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APPLICATIONS OF FIXED POINT THEOREMS TO BEST APPROXIMATIONS

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Let X be a real normed linear space and K any subset of X. An element g_0 in K is called a best K-approximant for an arbitrary element x in X, if $\| x-g_0 \| \leqq \| x-g \|$ all g in K. Let D be the set of all best K-approximants to x. If T is an operator on X with a fixed point x, then by imposing some conditions on T or the set K, it is possible to find another fixed point in the set D. Brosowski had obtained a result of this kind for a contractive linear operator T and a compact, convex subset K. In this paper, the linearity condition on T and the convexity of K are weakened to give rise to some generalizations.

1. Let X be a real normed linear space, K a subset of X, and x an element of X, not in the closure of K. The set of best K-approximants to x consists of those $g_0 \in K$ satisfying $\| x-g_0 \| = \inf\{\| x-g \| : g \in K\}$ and it is denoted as $P_K(x)$. Let T be a self-map on X. T is called a contraction, if $\| Ty - Tz \| \leqq \alpha \| y-z \|$ for $0 \leqq \alpha < 1$, $y, z \in X$. Banach's contraction principle states that in a complete metric space a contraction map has a unique fixed point. T is called contractive whenever $\| Ty-Tz \| < \| y-z \|$ for y, z in X.

A subset S of X is called star-shaped if there exists a point p called star-centre in S such that $\lambda p+(1-\lambda)z \in S$, for all z in S and $0 \leqq \lambda \leqq 1$. It is clear that every convex subset is star-shaped, but a star-shaped set need not be convex. A more general class of sets containing the star-shaped sets is called 'contractive'. A set S in X is contractive if there exists a sequence $\{f_n\}$ of contraction mappings of S into itself such that $f_n y \to y$, for each y in S. Brosowski [1] proved the following

Theorem A. *Let T be a contractive linear operator on a normed linear space X. Let K be a T-invariant subset X and x a T-invariant point. If the set of best K-approximants to x is nonempty, convex and compact, then it contains a T-invariant point.*

Following the method of Brosowski, Singh [6] has obtained a generalization of Theorem A.

Theorem B. *Let T be a contractive operator on a normed linear space X. Let C be a T-invariant subset of X and x a T-invariant point. If the set of best C-approximants to x is nonempty, compact and star-shaped, then it contains a T-invariant point.*

In this paper, some generalizations of Theorem A and Theorem B are obtained.

2. Let T be a self-map on X such that for all y, z in X,

$$(1)\quad \| Ty-Tz \| \leqq \alpha \| y-z \| + \beta\{\| y-Ty \| + \| z-Tz \|\} + \gamma\{\| y-Tz \| + \| z-Ty \|\},$$

where α, β, γ are non-negative numbers satisfying $\alpha+2\beta+2\gamma \leqq 1$.

Let M be a subset of X. A sequence $\{y_n\}$ in M is said to be minimizing for x, if $\lim \| y_n - x \| = d(x, M)$, where $d(x, M)$ is the distance of x from the set M. M is called approximatively compact, if for every x in X, each minimizing sequence $\{y_n\}$ in M has a convergent subsequence, converging to an element of M. It is well-known that M is approximatively compact implies that the set of best M-approximants to x, namely $P_M(x)$, is compact.

The following result concerns the fixed point of a map T satisfying the condition (1), instead of being a contractive (linear) map.

Theorem 1. *Let T be a continuous self-map on a Banach space X satisfying* (1). *Let C be an approximatively compact and T-invariant subset of X. Let $Tx = x$ for some x, not in the norm-closure of C. If the set of best C-approximants to x is nonempty and star-shaped, then it has a T-invariant point.*

Proof. Let D be the set of best C-approximants to x. Then

$$D = \{z \in C : \| z - x \| \leq \| y - x \| \text{ for all } y \text{ in } C\}. \tag{2}$$

Let $z \in D$. Then, by (2.1) and the hypothesis it is clear that

$$\| x - Tz \| = \| Tx - Tz \| \leq \alpha \| x - z \| + \beta \| z - Tz \| + \gamma \{ \| x - Tz \| + \| z - Tx \| \},$$

where $\alpha + 2\beta + 2\gamma \leq 1$.

That is, $\| x - Tz \| \leq (\alpha + \gamma) \| x - z \| + \beta \| z - Tz \| + \gamma \| x - Tz \| \leq (\alpha + \gamma) \| x - z \| + \beta \| z - x \| + (\beta + \gamma) \| x - Tz \|$, or $(1 - \beta - \gamma) \| x - Tz \| \leq (\alpha + \beta + \gamma) \| x - z \|$.

That is,

$$\| x - Tz \| \leq \| x - z \|, \text{ since } \alpha + 2\beta + 2\gamma \leq 1 \leq \| x - y \| \text{ for all } y \text{ in } C \text{ by (2)}$$

This means that $Tz \in D$. Therefore, T is a self-map on D.

Since D is nonempty and star-shaped, there exists a star-centre p in D such that $\lambda p + (1 - \lambda) z \in D$, for all z in D, $0 \leq \lambda \leq 1$. Now taking a sequence k_n of non-negative real numbers $(0 \leq k_n < 1)$ converging to 1, one can define $T_n : D \to D$, for $n = 1, 2, \ldots$, as follows: $T_n z = k_n Tz + (1 - k_n) p$, $z \in D$. Since T is a self-map on D, so is T_n, for each n. Also, for all y, z in D,

$$\| T_n y - T_n z \| = k_n \| Ty - Tz \|$$

$$\leq k_n \alpha \| y - z \| + k_n \beta \{ \| y - Ty \| + \| z - Tz \| \} + k_n \gamma \{ \| y - Tz \| + \| z - Ty \| \},$$

where $\alpha k_n + 2k_n \beta + 2k_n \gamma < 1$.

Therefore, by a theorem of Hardy and Rogers [3], T_n has a unique fixed point in D, for each n. Let $T_n z_n = z_n$.

Now the approximative compactness of C implies that D is compact. Therefore, there is a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \to z_0$ in D. Again,

$$z_{n_i} = T_{n_i} z_{n_i} = k_{n_i} T z_{n_i} + (1 - k_{n_i}) p.$$

Considering the assumption that T is continuous and the fact that $k_{n_i} \to 1$ as $i \to \infty$, it follows that $z_0 = Tz_0$. Thus z_0 is a T-invariant point in D. This completes the proof.

Remark. For the case when $\beta = \gamma = 0$, the map T in Theorem 1, becomes nonexpansive and hence contractive. In this case the result reduces to Theorem B. If T is linear, it reduces to Theorem A of Brosowski stated in the Introduction.

Kannan [4] studied the fixed points of the map T satisfying $\|Ty-Tz\| \leq \{\|y-Ty\|+\|z-Tz\|\}/2$. [This is the case when $\alpha=\beta=0$ in (1)]. If $\alpha=\beta=0$,(1) becomes $\|Ty-Tz\| \leq \{\|y-Tz\|+\|z-Ty\|\}/2$. Such maps were analysed by Yadav [7]. Therefore the conclusion of Theorem 1 not only generalizes Brosowski's result, but also extends it to the maps investigated by Kannan and Yadav.

The following results are given in the context of a metric space. These are almost direct consequences of Lemma 1 of E. Chandler and G. Faulkner [2], who exploited the properties of a contractive set.

Theorem 2. *Let E be a metric space with metric d. Let C be an approximatively compact subset of E. Let T be a nonexpansive self-map on C and $Tx=x$. If the set of best C-approximants to x is nonempty and contractive, then it contains another fixed point of E.*

Proof. Let $D=\{y\in C: d(x, y)\leq d(x, z)$ for all z in $C\}$. Since C is approximatively compact, D is nonempty. Let $y\in D$. Then $d(x, Ty)=d(Tx, Ty) \leq d(x, y)\leq d(x, z)$ for all z in C, so that $Ty\in D$. Therefore, $T(D)\subset D$. Since D is contractive, there exists a sequence $\{f_n\}$ of contractions on D such that $f_n z \to z$, for every z in D.

Clearly $f_n T$ is a contraction on the compact set D. Thus D is a complete metric space and Banach's contraction principle ensures the existence of a unique fixed point, say z_n, of $f_n T$, for each n. Now $\{z_n\}$ in D has a convergent subsequence $\{z_{n_i}\}$ such that $z_{n_i}\to z_0$ in D. The following argument proves that z_0 is a fixed point of T.

Let $\varepsilon>0$ be given. Then there exists a positive integer m such that

$$d(z_m, z_0)\leq \varepsilon/2 \text{ and } d(f_n T z_0, Tz_0)<\varepsilon/2.$$

Again

$$d(f_m T z_m, f_m T z_0)\leq d(z_m, z_0)<\varepsilon/2.$$

Hence

$d(f_m T z_m, Tz_0)\leq d(f_m T z_m, f_m T z_0)+d(f_m T z_0, Tz_0)<\varepsilon/2+\varepsilon/2=\varepsilon$. Thus there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $f_{n_i} T z_{n_i}\to Tz_0$. But $f_{n_i} T z_{n_i}=z_{n_i}\to z_0$ and therefore $Tz_0=z_0$.

Remark. This theorem generalizes Theorem A as well as Theorem B to contractive compact sets. Every star-shaped set is contractive but there exist contractive sets which are not star-shaped.

Theorem 3. *Let E be a complete metric space. Let T be a continuous self-map on an approximatively compact subset C of E such that for all y, z in C,*

$$d(Ty, Tz)\leq \alpha d(y, z)+\beta\{(d(y, Ty)+d(z, Tz)\}+\gamma\{d(y, Tz)+d(z, Ty)\},$$

where α, β, γ are non-negative real numbers satisfying $\alpha+2\beta+2\gamma\leq 1$. Let x be not in C such that $Tx=x$. If the set of best C-approximants to x is nonempty and contractive, then it contains another fixed point of T.

Proof. Let D be the set of best C-approximants to x. The map T is the same as in Theorem 1. Clearly T is a self-map on D. Since C is approximatively compact, D is compact. Since D is a contractive set, there exists a sequence f_n of contraction maps such that $f_n(T(D))\subset D$. For all y, z in $T(D)$, since f_n is a contraction for each n,

$$d(f_nTy,\ f_nTz)\leqq a_nd(Ty,\ Tz),\ 0\leqq a_n<1$$

$$\leqq a_n\alpha d(y,\ z)+a_n\beta\{d(y,\ Ty)+d(z,\ Tz)\}+a_n\gamma\{d(y,\ Tz)+d(z,\ Ty)\},$$

where $a_n\alpha+2a_n\beta+2a_n\gamma<1$.

Now, if T has a fixed point, say z', then f_nT also has z' as its fixed point. For,

$$d(f_nTz',\ z')\leqq d(f_nTz',\ Tz')+d(Tz',\ z')=d(f_nTz',\ Tz')=d(f_nTz',\ z)$$

and $d(f_nTz',\ z')\to 0$, since for every z in C, $f_nz\to z$.

Now by Hardy and Roger's theorem, the map T has a unique fixed point in D. Therefore, for every n, f_nT has a unique fixed point, say z_n. Now the sequence $\{z_{n_i}\}$ obtained from $\{z_n\}$ converges to z_0 in D, by the compactness of D. Proceeding as in Theorem 2, one easily proves that $Tz_0=z_0$.

Definition. *For each bounded subset D of a metric space E, the measure of non-compactness of* A, $\alpha[A]$ *is defined as*

$$\alpha[A]=\inf\{\varepsilon>0:\ A \text{ is covered by a finite number of closed balls centred at points of } X \text{ of radius } \leqq\varepsilon\}.$$

Definition. *A mapping $T: D\to D$ is called condensing if for bounded sets $D\subset E$ with $\alpha[D]>0$, $\alpha[T(D)]<\alpha[D]$, where $\alpha[D]$ is the measure of non-compactness of D.*

Theorem 4. *Let E be a complete, contractive metric space with contractions f_n. Let C be a closed bounded subset of E. If T is a nonexpansive and condensing self-map on E such that $Tx=x$, for some $x\in E$ and the set of best C-approximants to x is nonempty, then it has a T-invariant point.*

Proof. Let D be the set of best C-approximants of x. Then D is a closed and bounded subset of C and $T(D)\subset D$. A direct application of Theorem 1 of Chandler and Faulkner [2], will now give a T-invariant point in D.

Theorem 5. *Let E be a complete metric space, M an approximatively compact subset of E and $x\in E/M$. Let T be a self-map on X with $Tx=x$ and for some positive integer m, let T^m satisfy the condition*

$$d(T^my,\ T^mz)\leqq\alpha\{d(y,\ T^my)+d(z,\ T^mz)\},\quad 0<\alpha<1/2,\ y,\ z \text{ in } M.$$

If the set of best M-approximants to x is nonempty, then it has a unique fixed point of T.

Proof. Let $D=D_M(x)=\{y_0\in M:\ d(x,\ y_0)\leqq d(x,\ y)$ for all y in $M\}$. Now, $Tx=x$ implies that $T^mx=x$ for the same integer m prescribed in the hypothesis. Let $y_0\in D$. Then, for $0<\alpha<1/2$,

$$d(x,\ T^my_0)=d(T^mx,\ T^my_0)\leqq\alpha\{d(x,\ T^mx)+d(y_0T^my_0)\}=\alpha d(y_0,\ T^my_0)$$

$$\leqq\alpha d(y_0,\ x)+\alpha d(x,\ T^my_0).$$

That is, $d(x,\ T^my_0)\leqq\frac{\alpha}{1-\alpha}d(y_0,\ x)\leqq\frac{\alpha}{1-\alpha}d(x.\ y)$, for all y in M. Therefore $T^my_0\in D$ which implies that $T^m(D)\subset D$. Since T^m satisfies the conditions of Kannan map, T^m has a unique fixed point in D. This means that there is an x_0 in D such that $T^mx_0=x_0$. Now, $T^m(Tx_0)=T(T^mx_0)=Tx_0$ implies that Tx_0

is a fixed point of T^m. But the fixed point of T^m is unique and equals x_0. Therefore $Tx_0=x_0$ and hence x_0 is a unique fixed point of T in D.

Remark. This theorem extends Brosowski's result to a generalized form of Kannan map. It is interesting to note that this theorem gives a unique fixed point in the set D of best M-approximants to x.

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