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ON A CONVOLUTION STRUCTURE OF A GENERALIZED HERMITE TRANSFORMATION

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In this paper we consider a generalization of the Hermite transformation introduced by Debnath. The problem of the linearization of the product of two modified Hermite functions of the first kind is partially solved. By means of this result we develop the convolution structure of this transformation. This result can't be specialized to the case of the Hermite polynomials (owing to a restriction of the parameter) according to the fact, that a convolution theorem for the Hermite (-polynomial) transformation is known only in the case of odd degree.

The convolution structure for integral transformations with orthogonal polynomials as kernel were investigated by many authors. So for the case of the Jacobi transformation by Askey and Wainger [2], Gasper [10; 11], Flensted-Jensen and Koornwinder [9], who even considered the generalization on Jacobi functions of the first kind and in the case of the Laguerre transformation by Mc Cully [3], Debnath [4], Askey [1], and the author [13], who dealt with the case of Laguerre functions of the first kind.

The foundation of the convolution structure in all the cases above was the linearization of the product of the Jacobi functions and the Laguerre functions. Such a product formula does not exist in the case of the Hermit functions (see also [6]) $H_z(t)$ of arbitrary order z [15]. Debnath had proved in [5] a product formula for Hermite polynomials of odd order and in [12] the author has shown, how to extend this formula for Hermite polynomials of arbitrary order by means of the theory of generalized integral transformations (of distributions) in the sense of Zemanian [16].

The aim of this paper is the construction of a product formula for the Hermite functions $H_z(t)$ under restricting conditions on z namely $\mathrm{Re}(z)<0$ in the so-called kernel-form, to define a positive translation operator and a positive convolution with the help of this product formula and to prove a convolution theorem for the assigned integral transformation. In Section 1 we introduce the Hermite functions of the first kind, define a generalized Hermite transformation of a certain kernel function. Section 2 deals with properties of the kernel function. In Section 3 a translation operator is introduced and investigated, in Section 4 a convolution defined and a tor convolution theorem proved. On account of the condition $\mathrm{Re}(z)<0$ these results cannot be spezialized to the case of Hermite polynomials in correspondence with the results of Debnath [5].

Notations. In this paper we design with ε a positive number, η a real number with $0<\eta<1$, C the field of complex numbers, N the set of natural numbers $N=\{1, 2, 3, \ldots\}$ and $N_0=N\cup\{0\}$.

1. We consider a modification of the Hermite functions $H_z(t)$ of the first kind (see [15], § 29, p. 218), namely $\widetilde{H}_z(t)=t^{-1}H_z(t)$, that is

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(1.1) $$\widetilde{H}_z(t)=t^{-1}2^zG(-z/2,\ 1/2,\ t^2)\quad (t>0,\ z\in C)$$

(1.1′) $$=t^{-1}2^{z/2}e^{t^2/2}D_z(\sqrt{2}\,t),$$

where G is the confluent hypergeometric function of the first kind, sometimes designated by ψ and D_z is the function of the parabolic cylinder (see [7], 8.2). With ([15], § 29, (11), (5)) we have

(1.2) $$\widetilde{H}_z(t)=\frac{1}{2\Gamma(-z)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\Gamma\left(\frac{k-z}{2}\right)t^{k-1},\ 0<t<\infty,\quad z\in C\setminus N_0,$$

and if $z=n\in N_0$

(1.2′) $$\widetilde{H}_n(t)=t^{-1}H_n(t)=t^{-1}(-1)^ne^{t^2}\frac{d^n}{dt^n}(e^{-t^2}),$$

where H_n are the Hermite polynomials, and if $t\to+\infty$

(1.3) $$\widetilde{H}_z(t)=2^zt^{z-1}[1+O(t^{-2})],\qquad t\to+\infty.$$

Let

(1.4) $$E^\beta=\{f:f\in C[0,\infty),\ f(t)=O(t^\varepsilon)\ \text{if}\ t\to+0,\ f(t)=O(t^{-\beta}e^{t^2}),\ \text{if}\ t\to+\infty\}$$

and

(1.4′) $${}_\beta A=\{F:F\ \text{holomorphic in}\ \mathrm{Re}(z)<\beta\}.$$

Then we define as generalized Hermite transformation of $f\in E^\beta$

(1.5) $$F(z)=\mathfrak{T}[f](z)=\int_0^\infty f(t)e^{-t^2}\widetilde{H}_z(t)dt.$$

With (1.2)–(1.4) we get immediately

Theorem 1.1. *If $f\in E^\beta$ then $F=\mathfrak{T}[f]\in{}^\beta A$.*

The operational properties of this transformation were investigated in [14] Now we are going to the product formula:

Theorem 1.2. *Let $x, y>0$, $\mathrm{Re}(z)<0$. Then*

(1.6) $$\widetilde{H}_z(x)\widetilde{H}_z(y)=\int_0^\infty e^{-\tau^2}\widetilde{H}_z(\tau)K_z(x,y,\tau)d\tau=\mathfrak{T}[K_z(x,y,\cdot)](z),$$

with

(1.7) $$K_z(x,y,\tau)=\frac{(xy)^{z-1}}{\sqrt{\pi}\Gamma(-z)}\tau^{-z}\int_0^\infty\exp\left\{-t\left[1+\frac{(x^2+y^2+t)\tau^2}{x^2y^2}\right]\right\}t^{-z-1/2}dt.$$

Proof. From ([7] 6.15, (21), p. 275) we have

$$\Gamma(2a-c+1)G(a,c,x)G(a,c,y)=$$

$$\int_0^\infty e^{-t}t^{2a-c}[(x+t)(y+t)]^{-a}F(a,a;\ 2a-c+1;\ \theta_0)dt$$

with $x, y>0$, $\mathrm{Re}\,(2a-c+1)<0$ and $\theta_0=t(x+y+t)/(x+t)(y+t)$.

Let $a=-z/2$, $c=1/2$ and substituting x, y by x^2, y^2 respectively we have under the condition $\mathrm{Re}(z)<1/2$

$$(*)\qquad \widetilde{H}_z(x)\widetilde{H}_z(y)$$

$$=\frac{2^{2z}(xy)^{-1}}{\Gamma(1/2-z)}\int_0^\infty e^{-t}t^{-z-1/2}[(x^2+t)(y^2+t)]^{z/2}.F(-z/2,\ -z/2\,;\ 1/2-z\,;\ \theta)dt$$

and $\theta=t(x^2+y^2+t)/(x^2+t)(y^2+t)$.

From ([7], 8.3, (11), p. 127) we get

$$\int_0^\infty e^{-zt}t^{-1+\beta/2}D_{-\nu}(2\sqrt{kt})dt$$

$$=\frac{2^{1-\beta-\nu/2}\sqrt{\pi}\Gamma(\beta)}{\Gamma((\nu+\beta+1)/2)}(z+k)^{-\beta/2}F(\frac{\nu}{2},\ \frac{\beta}{2}\,;\ \frac{\nu+\beta+1}{2}\,;\ \frac{z-k}{z+k})$$

with $\mathrm{Re}(\beta)$, $\mathrm{Re}\,(z/k)>0$.

Let $\nu=\beta=-z$, $k=1/2$, $\frac{z-k}{z+k}=\theta$, (substituting $t=\tau^2$) and using (1.1) we get after a short calculation

$$\frac{2^{2z}}{\Gamma(1/2-z)}F(-\frac{z}{2},\ -\frac{z}{2}\,;\ \frac{1}{2}-z,\ \theta)$$

$$=\frac{[1+t(x^2+y^2+t)/x^2y^2]^{-z/2}}{\sqrt{\pi}\Gamma(-z)}\int_0^\infty \exp\{-[1+\frac{t(x^2+y^2+t)}{x^2y^2}]\}\tau^2\tau^{-z}\widetilde{H}_z(\tau)d\tau$$

under the condition $\mathrm{Re}\,(z)<0$. Putting this in (*) we have after changing the order of integration (which is allowed under the conditions above)

$$\widetilde{H}_z(x)\widetilde{H}_z(y)=\frac{(xy)^{z-1}}{\sqrt{\pi}\Gamma(-z)}\int_0^\infty e^{-\tau^2}\tau^{-z}\widetilde{H}_z(\tau)\int_0^\infty \exp\{-t[1+\frac{(x^2+y^2+t)\tau^2}{x^2y^2}]\}t^{-z-1/2}dtd\tau$$

and this is (1.6), (1.7).

Remark. It must be pointed to the circumstance, that the kernel function K_z of this linearization of $\widetilde{H}_z(x)\widetilde{H}_z(y)$ is depending on the 'order' z of the Hermite functions, such that the Hermite transform $\mathfrak{T}[K_z(x,y,\cdot)](z)$ is depending on z in a double manner: on the one hand from, the variable of the transform and on the other hand, from the parameter in the original function $K_z(x,y,\cdot)$ of the image.

2. Now we are going to another representation of the kernel function. The formula (1.7) can be written as Laplace transform L of a special function with respect to the variable t, namely

$$(2.1)\qquad K_z(x,y,\tau)=\frac{(xy)^{z-1}}{\sqrt{\pi}\Gamma(-z)}\tau^{-z}L[t^{-z-1/2}e^{-\tau^2t^2/x^2y^2}](1+\frac{x^2+y^2}{x^2y^2}\tau^2).$$

It is well known (see [8], 4.5, (24), p. 135) that

$$L[t^{\nu-1}e^{-t^2/8\alpha}](p)=\Gamma(\nu)2^{\nu}\alpha^{\nu/2}e^{\alpha p^2}D_{-\nu}(2\sqrt{\alpha}p),\quad \mathrm{Re}\,(\alpha),\ \mathrm{Re}(\nu)>0)$$

$$=\Gamma(\nu)2^{(3\nu+1)/2}\alpha^{(\nu+1)/2}p\,\widetilde{H}_{-\nu}(\sqrt{2\alpha}p),$$

in account of (1.1′). Putting $\nu=1/2-z$, $\alpha=x^2y^2/8\tau^2$, $v=1+\frac{x^2+y^2}{x^2y^2}\tau^2$ we get after a short calculation the

Theorem 2.1. *Under the conditions* $x, y, \tau>0$, $\operatorname{Re}(z)<0$ *we have*

$$(2.2)\qquad K_z(x, y, \tau)=\frac{\Gamma(1/2-z)[x^2y^2+(x^2+y^2)\tau^2]}{2\sqrt{\pi}(xy\tau)^{3/2}\Gamma(-z)}\tilde{H}_{z-1/2}\left(\frac{x^2y^2+(x^2+y^2)\tau^2}{2xy\tau}\right).$$

By means of (2.1) and (2.2) we get immediately the following qualities of the kernel, the symmetric relatively to the variables x, y, τ and the positivity:

Conclusion 2.1.

$$(2.3)\qquad K_z(x, y, \tau)=K_z(x, \tau, y)=K_z(\tau, y, x),$$

Conclusion 2.2. *If* $x, y, \tau>0$, $z>0$, *then*

$$(2.4)\qquad K_z(x, y, \tau)>0.$$

The behaviour of $K_z(x, y, \tau)$ if τ tends to $+\infty$ is easily derived from (2.2) and (1.3). We get

Conclusion 2.3. *If* $x, y>0$, $\operatorname{Re}(z)<0$, *then*

$$(2.5)\qquad K_z(x, y, \tau)=O(\tau^{z-1}),\quad \tau\to+\infty.$$

Owing to the fact, that the argument of $\tilde{H}_{z-1/2}$ in (2.2) tends to infinity like τ^{-1} if τ tends to $+0$ we have with (1.3):

Conclusion 2.4. *If* $x, y>0$, $\operatorname{Re}(z)<0$, *then*

$$(2.6)\qquad K_z(x, y, \tau)=O(\tau^{-z}),\quad \tau\to+0.$$

3. At first we are defining a generalized translation operator in the following way:

Definition 3.1. *If* $x, t>0$, $\operatorname{Re}(z)<0$, $f\in E^0$ *we define as generalized translation operator of* f *the operator* T_x *with*

$$(3.1)\qquad T_x f(t)=\int_0^\infty f(\tau)e^{-\tau^2}K_z(x, t, \tau)d\tau.$$

Obviously this operator is well defined under the conditions above owing to (1.4), (2.5) and (2.6). From (2.3) we get

Conclusion 1.

$$(3.2)\qquad T_x f(t)=T_t f(x).$$

Another formulation of Theorem 1.2 is

Conclusion 2.

$$(3.3)\qquad T_x\tilde{H}_z(t)=\tilde{H}_z(x)\tilde{H}_z(t).$$

The answer to the question how to find the Hermite transformation of $T_x f$ is the following:

Theorem 3.1. *If* $f\in E^0$ *and* $x>0$, $\operatorname{Re}(z)<0$, *then* $T_x f\in E^\beta$ *with arbitrary* β *and*

$$(3.4)\qquad \mathfrak{T}[T_x f(\cdot)](z)=\tilde{H}_z(x)F(z),\quad \operatorname{Re}(z)<0.$$

Proof. According to (2.6) and the symmetrie of K_z, (2.3), we have immediately

$$T_x f(t) = O(t^{-z}), \quad t \to +0. \tag{3.5}$$

Similar to (2.5) instead of (2.6) we see that

$$T_x f(t) = O(t^{z-1}), \quad t \to +\infty. \tag{3.6}$$

So $T_x f$ has the behaviour $O(t^\varepsilon)$ if t tends to $0+$ if and only if $\mathrm{Re}(z)<0$ and we have no restriction on the behaviour if t tends to plus infinity.

So we have with (1.6)

$$\widetilde{H}_z(x)F(z) = \int_0^\infty e^{-t^2}\widetilde{H}_z(x)\widetilde{H}_z(t)f(t)dt$$

$$= \int_0^\infty e^{-t^2}f(t)[\int_0^\infty e^{-\tau^2}\widetilde{H}_z(\tau)K_z(x,t,\tau)d\tau]dt = \int_0^\infty e^{-\tau^2}\widetilde{H}_z(\tau)[\int_0^\infty e^{-t^2}f(t)K_z(x,t,\tau)dt]d\tau$$

$$= \int_0^\infty e^{-\tau^2}\widetilde{H}_z(\tau)[\int_0^\infty e^{-t^2}f(t)K_z(x,\tau,t)dt]d\tau = \int_0^\infty e^{-\tau^2}\widetilde{H}_z(\tau)T_x f(\tau)d\tau = \mathfrak{T}[T_x f](z),$$

if we use (2.3) and note that the changing of the order of integration is allowed.

4. By means of the generalized translation operator T_x we are as usual defining the convolution of two functions of E^0 as follows:

Definition 4.1. *If $f, g \in E^0$, $\mathrm{Re}(z)<0$, then we define*

$$(f*g)(t) = \int_0^\infty e^{-\tau^2}T_t f(\tau)g(\tau)d\tau \tag{4.1}$$

as the convolution of f and g.

We have

$$(g*f)(t) = \int_0^\infty e^{-\tau^2}T_t g(\tau)f(\tau)d\tau = \int_0^\infty e^{-\tau^2}T_\tau g(t)f(\tau)d\tau$$

$$= \int_0^\infty e^{-\tau^2}f(\tau)\int_0^\infty g(u)e^{-u^2}K_z(\tau,t,u)dud\tau$$

$$= \int_0^\infty e^{-u^2}g(u)\int_0^\infty f(\tau)e^{-\tau^2}K_z(t,u,\tau)d\tau du = \int_0^\infty e^{-u^2}T_t f(u)g(u)du = (f*g)(t),$$

where the symmetry (2.3) of K_z and (3.2) are used and the interchanging of the order of integration is allowed under our assumptions. So we have proved the

Conclusion 4.1. $f*g = g*f$.

By means of the definition of the convolution we can prove a convolution theorem:

Theorem 4.1. *If $f, g \in E^0$ and $\mathrm{Re}(z)<0$, then $f*g \in E^\beta$, with arbitrary β and $\mathfrak{T}[f*g](z) = F(z)G(z)$.*

Proof. From (4.1) we have owing to Theorem 3.1 at once that $f*g\in E^{\beta}$ with arbitrary β. Moreover $(f*g)(t)$ has the same behaviour ((3.5), (3.6)) as $(T_x f)(t)$ when t tends to $+0$ or $+\infty$. By means of (3.2) and (3.4), we get

$$\mathfrak{T}[f*g](z)=\int_0^{\infty} e^{-t^2}\widetilde{H}_z(t)[\int_0^{\infty} e^{-\tau^2}T_t f(\tau)\,g(\tau)d\tau]dt$$

$$=\int_0^{\infty} e^{-\tau^2}g(\tau)[\int_0^{\infty} e^{-t^2}\widetilde{H}_z(t)T_\tau f(t)dt]d\tau$$

$$=\int_0^{\infty} e^{-\tau^2}g(\tau)\mathfrak{T}[T_\tau f](z)d\tau=F(z)\int_0^{\infty} e^{-\tau^2}\widetilde{H}_z(\tau)g(\tau)d\tau=F(z)G(z).$$

The interchanging of the order of integration is allowed under the assumptions.

5. This paper contains some investigations towards a convolution structure for a generalized Hermite transformation. The kernel function $\widetilde{H}_z$ is a modification of the Hermite function H_z, which agrees with the Hermite polynomials if $z=n\in N_0$. Our convolution solves the problem only partially, because the convolution has two disadvantages:

At first, it is defined only for $\mathrm{Re}(z)<0$, so that a specialization to the case of Hermite polynomials is impossible. Secondly, the generalized translation operator and so also the convolution of two functions is depending on the complex variable z as an additional parameter, the same variable from which the Hermite transform is depending on.

That means, that it is impossible to use the inversion formula (see [14]) of the Hermite transformation because the variable z in $F(z)G(z)$ is derived partly from the variable z of the function $\widetilde{H}_z(t)$ in the transformation (1.5), partly from the parameter in the definition of $f*g$, that is one "index" z of K_z. So the formulas can be used only in one direction and our results are only a first step in the solution of the problem.

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