompanied by parallel inductive reasonings, proving the properties of this matrix (Induction 2 and Induction 3).

For this section K is a field, $H=(s_0,\dots,s_m)$ is a standard description and $n=s_h$ or $n=s_h+1=2h+2$, $0\leq n\leq s=s_m$.

Induction 1.

1. Beginning of the induction: for $m=0$ we set $U=0$.

2. Induction hypothesis: For each $m'<m$, the $m'\times m'$ matrix U is constructed.

3. Construction of U.

Note. At the time of constructing U one may notice that $U(s',s')$ where $s'=s_{m-1}$, is the matrix corresponding to the description $H'=(s_0,\dots,s_{m-1})$, $n'=\min\{n, s'\}$.

3.1. Classification of the columns: Let $D=\{s_0+1, s_1+1,\dots, s_{m-1}+1\}\subset C=\{2, 3,\dots, s\}$. Inductively on r, we construct a function $f(r)$ with integer values:

$f(0)=\min C\setminus D$,

$f(r)=\min\{k: f(r-1)<k \text{ and } k\in C\setminus D\}$ for $0<r\leq m-1$.

Note. If $s=2m$, we define f only for $r=0,1,\dots, m-2$. The jth column (resp. row) is called a column (resp. row) with indeterminates, if either $j\in D$ or $j=f(r)$ for some r. The remaining columns and rows are called ordinary columns or rows $(j\in C)$.

3.2. Property A: The ordinary columns of U are linear combination of the columns with indeterminates with coefficients in $K(u_{ij})$ where u_{ij} are the indeterminates over K which are entries of U.

Property A will be proved by Induction 2. Here we shall use only the hypothesis of Induction 2.

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3.3. Filling-in the columns with indeterminates: First of all place 0 on the diagonal places. After this:

3.3.1. $j \in D$. When $j = s_k + 1$ we fill-in the column by indeterminates u_{ij} when $i \neq s_0 + 1, s_1 + 1, \ldots, s_k + 1$; in case $n > s_k$, $i \neq 1$ we put 0 as first entry. The $s_k + 1$ — st row is filled-in skew-symmetrically.

3.3.2. $j = f(r)$ for some r. The jth rows and columns for $j \in D$ have already been filled-in.

3.3.2.a) If $s_r > 2r + 1$, then $j = f(r) \leq s_r$. Fill-in the jth column new indeterminates u_{ij} where $i > s_r + 1$, $i \notin D$. The remaining vacant places will be filled as follows: Consider the entries $\{u_{ij} : i \leq s_r\}$ as a vector-column of the matrix $U(s_r, s_r)$ and fill it as an ordinary column in $U(s_r, s_r)$. Hence, by the induction hypothesis for property A, these entries filling the column are not new indeterminates. By skew-symmetry we fill-in the corresponding row.

3.3.2.b) If $s_r = 2r + 1$, then $j = s_r + 2 = 2r + 3$. In this case the first j places of the column (except for the first one) are already filled-in. Let u_{ij} for $i > j$, $i \notin D$ be new indeterminates over K. In addition if $n > s_r + 1$, place 0 in the first place; otherwise let u_{1j} be a new indeterminate over K. After that fill-in the jth row skew-symmetrically.

3.3.3. Thus the matrix U is constructed in the case $s = 2m$ and the following arguments don't concern this case.

N o t e s. 1. On each row (resp. column) filled-in up to now the indeterminate entries (represented in Fig. 1 by points) lie on the right part of the row (resp. on the lower part of the column).

2. Call primary entries of U those with indices $(f(r), s_r + 1)$ $(r = 0, 1, \ldots$ $m-1)$ and the symmetric to them. They are represented in Fig. 1 by $\oplus$' For every row (column) with indeterminates exactly one primary entry lies'

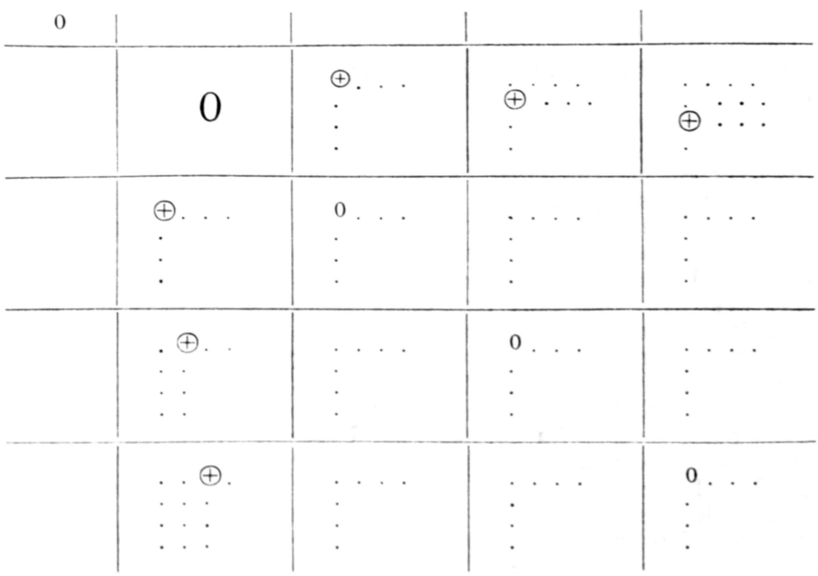


Fig. 1

on it. All the indeterminates on such a row (resp. column) lie to the right (resp. downwards) of its primary entry. Every primary entry is an indeterminate over K. The number of the primary entries which lie in the submatrix $U(s_t, s_t)$ is even $(0 \leqq t \leqq m)$.

3.4. Definition of property B: Denote by u the set of all the indeterminates over K which are entries of U. Specialize the above diagonal primary entries to 1 (their symmetric ones to -1) and the other indeterminates to 0. We denote this specialization by $u = u_0$. Now we can formulate a property which we shall name property B of the matrix U.

Property B: For every i and j let the entry $f_{ij}(u)/g_{ij}(u)$ where $f_{ij}, g_{ij} \in K[u]$ lie on the ith row and the jth column of U. Under the specialization $u = u_0$ we have: $g_{ij}(u_0) \neq 0$ for each (i, j); $f_{ij}(u_0) = 0$ for each (i, j) except in the cases when u_{ij} is a primary entry.

This property will be proved by Induction 3. Now we shall use only the hypothesis of the Induction 3.

3.5. Filling-in the ordinary columns: We shall fill-in the ordinary columns in order of increasing of the index. We shall construct them as linear combinations of the columns with indeterminates with coefficients in $K(u)$.

Up to this point property B follows for the entries of the columns with indeterminates by the hypothesis of Induction 3.

3.5.1. Conditions for the linear combination: Now we shall fill-in the jth column (an ordinary column) supposing all the columns to the left of it (except the first one) already filled-in. Some entries on the jth column have already been written and a part of them are indeterminates over K. These indeterminates are represented in Fig. 2 by **x**, and the rest of the column by●. By note 3.3.3.2. the number of these indeterminates equals the number of the primary entries which lie to the left of the jth column and, hence, equals the number of the columns with indeterminates which lie to the left of the jth column. Therefore the intersection of these columns with the rows, on which lie the **x**-entries from Fig. 2, gives a square matrix M_j whose entries are represented by x.

For every column and row of M_j there is just one primary entry which lies on it. A part of the entries of M_j, including the primary ones, are indeterminates over K, and the other entries depend on them.

3.5.2. Uniqueness of the linear combination: We shall show that M_j is an invertible matrix. Put $u = u_0$. Thus, because for the entries on the columns with indeterminates we have property B, the primary entries of M_j become equal to 1 or -1 and the other ones — to 0. We get a matrix in which every column and every row has only one entry not equal to 0. Hence, it is an invertible matrix. Consequently M_j is invertible.

We want the jth column to be a linear combination of all the columns with indeterminates which lie to the left of it. For the coefficients $k_1, k_2, \ldots, k_p$ of this combination we have a system of linear equations: one for each filled-in place on the jth column. The entries represented by **x** in Fig. 2 give so many equations as the desired coefficients, and the matrix of this subsystem is M_j. Hence where is a unique solution which satisfies the subsytem. The other equations of the system, except for the one given by the diagonal entry, are parts of ordinary rows (more precisely: the coefficients of such an equation and its free term are entries of an ordinary row — the j'th row, where $j' < j$). But each such row is a linear combination of the rows with in-

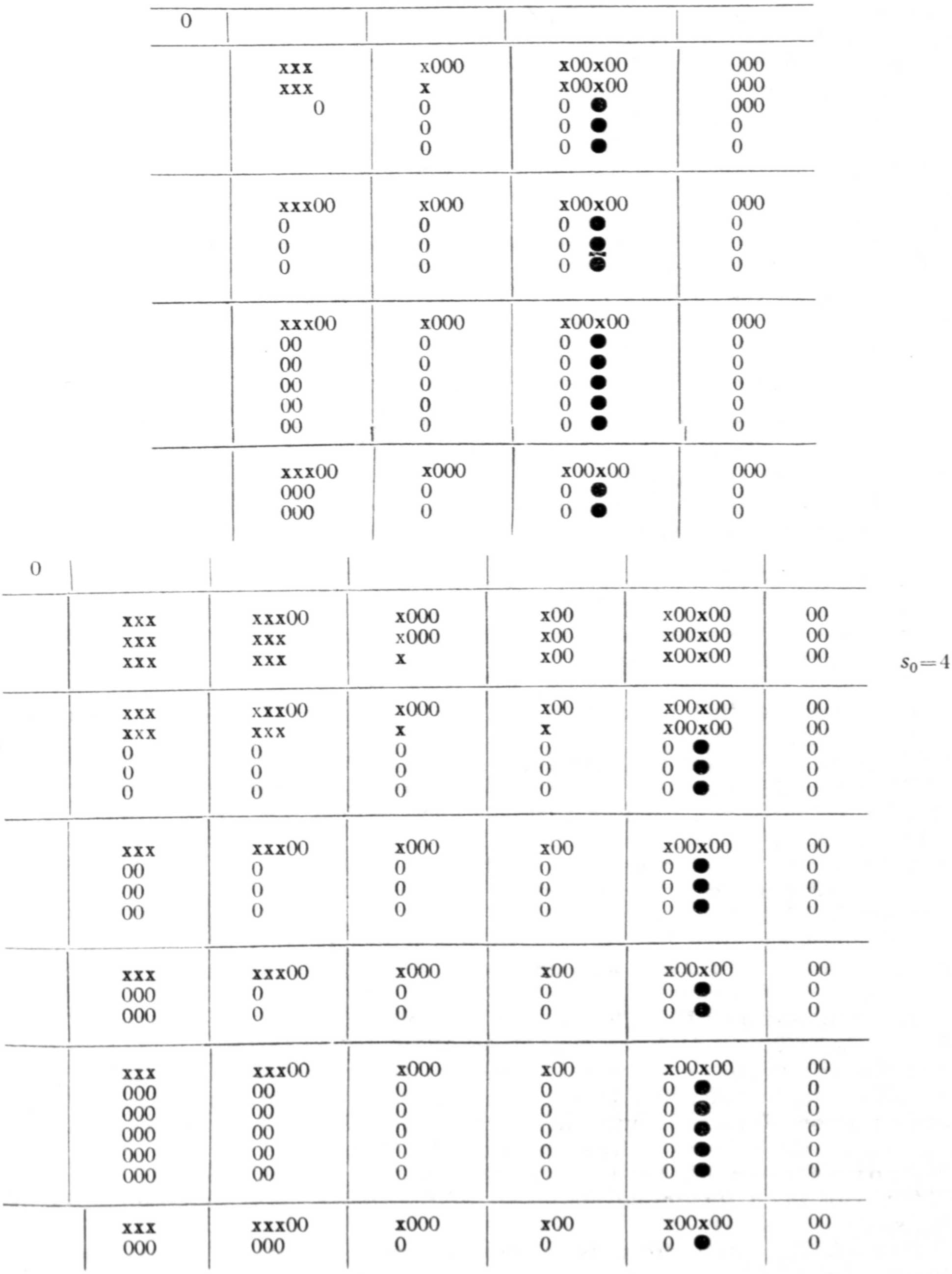


Fig. 2

determinates which lie above it. Therefore every such equation is a linear combination of the previous equations and, hence, the obtained solution $(k_1, \ldots, k_p)$ satisfies it. Thus the jth column is determined.

3.5.3. Existence of the linear combination (verification of the diagonal entry): Now we shall verify that the diagonal entry of the jth column is 0. Let $s_{t-1}<j\leq s_t$ and let us consider the first s_t entries of the jth column. They form the jth column of $U(s_t, s_t)$ and the diagonal entry of the jth column of U is a diagonal of the jth column of $U(s_t, s_t)$.

If $s_t<s$, together with construction of the jth column of U we make the construction of the jth column of $U(s_t, s_t)$, Therefore, by the induction hypothesis, the diagonal entry of the jth column of $U(s_t, s_t)$ is zero, as required.

If $s_t=s$, all the primary entries are to the left of the jth column. In addition, their number is even. The matrix M_j is alternating. Consider again the system of equalities by which we determined $(k_1, k_2, \ldots, k_p)$. It is of the form $a_{i1}k_1+a_{i2}k_2+\ldots+a_{ip}k_p=b_i\ (1\leq i\leq p)$, where $M_j=(a_{kr})$ is alternating. We want to verify

whether the equality determined by the diagonal entry is satisfied, i. e. whether $-b_1k_1-b_2k_2-\ldots-b_pk_p=0$. This follows from the computation: $b_1k_1+b_2k_2+\ldots+b_pk_p=(a_{11}k_1+a_{12}k_2+\ldots+a_{1p}k_p)k_1+(a_{21}k_1+a_{22}k_2+\ldots+a_{2p}k_p)k_2+\ldots+(a_{p1}k_1+a_{p2}k_2+\ldots+a_{pp}k_p)k_p=\Sigma_{1\leq i<j\leq p}(a_{ij}+a_{ji})k_ik_j+\Sigma_{i=1}^{p}a_{ii}k_i^2=0$. After determination of the jth column fill-in the jth row skew-symmetrically.

3.6. Filling-in the first column: After determining all the columns except the first one, the first row is determined. Note that the first n entries of the first row are 0. Fill-in the first column skew-symmetrically to the first row

Induction 2 (for property A).

We make this induction on m. For $m=0$ the property is obvious, so let $m>0$ and suppose that for each $m'<m$ the property holds. The induction step is made in 3.5. of Induction I.

Induction 3 (for property B).

We make this induction on m. For $m=0$ $U=0$ and the property is obvious, so let $m>0$ and suppose that for each $m'<m$ the property holds. For the entries of the columns and rows with indeterminates we have required property. Therefore, all other entries which now interest us lie on an ordinary column. Having in mind the method for determination the $(k_1, k_2, \ldots, k_p)$ in Induction 1, 3.5.2., the condition on the denominators follows immediately from Kramer's formulas. Also by Kramer's formulas the condition on the numerators follows, because with $u=u_0$ all free terms of the system of equations specialize to 0.

7. Computation of depth $A_{H,n}$. In the following two propositions K is a field, $H=(s_0, \ldots, s_m)$ is a standard description, $n=s_h$ or $n=s_h+1=2h+2$, U is the matrix for $A_{H,n}$ constructed in 6. We use in the proofs of the propositions all consepts and symbols introduced in 6, and also introduce one more concept: a row (column) with indeterminates in $U(s_t, s_{t'})$ is a row (column) on which lie the primary entries contained in $U(s_t, s_{t'})$.

Proposition 17. *Let $0\leq t\leq t'\leq m$ and $t+t'+1\leq s_t$. Then the first row of $U(s_t, s_{t'})$ is a linear combination of the rows with indeterminates of $U(s_t, s_{t'})$.*

Proof. The proposition is obviously true if either $s=2m$ or $m=0$. By induction on m, we suppose that for each $m'<m$ the proposition is true. If

$s_{t'} < s$, i. e. $t' < m$, the question reduces to the $U(s_{t'}, s_{t'})$ and we have the proposition by the induction hypothesis. If $s_{t'} = s$, i. e. $t' = m$. the condition $t + t' + 1 = t + m + 1 \leqq s_t$ give us $f(m-1) \leqq s_t$. Indeed, if we express the function f explicitely, we see that: $f(r) = r + q + 2$ for $s_{q-1} \leqq r \leqq s_q - q - 2$; $q = 0, 1, \ldots$ $(s_{-1} = 0)$. Hence, we see that $t + m + 1 \leqq s_t$ implies $m - 1 \leqq s_t - t - 2$. Let $u = \min \{v : m - 1 \leqq s_v - v - 2\}$, hence $u \leqq t$. Consequently $f(m-1) = m - 1 + u + 2 = m + u + 1 \leqq m + t + 1 \leqq s_t$. Let t be the smallest number such that $t + m + 1 \leqq s_t$. Obviously, if we prove the proposition for t, it will follow immediately for all integers up to m.

Let us turn to the last $q = s - s_{m-1} - 1$ columns. In the case $t = m$ and $s_{m-1} = 2(m-1) + 1$ let $q = s - s_{m-1} - 2$. These q columns form a submatrix N of $U(s_t, s)$. The columns of N lie to the right of all the columns with indeterminates of U and, by property A, are their linear combinations. These combinations are determined by the submatrix F of $U(s_t, s)$ whose entries are denoted in Fig. 3 by x. F is formed by the rows and columns of the primary entries in $U(s_t, s)$.

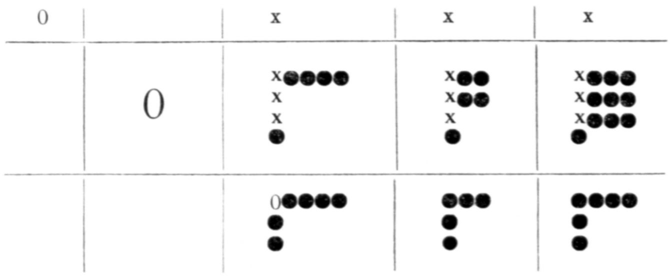


Fig. 3

By property B of U we see that F is an invertible matrix. Let the ith column of M $(1 \leqq i \leqq q)$ be a linear combination of the columns with indeterminates in $U(s_t, s)$ with coefficients $(y_{1i}, y_{2i}, \ldots, y_{pi})^T$. Here p is the size of F.

Let $Y = (y_{ij})$. Therefore Y is a $p \times q$ matrix. Let G be the intersection of M with the rows with indeterminates in $U(s_t, s)$. Hence G is a $p \times q$ matrix and $G = FY$. Let c be the first vector-row of N and d be the intersection of the first row with columns with indeterminates in $U(s_t, s)$. Hence c is an $1 \times q$ matrix, d is an $1 \times p$ matrix, and $c = dY$. The entries of d are represented in Fig. 3 by x.

By Lemma 15 (cf. section 5), c is a linear combination of the vector-rows of G. Since F is an invertible $p \times p$ matrix, d is a uniquely defined linear combination over $K(u)$ of the vector-rows of F. Then by Lemma 15 c is the same linear combination of the vector-rows of G. Therefore the first row of $U(s_t, s)$ is a linear combination of the rows with indeterminates in $U(s_t, s)$, at least for these entries, which form c and d. If there are no other entries (except these on c and d) on the first row, the proposition is proved.

Now we shall consider the case when other entries do exist. More than one primary entry can occur on the last $s - s_{m-1}$ columns of U (in particular on the last $s - s_{m-1}$ columns of $U(s_t, s)$) only if $t = m$ and $s_{m-1} = 2(m-1) + 1$. This is seen from the definition of primary entry, and from the fact that $f(r) > s_r$ is satisfied only if $s_r = 2r + 1$, $0 \leqq r \leqq m - 1$ (see 6, 3.3.2.a)). But in this

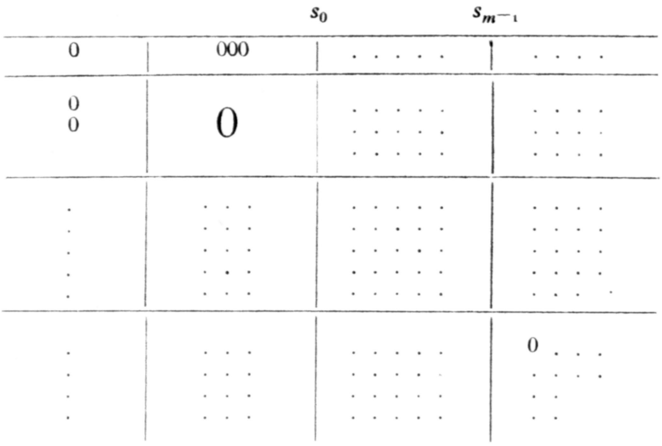


Fig. 4

case the proposition is proved, because the entries on c and d are the only ones on the first row of U (see Fig. 4).

Therefore in the rest of the proof we suppose there is just one primary entry on the last $s-s_{m-1}$ columns of $U(s_t, s)$, namely the entry with indices $(f(m-1), s_{m-1}+1)$. Consequently there are one less primary entries and one less rows with indeterminates in $U(s_t, s_{m-1})$, namely all the rows with indeterminates except the $f(m-1)$th row.

Let F' be obtained by deleting the last column of F and d' be obtained by deleting the last entry of d. Hence d' is a linear combination of the vector-rows of F' with the same coefficients which express d as a combination of the rows of F. It is seen by the construction of $U(s_{m-1}, s_{m-1})$ that the $f(m-1)$ th row (which is ordinary there) is a linear combination of the rows with indeterminates which lie above it. Moreover the first row of $U(s_{m-1}, s_{m-1})$ is, by the induction hypothesis, a linear combination of the same rows. If we choose these entries on the first row which correspond to the columns with indeterminates in $U(s_t, s_{m-1})$, we get the vector d'. Then by Lemma 16 the first row of $U(s_t, s_{m-1})$ is such a linear combination of the rows with indeterminates and of the $f(m-1)$th row as d' is of the vector-rows of F'. Therefore this linear combination gives the entire first row of $U(s_t, s)$. Thus the proposition is proved.

Note. In the above proof we have shown that with the same hypothesis as in proposition 17, $f(m-1)\leq s_t$ if $t'=m$.

Definition. *By a generic point for a prime ideal we mean a homomorphism from the given ring to a domain whose kernel is the ideal in question.*

Proposition 18. *Let X be a generic alternating $s\times s$ matrix, and $R=K[X]$. Consider $K[U]\subset K(u)(=K(u_{ij})$, where u_{ij} are all the indeterminates entries of U). Then the K-homomorphism Φ mapping $K[X]$ onto $K[U]$, which takes each entry of X to the corresponding entry of U, is a generic point for $A_{H,n}(X)$. Moreover* coht $A_{H,n}(X)=2sm-m(m+1)-s_0-s_1-\dots-$

$s_{m-1}-h-\Sigma_h$ *for* $n=s_h$ *and* coht $A_{H,n}(X)=$ coht $A_{H,n-1}(X)-1$ *for* $n=s_h+1$ $=2h+2$.

Proof. 1. We shall prove $A_{H,n}(X)\subset \ker \Phi$. Hence it is sufficient to verify by induction on m that $A_{H,n}(U)=0$. If $m=0$, $U=0$, hence $A_{H,n}(U)=0$. So suppose $m>0$ and that for each $m'<m$ the proposition holds. Let $1\leqq j\leqq k\leqq m$. If $k\leqq m-1$ by the induction hypothesis $I_{j+k+1}(U'(s_j, s_k))\subset A_{H',n'}(U')=0$ where $U'=U(s_{m-1}, s_{m-1})$, $H'=(s_0, \ldots, s_{m-1})$, $n'=\min\{n, s_{m-1}\}$. But $I_{j+k+1}(U(s_j, s_k))$ is the extention of $I_{j+k+1}(U'(s_j, s_k))$ with respect to the inclusion $K[U']\subset K[U]$. Hence $I_{j+k+1}(U(s_j, s_k))=0$.

If $k=m$, there are two cases:

a) When $j+m+1>s_j$, $I_{j+m+1}(U(s_j, s))=0$.

b) When $j+m+1\leqq s_j$, by Proposition 17 the first row, and by Property A for U (section 6) all the ordinary rows of $U(s_j, s)$, are linear combinations of the rows with indeterminates in $U(s_j, s)$. By the note after Proposition 17 $f(m-1)$ $\leqq s_j$. Having in mind this and the definition of primary entry we see that there are r primary entries more in $U(s_{j+r}, s)$ than there are in $U(s_j, s)$. We see also that the number of all primary entries (i. e. for $j=m$) is $2m$. Hence there are $j+m$ primary entries in $U(s_j, s)$ and $j+m$ rows with indeterminates. Consequently, rank $U(s_j, s)=j+m$ and $I_{j+k+1}(U s_j, s_k))=I_{j+m+1}(U(s_j, s))=0$.

Now consider $Pf_{2j+2}(U(s_j, s_j))$. If $j<m$, $Pf_{2j+2}(U(s_j, s_j))=0$ in $K[U(s_j, s_j)]$ $\subset K[U]$ by the induction hypothesis. On the other hand, $Pf_{2j+2}(U(s_j, s_j))$ is its extension in $K[U]$, hence it is 0 as well. If $j=m$, $Pf_{2j+2}(U)\subset \operatorname{rad}(I_{2j+1}(U))$ $=0$ by Lemma 3. Therefore $A_{H,n}(U)=0$

2. Now we shall prove $\ker \Phi\subset A_{H,n}(X)$, i. e. $A_{H,n}=\ker \Phi$.

Let $D=K[X]/A_{H,n}(X)$; D is a domain by Proposition 14. Let $\bar{X}$ be the image of X in D and L be the fraction field of D. It will be shown that $\bar{X}$ can be "factored" in the same form as U over L.

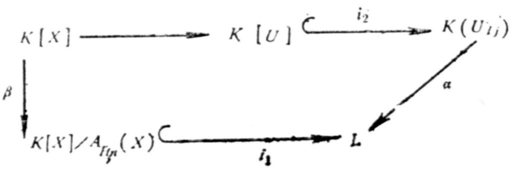
First let us specialize the entries of X in the following way: set 1 for the above-diagonal entries which correspond to the primary entries of U, and 0 for the others. We get a matrix X_0 for which $A_{H,n}(X_0)=0$ and the columns of X_0 corresponding to the columns with indeterminates of U are linearly independant. This follows by Property B of U (section 6) (X_0 corresponds to the specialization $u=u_0$ of U). Because $A_{H,n}(X_0)=0$. X_0 is a homomorphic image of $\bar{X}$ in some ring. Therefore the columns of X corresponding to the columns with indeterminates of U are linearly independant over L. The corresponding statement for the row of $\bar{X}$ follows in a similar way.

For each pair (i, j) such that $0\leqq i\leqq j\leqq m$ by definition rank $(\bar{X}\ (s_i, s_j))$ $\leqq i+j$. But as in 1, an easy count shows that the number of primary entries of $U(s_j, s_j)$ is $2j$. Hence their number in $U(s_i, s_j)$ is $i+j$ (in case $i+j+1\leqq s_i$).

Consequently there are $i+j$ linearly independant rows in $\bar{X}(s_i, s_j)$ (corresponding to the rows with indeterminates in $U(s_i, s_j)$) and the other rows are their linear combinations. Moreover, the combinations are uniquely defined, which is seen by specializing to X_0. Therefore the other rows can be constructed from the linearly independant rows in the same way as those corresponding to them were constructed in U.

In case $i+j+1>s_i$, $i+k+1=s_i$ for some k, and we know that in $\bar{X}$ (s_i, s_k) there are $i+k$ linearly independant rows and the first row, which is their linear combination. The first row of $\bar{X}$ is a linear combination of linearly independent rows corresponding to the ones described in Proposition 17. Therefore all the rows in $\bar{X}(s_i, s_j)$ are defined and linearly independent.

Let u_{ij} be all indeterminate over K entries of U. Let $\alpha : K(u_{ij}) \to L$ be a K-homomorphism defined as follows: $\alpha(u_{ij}) = \bar{x}_{ij} = x_{ij} + A_{H,n}(X)$. Thus U is



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taken onto $\bar{X}$ and the following diagram is commutative: Hence $i_1 \beta(\ker \Phi) = \alpha i_2 \Phi(\ker \Phi) = 0$. Consequently $\beta(\ker \Phi) = 0$. Hence $\ker \Phi \subset A_{H,n}(X)$ and, therefore, $\ker \Phi = A_{H,n}(X)$, as required.

3. We have only to compute coht $A_{H,n} = \dim K[U]$. But $K[u_{ij}] \subset K[U] \subset K(u_{ij})$, and $\dim K[U] = \mathrm{trdeg}_K K[U]$ which is just the number of indeterminates u_{ij}. For $n = s_h$ it is equal to: $[(s-1-1)+(s-2-1)+\ldots+(s-h-1)] + (s-(h+1)) + \ldots + (s-m) + (s-s_0-m) + (s-s_1-m+1) + (s-s_2-m+2) + \ldots + (s-s_{m-1}-m+m-1) - h - \Sigma_h = 2sm - m(m+1) - s_0 - s_1 - \ldots - s_{m-1} - h - \Sigma_h$. For $n = s_h + 1 = 2h+2$ it is one less than for $n = s_h$.

Ler for the rest of the paragraph K be a field, X be a generic alternating $s \times s$ matrix over K, $R = K[X]$ and $H = (s_0, \ldots, s_m)$ be a standard description. In addition $0 \leq n \leq s$ and $A_{H,n} = A_{H,n}(X)$.

Proposition 19. *For a field K and (H, n) such that $A_{H,n}$ is prime, the depth of $A_{H,n}$ is equal to $g_{H,n}$, where*

$$g_{H,n} = \begin{cases} \frac{s(s-1)}{2} - 2sm + m(m-1) + s_0 + s_1 + \ldots + s_{m-1} + h + \Sigma_h \text{ if } n = s_h, \\ g_{H,\, n-1} + 1 \text{ if } n = s_h + 1 = 2h + 2. \end{cases}$$

Proof. By standard properties of ideals in polynomial rings (cf. [9], [2]) we have depth $A_{H,n} = \mathrm{ht}\, A_{H,n} = \dim K[X] - \mathrm{coht}\, A_{H,n}$ and $\dim K[X] = \mathrm{tr\,deg}_K K[X] = s(s-1)/2$. Hence depth $A_{H,n} = s(s-1)/2 - \mathrm{coht}\, A_{H,n}$ and the result follows by Proposition 18.

Corollary. If *$A_{H,n}$ is not prime*, depth $A_{H,n} = g_{H,n}$ *with* $h = \min\{t : s_t \geq n\}$.

This follows from the formula depth $\mathbf{a} \cap \mathbf{b} = \min\{\text{depth } \mathbf{a}, \text{depth } \mathbf{b}\}$ and Propositions 14 and 19.

8. Perfection. In this section X is a generic alternating $s \times s$ matrix over the Noetherian ring R, $H = (s_0, \ldots, s_m)$ is a standard description. We prove the perfection of the "fundamental ideals" in Propositions 21 and 22.

Proposition 20. *Let $R = \mathbf{Z}$ and $0 \leq n \leq s$. Then $\mathbf{Z}[X]/A_{H,n}(X)$ is is an abelian group without torsion.*

Proof. By Proposition 14, 2) we see it is sufficient to suppose $A_{H,n}$ prime, and in this case the result follows from the relation $A_{H,n}(X) \cap \mathbf{Z} = 0$.

Proposition 21. *Let $R = K$ be a field and $n = s_h$ or $n = s_h + 1$ or $n = s_h + 2 = 2h + 3$. Then $A_{H,n}(X)$ is perfect and* depth $A_{H,n}(X) = g_{H,n}$.

Proof. We can assume the result inductively, for smaller matrices, and for larger ideals of the form $A_{H', n'}$, $n' = s_{h'}$ or $n' = s_{h'} + 1$ or $n' = s_{h'} + 2 = 2h' + 3$. By Lemma 5 we can reduce to the case, where $n < s$.

If $n=s_h$ or $n=s_h+1=2h+2$, $A_{H,n}$ is prime, homogeneous and $x=x_{1,n+1}$ is a form which is not a zero divisor on $A_{H,n}$; hence, $A_{H,n}+(x)=A_{H,n+1}$ and $A_{H,n}$ are perfect or not alike, by [6. Corollary to Proposition 19]. But $A_{H,n+1}$ is perfect by the induction hypothesis, and our proof is complete in this case.

If $n=s_h+1<s_{h+1}$ where $s_h \neq 2h+1$, or if $n=s_h+2=2h+3<s_{h+1}$, we have $A_{H,n}=A_{H',n}\cap A_{H,n'}$, as in Proposition 14 and by the induction hypothesis $A_{H',n}$, $A_{H,n'}$ and $A_{H',n}+A_{H,n'}=A_{H',n'}$ are all perfect. By Proposition 19 we see $A_{H',n}\not\subset A_{H,n'}$ and $A_{H,n'}\not\subset A_{H',n}$. For the depth we have depth $A_{H',n'}\leq$depth $A_{H',n}+1=$depth $A_{H,n'}+1$. Hence, by [6, Proposition 18] it follows that $A_{H,n}$ is perfect.

Proposition 22. *Let* $n=s_h$ *or* $n=s_h+1$ *or* $n=s_h+2=2h+3$. *Then* $A_{H,n}(X)$ *is perfect and* depth $A_{H,n}=g_{H,n}$.

Proof. According to [7, Proposition 20], the conclusions of Propositions 20 and 21 are sufficient to prove the desired result.

9. Completion of the proof of theorem 1. First consider the case $R=K[X]$, X a generic alternating matrix. Let $n=s_t$ or $n=s_t+1$ or $n=s_t+2=2t+3$. Then the perfection of $A_{H,n}$ and the equation (1): depth $A_{H,n}=g_{H,n}$ are proved in Proposition 22. Proposition 13 shows that $A_{H,n}$ is radical (K is a domain), and by Propositions 14, 2) and 22 it follows that depth $A_{H,n}=\min$ {depth $A_{H',n}$, depth $A_{H,n'}$}$=g_{H,n}$; thus (2) is proved. Part (3) is found in Proposition 14, 1) for $n<s$ and by Lemma 5 for $n=s$.

Now let R and M be arbitrary and $n'=s_h$. Suppose first that none of the relations $n=s_t$ or $n=s_t+1$ or $n=s_t+2=2t+3$ is satisfied. We have

depth $A_{H,n}(M)\leq$depth $A_{H,n'}(M)$ (obvious)

$\leq pd_{\mathbb{Z}[X]}(\mathbb{Z}[X]/A_{H,n'}(X))$ (by Proposition 20 and [5, Proposition 4])$=$depth $A_{H,n'}(X)=g_{H,n'}$ (by the particular case already considered)$=g_{H,n}$.

If $n=s_t$ or $n=s_t+1$ or $n=s_t+2=2t+3$, a similar argument shows depth $A_{H,n}(M)\leq \mathrm{pd}_{\mathbb{Z}[X]}(\mathbb{Z}[X]/A_{H,n}(X))=$depth $A_{H,n}(X)=g_{H,n}$; moreover [5, Proposition 4] shows that in the case depth $A_{H,n}(M)=g_{H,n}$, $A_{H,n}(M)$ is perfect.

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