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ON EQUIVALENT LATTICE NORMS WHICH ARE UNIFORMLY CONVEX OR UNIFORMLY DIFFERENTIABLE IN EVERY DIRECTION IN BANACH LATTICES WITH A WEAK UNIT

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An equivalent lattice norm which is uniformly convex in every direction is introduced in $L_1(S, \Sigma, \mu)$. As an application several results concerning the existence of eguivalent norms which are uniformly convex (resp. uniformly differentiable) in every direction in Banach lattices are obtained.

1. In [3] it is shown that a Banach space X, which is separable or conjugate to a separable space, admits an equivalent norm, uniformly convex in every direction. From a result of Shmulyan [8] and the construction of the norm in [3], it follows that each separable space has an equivalent norm, uniformly differentiable in every direction,

The paper [7] contains necessary and sufficient conditions for existence of an equivalent norm which is uniformly convex (resp. uniformly differentiable) in every direction. Later on an example is given in [5] of a reflexive Banach space with an (uncountable) unconditional basis which fails to have either an equivalent norm that is uniformly convex in every direction or an equivalent norm that is uniformly differentiable in every direction.

In the present paper we obtain several results concerning the existence of equivalent norms which are uniformly convex (resp. uniformly differentiable) in every direction in Banach lattices.

2. Definitions and notations. The norm of a Banach space is said to be uniformly convex in every direction if the conditions $x_n, y_n, z \in X$, $\|x_n\| \to 1$, $\|y_n\| \to 1$, $\|x_n+y_n\| \to 2$ and $x_n-y_n=\lambda_n z$ imply $\|x_n-y_n\| \to 0$.

The norm of a conjugate Banach space X^* is said to be w*-uniformly convex if the conditions $x_n^*, y_n^* \in X^*$, $\|x_n^*\| \to 1$, $\|y_n^*\| \to 1$, and $\|x_n^*+y_n^*\| \to 2$ imply that $x_n^*(x)-y_n^*(x) \to 0$ for each $x \in X$.

The norm of a Banach space X is Gateaux differentiable if for any $x, y \in X$ with $\|x\|=\|y\|=1$, $\lim_{\tau\to 0} \tau^{-1} (\|x+\tau y\|+\|x-\tau y\|-2)=0$.

The norm of a Banach space X is uniformly differentiable in every direction if for any $x, y \in X$ with $\|y\|=1$, $\lim_{\tau\to 0} \tau^{-1}. \sup_{\|x\|=1} (\|x+\tau y\|+\|x-\tau y\|-2)=0$.

A partially ordered Banach space X over the reals is a Banach lattice provided

(i) $x \leq y$ implies $x+z \leq y+z$, for every $x, y, z \in X$.

(ii) $ax \geq 0$, for every $x \geq 0$ in X and every non-negative real a.

(iii) for all $x, y \in X$ there exists a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$.

(iv) $\| x \| \leq \| y \|$ whenever $|x| \leq |y|$, where the absolute value $|x|$ of $x \in X$ is defined by $|x| = x \vee (-x)$.

A Banach lattice X is σ-order complete if every order bounded sequence in X has a least upper bound.

A Banach lattice X has an order (σ-order) continuous norm if for every downward directed set (sequence) $\{x_\alpha\}_{\alpha \in A}$ in X with $\bigwedge_{\alpha \in A} x_\alpha = 0$, $\lim_\alpha \| x_\alpha \| = 0$.

An element $e \geq 0$ of a Banach lattice X is a weak unit of X if $x \in X$, $e \wedge |x| = 0$ imply $x = 0$.

Let (S, Σ, μ) be a measurable space. A Banach space X consisting of equivalence classes of μ-measurable real valued functions on S is a Köthe function space if X is a Banach lattice in the obvious order ($f \geq 0$ if $f(s) \geq 0$ almost everywhere) and the following conditions hold:

(i) If $|f(s)| \leq |g(s)|$ almost everywhere (a.e.) on S with f measurable and $g \in X$, then $f \in X$.

(ii) If $f \in X$ then f is locally integrable, i. e. for every $A \in \Sigma$ with $\mu(A) < \infty$ there exists $\int_A f(s)\, d\mu$.

(iii) For every $A \in \Sigma$ with $\mu(A) < \infty$ the characteristic function X_A of A belongs to X.

3. Main results. Theorem 3.1. *For every measuable-space (S, Σ, μ) the space $L_1(S, \Sigma, \mu)$ admits an equivalent lattice norm that is uniformly convex in every direction.*

Corollary 3.2. *Let X be a Banach lattice and there exists an element $e^* \in X$ with $e^* \geq 0$ so that $e^*(|x|) = 0$ for $x \in X$ implies $x = 0$. Then X has an equivalent lattice norm which is uniformly convex in every direction.*

Proof. Let $\| x \|_1 = e^*(|x|)$ and $\tilde{X}$ be the completion of X in the norm $\| \cdot \|_1$. Since $\| \cdot \|_1$ is additive on the positive cone, there exists a measurable space (S, Σ, μ) and an operator T such that T is an order isometry from $\tilde{X}$ onto $L_1(S, \Sigma, \mu)$ (cf. e.g, [6. p. 15]). By Theorem 3.1 there exists in $L_1(S, \Sigma, \mu)$ an order equivalent norm $\| \cdot \|_2$, uniformly convex in every direction. Put $||| x ||| = (\| x \|^2 + \| Tx \|_2^2)^{1/2}$. We shall show that $||| \cdot |||$ is uniformly convex in every direction. Let $||| x_n ||| \to 1$, $||| y_n ||| \to 1$, $||| x_n + y_n ||| \to 2$, $x_n, y_n, z \in X$ and $x_n - y_n = \lambda_n z$. It is no loss of generality to consider $\| Tx_n \|_2 \to a$. By the uniform convexity of the space l_2 and the triangle inequality, we get that $\| Ty_n \|_2 \to a$ and $\| T(x_n + y_n) \|_2 \to 2a$. Hence, we have that $e^*(|\lambda_n z|) \to 0$. Then, $\| \lambda_n z \| \to 0$, which completes the proof.

Corollary 3.3 *Let X be a Banach lattice with a weak unit. Then, if the norm is order continuous, X admits an equivalent lattice norm, uniformly convex in every direction.*

In order to prove this, it suffices to observe that the assumptions of Corollary 3.3 imply the existence of an element $e^* \in X^*$ satisfying the assumption of Corollary 3.2 (cf. e.g. [6. p. 25]).

Corollary 3.4. *Let (S, Σ, μ) be a space with a σ-finite measure. Then, if X is a Köthe function space on (S, Σ, μ), it has an equivalent lattice norm, uniformly convex in every direction.*

In order to prove this, it suffices to observe that there exists an element $e^* \in X^*$ satisfying the assumption of Corollary 3.2.

T h e o r e m 3.5. *Let (S, Σ, μ) be a probability space and X be a Köthe function space on (S, Σ, μ) with σ-order continuous norm. Then X has an equivalent lattice norm, uniformly differentiable in every direction.*

C o r o l l a r y 3.6. *Let X be a σ-order complete Banach lattice with a weak unit. Then the following conditions are equivalent:*

(i) *X has an equivalent Gateaux differentiable norm.*

(ii) *X has an equivalent lattice norm that is uniformly differentiable in every direction.*

(iii) *X has σ-order continuous norm.*

(iv) *X does not contain a subspace isomorphic to l_∞.*

P r o o f. Let (iii) holds. Then, there exists a probability space (S, Σ, μ) and a Köthe function space $\tilde{X}$ on (S, Σ, μ) that is order isometric to X (cf. e.g. [6. p. 25]). Hence, by Theorem 3.5, (ii) holds. The implication (ii) $\Rightarrow$ (i) is trivial. Since l_∞ does not admit an equivalent Gateaux differentiable norm [2], (i) implies (iv). By the theorem of Lozanovskii (see e.g. [6. p. 7]), (iv) implies (iii).

C o r o l l a r y 3.7. *Let (S, Σ, μ) be a space with a σ-finite measure and X be a Köthe function space on (S, Σ, μ). Then the following conditions are equivalent:*

(i) *X has an equivalent Gateaux differentiable norm.*

(ii) *X has an equivalent lattice norm that is uniformly differentiable in every direction.*

(iii) *X has σ-order continuous norm.*

(iv) *X does not contain a subspace isomorphic to l_∞.*

This assertion results immediately from Corollary 3.6 because X is σ-order complete lattice (cf. e.g. [6. p. 29]) with a weak unit.

4. Auxiliary results and proof of Theorem 3.1. L e m m a 4.1. *Let X be a Banach space such that the conditions $x_n, y_n, z \in X$, $\|x_n\| \to 1$, $\|y_n\| \to 1$, $\|x_n+y_n\| \to 2$ and $x_n-y_n=z$ imply $z=0$. Then the norm is uniformly convex in every direction.*

L e m m a 4.2. *Every uncountable set H of real numbers has a point, approximable from the left and the right by points of H.*

L e m m a 4. 3. *Let $\{f_n\}_{n=1}^\infty$ be non-increasing left continuous functions on $(0, \infty)$ which are uniformly bounded in every interval (c, ∞), $c>0$. Then there exists a subsequence $\{n\}$ of indices so that for every $t \in (0, \infty)$, $\lim_{n\to\infty} f_n(t)=f(t)$ and the following condition holds:*

(*) *For any finite interval (a, b), $a, b>0$, there exists $\lambda \in (a, b)$ and $\lambda_i \in (a, b)$, $i=1, 2, \ldots$, with $\lambda_1<\lambda_2< \ldots <\lambda_i < \ldots$, $\lim_{i\to\infty} \lambda_i=\lambda$, such that for each $\delta>0$ there is N so that $n>N$ and $i>N$ imply*
$$|f_n(\lambda_i)-f_n(\lambda)|<\delta.$$

P r o o f. Since f_n are non-increasing and uniformly bounded in (c, ∞), $c>0$, then, by a known theorem, there exists a subsequence $f_n(t)$ which is convergent for any $t \in (1/k, \infty)$. By the diagonal procedure, we choose a subsequence $\{n\}$ of indices, so that $\lim_{n\to\infty} f_n(t)=f(t)$ for every $t \in (0, \infty)$.

Let now $0<a<b<\infty$. Since f_n are Lebesgue measurable functions, by the theorem of Egorov, we obtain that $f_n(t) \to f(t)$ almost uniformly in (a, b). Then, there exists a set $H \subset (a, b)$ with positive Lebesgue measure such that $\{f_n\}_{n=1}^\infty$ converges uniformly on H. It follows from Lemma 4.2 that there is $\lambda \in (a, b)$, approximable from the left by points of H. Therefore, there exist $\lambda_i \in (a, b)$, $i=1, 2, \ldots$ with $\lambda_i \uparrow \lambda$ and $\lambda_i \in H$.

Let $\delta>0$. Since $\lambda_i \in H$, there is M such that $|f_n(\lambda)-f(\lambda)|<\delta$ and $|f_n(\lambda_i)-f(\lambda_i)|<\delta$, $i=1,2,\ldots$, whenever $n\geq M$. By the assumption, f_n are left continuous. It follows from $\lambda_i \uparrow \lambda$ that there is $N\geq M$ so that $|f_M(\lambda_i)-f_M(\lambda)|<\delta$ for each $i>N$. Then, for each $i, n>N$ we have

$$|f(\lambda_i)-f(\lambda)|\leq|f(\lambda_i)-f_M(\lambda_i)|+|f_M(\lambda_i)-f_M(\lambda)|+|f_M(\lambda)-f(\lambda)|<3\delta,$$

whence

$$|f_n(\lambda_i)-f_n(\lambda)|\leq|f_n(\lambda_i)-f(\lambda_i)|+|f(\lambda_i)-f(\lambda)|+|f(\lambda)-f_n(\lambda)|<5\delta.$$

L e m m a 4.4. *Let $x_n, y_n \in L_1(\Omega, \Sigma, \mu)$, $\|x_n\|$, $\|y_n\|$, $\|(x_n+y_n)/2\|\to 1$ and the following conditions hold:*
(**) *For each a, b, $0<a<b<\infty$,*

$$\mu(\{|x_n|<a, |y_n|\geq b\})\to 0,\ \mu(\{|x_n|\geq b, |y_n|<a\})\to 0.$$

Then, $\{x_n-y_n\}_{n=1}^{\infty}$ tends to zero in measure.

P r o o f. Fix $\theta>0$. Denote

$$E_n=\{|x_n|-|y_n|>\theta/4,\ |x_n|\geq\theta/2\},\ F_n=\{|y_n|-|x_n|>\theta/4,\ |y_n|\geq\theta/2\}.$$

We shall prove that

(1) $\lim_{n\to\infty}\mu(E_n)=0.$

Let $\varepsilon>0$. Choose K such that $\|x_n\|<K$, $n=1,2,\ldots$ Put $M=K/\varepsilon$. Since $M\mu(\{|x_n|>M\})\leq\int_{\{|x_n|>M\}}|x_n|\,d\mu\leq\|x_n\|$, then

(2) $\mu(\{|x_n|>M\})<\varepsilon$

Consider $E_n\cap\{|x_n|\leq M\}$. Choose $\{j_i\}_{i=0}^{k}$ so that $\theta/2=j_0<j_1<\ldots<j_k=M$ and $j_i-j_{i-1}<\theta/8$, $i=1,2,\ldots,k$. We shall prove that

$$\bigcup_{i=0}^{k}\{|x_n|\geq j_i, |y_n|<j_i-\theta/8\}\supset E_n\cap\{|x_n|\leq M\}. \tag{3}$$

Let s belong to the right hand set. Then, there exists i such that $j_i\leq|x_n(s)|\leq j_{i+1}$, whence $|y_n(s)|<|x_n(s)|-\theta/4\leq j_{i+1}-\theta/4<j_i-\theta/8$. Therefore, (3) holds.

By the assumptions of Lemma 4.4 and (3), we get $\lim_{n\to\infty}\mu(E_n\cap\{|x_n|\leq M\})=0$. Hence, by (2), we obtain (1). Symmetrically, we have $\lim_{n\to\infty}\mu(F_n)=0$.

Let $A_n=\{|x_n-y_n|>\theta\}$, $P_n=\{|x_n|\geq|y_n|, |x_n|>\theta/2\}$, $Q_n=\{|y_n|\geq|x_n|, |y_n|>\theta/2\}$. Obviously, $A_n\subset P_n\cup Q_n$. We shall prove that

$$\lim_{n\to\infty}\mu(P_n\cap A_n)=0. \tag{4}$$

Clearly, $P_n\subset T_n\cup E_n$, where $T_n=\{|x_n|\geq|y_n|, |x_n|>\theta/2, |x_n|-|y_n|\leq\theta/4\}$. Let $T'_n=\{|x_n-y_n|=|x_n|-|y_n|\}\cap T_n$, $T''_n=T_n\setminus T'_n$. Obviously, $T'_n\subset S\setminus A_n$. Let $s\in T''_n$. Then, since

$$|x_n(s)+y_n(s)|=|x_n(s)|-|y_n(s)|\leq\theta/4\leq|x_n(s)|-\theta/4,$$

we get that

$$\|x_n+y_n\|\leq\int_{S\setminus T''_n}(|x_n|+|y_n|)\,d\mu+\int_{T''_n}|x_n+y_n|\,d\mu$$
$$\leq\|x_n\|+\|y_n\|-\mu(T''_n)\,\theta/4.$$

Hence, $\lim_{n\to\infty} \mu(T_n'')=0$. Therefore, (4) holds. Similarly, we obtain

$$\lim_{n\to\infty} \mu(Q_n \cap A_n)=0. \tag{5}$$

It follows from (4) and (5) that $\lim_{n\to\infty} \mu(A_n)=0$, which completes the proof of Lemma 4.4.

Let (S, Σ, μ) be a measurable space with a non-negative measure μ and consider $L_1(S, \Sigma, \mu)$. Let $\Sigma_1 \subset \Sigma$ consist of all μ-measurable sets, free of atoms. Define for any $x \in L_1(S, \Sigma, \mu)$

$$\tilde{x}(t) = \sup_{A \in \Sigma_1, \mu(A) \leq t} \int_A |x(s)|\, d\mu, \quad t \in [0, \infty).$$

This function is introduced in [1] for a probability measure μ, free of atoms.

Lemma 4.5. *Let $x \in L_1(\Omega\ \Sigma, \mu)$ and $t, u \in (0, \mu(\Omega)]$ with $t \leq u < \infty$. Let B be a measurable subset of Ω such that B is free of atoms, $\mu(B)=u$ and $\int_B |x(s)|\, d\mu > \tilde{x}(u)-\delta$, where $\delta>0$. Then there exists $Q \subset B$ with $\mu(Q)=t$ and $\int_Q |x(s)|\, d\mu > \tilde{x}(t)-3\delta$.*

Proof. Without loss of generality we may assume that Ω is free of atoms. Then, there exists $A \subset \Omega$ such that $\mu(A)=t$ and $\int_A |x(s)|\, d\mu > \tilde{x}(t)-\delta$. Put $C = A \cap B$, $\mu(C)=v$ and $D=A \setminus C$. Choose $E \subset B \setminus C$ with $\mu(E)=t-v$. We shall prove that $\int_E |x(s)|\, d\mu > \int_D |x(s)|\, d\mu - 2\delta$. Suppose the contrary. Then, setting $F=B\setminus(C \cup E)$ and $G=A \cup F$, we have $\mu(G)=u$ and

$$\begin{aligned}\tilde{x}(u) &\geq \int_G |x|\, d\mu = \int_C |x|\, d\mu + \int_D |x|\, d\mu + \int_F |x|\, d\mu \\ &\geq \int_C |x|\, d\mu + \int_E |x|\, d\mu + \int_F |x|\, d\mu + 2\delta \\ &= \int_B |x|\, d\mu + 2\delta > \tilde{x}(u)+\delta.\end{aligned}$$

The contradiction implies $\int_E |x|\, d\mu > \int_D |x|\, d\mu - 2\delta$. Put $Q=C \cup E$. Thus, $\mu(Q) = t$ and

$$\begin{aligned}\int_Q |x|\, d\mu &= \int_C |x|\, d\mu + \int_E |x|\, d\mu > \int_D |x|\, d\mu - 2\delta + \int_C |x|\, d\mu \\ &= \int_A |x|\, d\mu - 2\,\delta > \tilde{x}(t) - 3\delta.\end{aligned}$$

Lemma 4.6. *For each $x \in L_1(\Omega, \Sigma, \mu)$ the function $\tilde{x}$ is concave in $[0, \infty)$.*

Proof. Without affecting the generality we may assume that Ω is free of atoms. Suppose that the assertion of Lemma 4.6 is false, i. e. there exists $\varepsilon>0$ and t_1, t_2 with $0 \leq t_1 < t_2 \leq \mu(\Omega)$, $t_2 < \infty$ so that

$$\tilde{x}((t_1+t_2)/2) = 1/2\ (\tilde{x}(t_1)+\tilde{x}(t_2)-\varepsilon). \tag{6}$$

Choose $\delta<\varepsilon/10$. Since Ω is free of atoms, it follows that there exists $C\subset\Omega$ with $\mu(C)=t_2$ and $\int_C |x|\, d\mu>\tilde{x}(t_2)-\delta$. By Lemma 4.5 we get that there exists $B<C$ with $\mu(B)=(t_1+t_2)/2$ and $\int_B |x|\, d\mu>\tilde{x}\ ((t_1+t_2)/2-3\delta$. Once again, by Lemma 4.5, we find $A\subset B$ such that $\mu(A)=t_1$ and $\int_A |x|\, d\mu>\tilde{x}\ (t_1)-9\delta$. Denote $D=C\setminus B$. Therefore, $\mu(D)=(t_2-t_1)/2$ and $\mu(A\cup D)=(t_1+t_2)/2$. Next, we have

$$\int_{A\cup D} |x|\, d\mu=\int_A |x|\, d\mu+\int_D |x|\, d\mu>\tilde{x}(t_1)-9\delta+\int_D |x|\, d\mu$$

and

$$\int_D |x|\, d\mu=\int_C |x|\, d\mu-\int_B |x|\, d\mu\geqq\tilde{x}(t_2)-\tilde{x}((t_1+t_2)/2)-\delta.$$

Thus, by (6),

$$\int_D |x|\, d\mu>\tilde{x}((t_1+t_2)/2)-\tilde{x}(t_1)-\delta+\varepsilon.$$

Hence,

$$\tilde{x}((t_1+t_2)/2)\geqq\int_{A\cup D} |x|\, d\mu>\tilde{x}((t_1+t_2)/2)+\varepsilon-10\,\delta,$$

which is a contradiction.

Therefore, $\tilde{x}$ is concave in $[0,\ \mu(\Omega)]$ provided $\mu(\Omega)<\infty$. Since $\tilde{x}(t)=\tilde{x}(\mu(\Omega))$ whenever $t>\mu(\Omega)$ and $\tilde{x}$ is increasing in $[0,\infty)$, it follows that $\tilde{x}$ is concave in $[0,\infty)$.

Lemma 4.7. *Let $x\in L_1(\Omega,\Sigma,\mu)$ with $x\geqq 0$ and Ω be free of atoms. For every $\lambda>0$ define $D_\lambda=\{x\geqq\lambda\}$. Then for each d with $0\leqq d\leqq\mu(D_\lambda)$, the inequality $\lambda d\leqq\tilde{x}(\mu(D_\lambda))-\tilde{x}(\mu(D_\lambda)-d)$ holds.*

Proof. Let $\varepsilon>0$. Since $x(s)<\lambda$ off the set D_λ, then $\int_{D_\lambda} x\, d\mu=\tilde{x}\ (\mu\ (D_\lambda))$. Since Ω is free of atoms, by the same argument, we may choose B so that $B\subset D_\lambda$, $\mu(B)=\mu(D_\lambda)-d$ and $\int_B x\, d\mu>\tilde{x}(\mu(D_\lambda)-d)-\varepsilon$. Also, we have that

$$\int_{D_\lambda\setminus B} x\, d\mu\geqq\lambda\mu(D_\lambda\setminus B)=\lambda d.$$

Thus,

$$\tilde{x}(\mu(D_\lambda))=\int_{D_\lambda} x\, d\mu=\int_B x\, d\mu+\int_{D_\lambda\setminus B} x\, d\mu>\tilde{x}\ (\mu(D_\lambda)-d)-\varepsilon+\lambda d.$$

Since ε is chosen arbitrarily, this then yields $x(\mu(D_\lambda))\geqq\tilde{x}(\mu(D_\lambda)-d)+\lambda d$.

Lemma 4.8. *Let $x, y\in L_1(\Omega,\Sigma,\mu)$ with $x, y\geqq 0$ and Ω be free of atoms. Let $0<\sqrt{2\gamma}<\alpha<\beta<\lambda<\infty$. Put $D_\lambda=\{x\geqq\lambda\}$, $E_\lambda=\{y\geqq\lambda\}$, $K_\lambda=\{x<\lambda\}$, $M_\lambda=\{y<\lambda\}$ and $t=\mu(D_\lambda)$. Let $|\tilde{x}(t)-\tilde{y}(t)|<\gamma/2$, $|\widetilde{x+y}(t)-2\tilde{x}(t)|<\gamma/2$ and A be such that $\mu(A)=t$ and $\int_A (x+y)\, d\mu>\widetilde{x+y}(t)-\gamma/2$. Then, $\mu\ (A\cap K_{\lambda-\alpha})<\alpha$. If moreover $|\tilde{x}(u)-\tilde{y}(u)|<\gamma/2$, where $u=\mu(A\cap E_{\lambda-\beta})$, then $\mu\ (A\cap M_{\lambda-\beta})<\beta$.*

Proof. It follows from $|\overline{x+y}(t)-2\tilde{x}(t)|<\gamma/2$ and $\int_A (x+y)\,d\mu>\overline{x+y}(t)-\gamma/2$ that $2\tilde{x}(t)-\gamma/2<\int_A (x+y)\,d\mu+\gamma/2$. Thus,

$$2\tilde{x}(t)-\gamma<\int_A (x+y)\,d\mu\leq\int_A x\,d\mu+\tilde{y}(t)\leq\int_A x\,d\mu+\tilde{x}(t)+\gamma/2.$$

Therefore,

$$\tilde{x}(t)-2\gamma<\int_A x\,d\mu. \tag{7}$$

On the other hand, we have

$$\begin{aligned}\int_A x\,d\mu&=\int_{A\cap D_{\lambda-\alpha}} x\,d\mu+\int_{A\cap K_{\lambda-\alpha}} x\,d\mu\\ &\leq\tilde{x}(\mu(A\cap D_{\lambda-\alpha}))+(\lambda-\alpha)\,\mu(A\cap K_{\lambda-\alpha})\\ &=\tilde{x}(\mu(A)-\mu(A\cap K_{\lambda-\alpha}))+(\lambda-\alpha)\,\mu(A\cap K_{\lambda-\alpha}),\end{aligned}$$

i.e.

$$\int_A x\,d\mu\leq\tilde{x}(\mu(A)-\mu(A\cap K_{\lambda-\alpha}))+(\lambda-\alpha)\,\mu(A\cap K_{\lambda-\alpha}). \tag{8}$$

By (7) and (8), we obtain

$$\tilde{x}(t)-\tilde{x}(t-\mu(A\cap K_{\lambda-\alpha}))-2\gamma<(\lambda-\alpha)\,\mu(A\cap K_{\lambda-\alpha}). \tag{9}$$

Since $0\leq\mu(A\cap K_{\lambda-\alpha})\leq\mu(A)=t=\mu(D_\lambda)$, by Lemma 4.7 with $d=\mu(A\cap K_{\lambda-\alpha})$ we get that

$$\lambda\mu(A\cap K_{\lambda-\alpha})\leq\tilde{x}(t)-\tilde{x}(t-\mu(A\cap K_{\lambda-\alpha})).$$

Putting this together with (9), we deduce

$$\lambda\mu(A\cap K_{\lambda-\alpha})-2\gamma<(\lambda-\alpha)\,\mu(A\cap K_{\lambda-\alpha}).$$

Therefore, $\alpha\mu(A\cap K_{\lambda-\alpha})<2\gamma$, i.e. $\mu(A\cap K_{\lambda-\alpha})<2\gamma/\alpha<\alpha$. Next we have $2\tilde{x}(t)-\gamma<\int_A (x+y)\,d\mu\leq x(t)+\int_A y\,d\mu$. Hence,

$$\int_A y\,d\mu>\tilde{x}(t)-\gamma. \tag{10}$$

On the other hand, we get

$$\int_A y\,d\mu\int_{A\cap E_{\lambda-\beta}} y\,d\mu+\int_{A\cap M_{\lambda-\beta}} y\,d\mu\leq\tilde{y}(u)+(\lambda-\beta)\,\mu(A\cap M_{\lambda-\beta}).$$

By combining this with (10), we obtain $\tilde{x}(t)-\gamma<\tilde{y}(u)+(\lambda-\beta)\,\mu(A\cap M_{\lambda-\beta})$. Since $|\tilde{x}(u)-\tilde{y}(u)|<\gamma/2$, it follows that $\tilde{x}(t)-2\gamma<\tilde{x}(u)+(\lambda-\beta)\,\mu(A\cap M_{\lambda-\beta})$. We have that $u=t-\mu(A\cap M_{\lambda-\beta})$ and therefore,

$$\tilde{x}(t)-2\gamma<\tilde{x}(t-\mu(A\cap M_{\lambda-\beta}))+(\lambda-\beta)\,\mu(A\cap M_{\lambda-\beta}). \tag{11}$$

By Lemma 4.7 with $d=\mu(A\cap M_{\lambda-\beta})$, we get

$$\lambda\mu(A\cap M_{\lambda-\beta})\leq\tilde{x}(t)-\tilde{x}(t-\mu(A\cap M_{\lambda-\beta})). \tag{12}$$

It follows from (11) and (12) that $\lambda\mu(A\cap M_{\lambda-\beta})-2\gamma<(\lambda-\beta)\mu(A\cap M_{\lambda-\beta})$. Consequently, $\beta\mu(A\cap M_{\lambda-\beta})<2\gamma$, whence $\mu(A\cap M_{\lambda-\beta})<2\gamma/\beta<\beta$, which completes the proof.

Definition 4.9. (of an equivalent lattice norm in $L_1(S,\Sigma,\mu)$). *For each* $x\in L_1(S,\Sigma,\mu)$ *put* $p(x)=(\int_0^\infty \tilde{x}^2(t)e^{-t}\,dt)^{1/2}$. *Obviously,* $p(x)\leq\|x\|$. *Denote* $q(x)=\sup_{A\in\Sigma_1}\int_A|x(s)|\,d\mu$. *Put* $r(x)=\sup(\Sigma_{i=1}^n x^2(a_i)\mu^2(A_i))^{1/2}$, $a_i\in A_i$, *where the supremum is taken over all finite systems of atoms* $\{A_i\}_{i=1}^n$, $n=1,2,\ldots$, *with* $A_i\neq A_j$ *if* $i\neq j$. *Thus,* $r(x)\leq\|x\|$.

We introduce a new norm in $L_1(S,\Sigma,\mu)$ *by the formula*

$$|||x|||=(\|x\|^2+p^2(x)+q^2(x)+r^2(x))^{1/2}.$$

It is clear that this is a lattice norm, order equivalent to the $L_1(S,\Sigma,\mu)$ norm $\|\cdot\|$.

Lemma 4.10. *Let* $x_n,\ y_n\in L_1(S,\Sigma,\mu)$, $n=1,2,\ldots$, *with* $\sup_n(\|x_n\|,\|y_n\|)<\infty$. *Let* $\lim_{n\to\infty}p(x_n)=1$, $\lim_{n\to\infty}p(y_n)=1$ *and* $\lim_{n\to\infty}p(x_n+y_n)=2$. *Then there exists a function* $v(t)$, $t\in[0,\infty)$ *and a subsequence* $\{n\}$ *of indices so that* $\lim_{n\to\infty}\tilde{x}_n(t)=v(t)$, $\lim_{n\to\infty}\tilde{y}_n(t)=v(t)$, $\lim_{n\to\infty}\widetilde{x_n+y_n}(t)=2v(t)$ *for each* $t\in[0,\infty)$ *and the convergence is uniform for* $t\in[\xi,\eta]$, *wbere* ξ,η *are arbitrary positive numbers.*

Proof. Using the triangle inequality, we see that $[\int_0^\infty(\tilde{x}_n(t)+\tilde{y}_n(t))^2e^{-t}dt]^{1/2}\to 2$. Therefore, by the uniform convexity of $L_2([0,\infty))$, we obtain tha $[\int_0^\infty(\tilde{x}_n(t)-\tilde{y}_n(t))^2e^{-t}\,dt]^{1/2}\to 0$. Thus, there exists a subsequence $\{n\}$ of indices, so that $\lim_{n\to\infty}(\tilde{x}_n(t)-\tilde{y}_n(t))=0$ almost everywhere in $[0,\infty)$. Since $\tilde{x}_n$ are increasing and uniformly bounded, there exists a subsequence such that $\lim_{n\to\infty}\tilde{x}_n(t)=v(t)$ for every $t\in[0,\infty)$, where v is an increasing function. By Lemma 4.6, $\tilde{x}_n$ are concave in $[0,\infty)$ and therefore v is concave in $[0,\infty)$. Consequently, v is continuous in $(0,\infty)$.

As above, we may choose once again a subsequence $\{n\}$ of indeces so that $\lim_{n\to\infty}\tilde{y}_n(t)=w(t)$ for each $t\in[0,\infty)$, where w is increasing in $[0,\infty)$ and continuous in $(0,\infty)$. It follows from $\lim_{n\to\infty}(\tilde{x}_n(t)-\tilde{y}_n(t))=0$ a.e. that $v(t)=w(t)$ a.e. in $[0,\infty)$. Since v,w are continuous in $(0,\infty)$ and $v(0)=0=w(0)$, we get that $v(t)=w(t)$ for any $t\in[0,\infty)$.

The functions $\tilde{x}_n^2$ are uniformly bounded and therefore,

$$\int_0^\infty v^2(t)e^{-t}\,dt=\lim_{n\to\infty}\int_0^\infty \tilde{x}_n^2(t)e^{-t}\,dt=\lim_{n\to\infty}p^2(x_n)=1. \tag{13}$$

As above, there is a new subsequence so that $\lim_{n\to\infty}\widetilde{x_n+y_n}(t)=u(t)$ for each $t\in[0,\infty)$ where u is increasing in $[0,\infty)$ and continuous in $(0,\infty)$. Similarly,

$$\int_0^\infty u^2(t)e^{-t}\,dt=\lim_{n\to\infty}p^2(x_n+y_n)=4. \tag{14}$$

By the triangle inequality, we get $0\leq\widetilde{x_n+y_n}(t)\leq\tilde{x}_n(t)+\tilde{y}_n(t)$, $t\in[0,\infty)$, whence $0\leq u(t)\leq 2v(t)$. Thus, by (13) and (14), we obtatin that $u(t)=2v(t)$ a.e. in $[0,\infty)$. Since u and v are continuous in $(0,\infty)$ and $u(0)=0$, it follows that $u(t)=2v(t)$ for every $t\in[0,\infty)$.

In any interval $[\xi,\eta]$ with $0<\xi<\eta<\infty$ the functions $\widetilde{x}_n$, $\widetilde{y}_n$, $\widetilde{x_n+y_n}$, v are increasing and continuous, whence, by a known theorem, $\lim_{n\to\infty}\widetilde{x}_n(t)=v(t)$, $\lim_{n\to\infty}\widetilde{y}_n(t)=v(t)$ and $\lim_{n\to\infty}\widetilde{x_n+y_n}(t)=2v(t)$ uniformly for $t\in[\xi,\eta]$.

Lemma 4.11. *Let* $x_n, y_n\in L_1(S,\Sigma,\mu)$ *be such that* $p(x_n)\to p$, $p(y_n)\to p$, $d(x_n+y_n)\to 2p$, $q(x_n)\to q$, $q(y_n)\to q$, $q(x_n+y_n)\to 2q$ *and*

$$\lim_{\mu(C)\to 0}\sup_n\{|\int_C(|x_n|-|y_n|)\,d\mu|\}=0.$$

Denote by Ω *the non-atomic part of* $\cup_{n=1}^{\infty}(\operatorname{supp}x_n\cup\operatorname{supp}y_n)$. *Then, there exists a subsequence* $\{n\}$ *of indices so that* $(x_n-y_n)\chi_\Omega\to 0$ *in measure as* $n\to\infty$.

Proof. Let x be one of the functions x_n, y_n, x_n+y_n, $n=1,2,\ldots$ Then, we have that

$$q(x)=\int_\Omega|x|\,d\mu,\quad \widetilde{x}(t)=\sup_{A\subset\Omega,\,\mu(A)\le t}\int_A|x|\,d\mu.$$

Thus, without loss of generality we may assume that $S=\Omega$.

Applying Lemma 4.10, we choose a subsequence $\{n\}$ of indices with the following property:

$$(***)\qquad \lim_{n\to\infty}\widetilde{x}_n(t)=v(t),\ \lim_{n\to\infty}\widetilde{y}_n(t)=v(t),\ \lim_{n\to\infty}\widetilde{x_n+y_n}(t)=2v(t)$$

uniformly for $t\in[\xi,\eta]$ whenever $0<\xi<\eta<\infty$.

For any $\lambda>0$ put

$$D_{n,\lambda}=\{|x_n|\ge\lambda\},\quad K_{n,\lambda}=\{|x_n|<\lambda\},$$
$$E_{n,\lambda}=\{|y_n|\ge\lambda\},\quad M_{n,\lambda}=\{|y_n|<\lambda\}.$$

Let f_n be the distribution function of x_n, $n=1,2,\ldots,$ i.e. $f_n(\lambda)=\mu(D_{n,\lambda})$, $0<\lambda<\infty$. We have that f_n are non-increasing and left continuous. Since $\{\int_\Omega|x_n|\,d\mu\}_{n=1}^{\infty}$ is convergent, the functions f_n are uniformly bounded in (c,∞) for each $c>0$. Thus, by Lemma 4.3, we may choose a subsequence which is convergent for any $\lambda\in(0,\infty)$ and satisfies condition $(*)$ of Lemma 4.3.

In order to prove that $\{x_n-y_n\}_{n=1}^{\infty}$ tends to zero in measure, by Lemma 4.4, it suffices to show that $\lim_{n\to\infty}\mu(\{|x_n|\ge b,|y_n|<a\})=0$ for every $0<a<b<\infty$.

It follows from $\widetilde{x}=\widetilde{|x|}$ for any $x\in L_1(S,\Sigma,\mu)$, $(***)$ and $\widetilde{x_n+y_n}\le|\widetilde{x}_n|+|\widetilde{y}_n|$ that $\lim_{n\to\infty}\widetilde{|x_n|+|y_n|}(t)=2v(t)$ uniformly in every finite interval which does not contain zero. Moreover, the condition $(**)$ of Lemma 4.4 involves only the absolute values $|x_n|$ and $|y_n|$ and therefore, without affecting the generality we may assume in the sequel that $x_n, y_n\ge 0$ a.e., $n=1,2\ldots.$

Fix $0<a<b<\infty$. Applying condition $(*)$ of Lemma 4.3 to f_n and (a,b), we obtain some λ and λ_i, $i=1,2,\ldots$ and denote $\beta_i=\lambda-\lambda_i$. Consequently, $\beta_i\downarrow 0$ and $\lambda-\beta_i\in(a,\lambda)$, $i=1,2,\ldots.$ In particular, $\beta_i<\lambda$. Since $\lambda<b$ and $\lambda-\beta_i>a$, then it suffices to show that

$$(15)\qquad \lim_{n\to\infty}\mu(D_{n,\lambda}\cap M_{n,\lambda-\beta_i})=0.$$

Let $\varepsilon<0$. Since $\beta_i\downarrow 0$, choose $\beta_{i_0}<\varepsilon$ and put $\beta=\beta_{i_0}$. Denote

(16) $$\delta=\min\,(\beta\varepsilon/4\lambda,\ \sqrt{\beta\varepsilon},\ \beta).$$

By the assumption, there is $\xi<\delta$ so that

(17) $$\left|\int_C x_n\,d\mu-\int_C y_n\,d\mu\right|<\beta\varepsilon,\ n=1,2,\dots \text{ whenever } \mu(C)<\xi.$$

Choose and fix such ξ. Put

(18) $$\eta=\sup_n f_n(\lambda).$$

By condition (*) of Lemma 4.3, select N_1 so that $n>N_1$ and $i>N_1$ imply $|f_n(\lambda-\beta_i)-f_n(\lambda)|<\delta/2$. Since $\beta_i\downarrow 0$, choose $i_1>N_1$ such that $\beta_{i_1}<\delta/2$. Put $\alpha=\beta_{i_1}$. Thus,

(19) $$0<\alpha<\delta/2.$$

Moreover, $|f_n(\lambda-\alpha)-f_n(\lambda)|<\delta/2$ whenever $n>N_1$. Putting this together with $\mu(D_{n,\lambda-\alpha}\cap K_{n,\lambda})=f_n(\lambda-\alpha)-f_n(\lambda)$, we get

(20) $$\mu(D_{n,\lambda-\alpha}\cap K_{n,\lambda})<\delta/2 \text{ if } n>N_1.$$

Fix

(21) $$0<\gamma<\alpha^2/2.$$

Applying (***) to the interval $[\xi,\eta]$, ξ, η chosen above, select $N>N_1$ so that for each $n>N$ the following inequalities hold:

(22) $$|\widetilde{x}_n(t)-v(t)|<\gamma/8,\quad |\widetilde{y}_n(t)-v(t)|<\gamma/8 \text{ and}$$
$$|\overline{x_n+y_n}(t)-2v(t)|<\gamma/4 \text{ whenever } t\in[\xi,\eta].$$

Fix $n>N$. It follows from (22) that

(23) $$|\widetilde{x}_n(t)-\widetilde{y}_n(t)<\gamma/2\cdot|\overline{x_n+y_n}(t)-2\widetilde{x}_n(t)|<\gamma/2 \text{ for each } t\in[\xi,\eta].$$

In order to prove (15), it suffices to show that $\mu(D_{n,\lambda}\cap M_{n,\lambda-\beta})<2\varepsilon$. Suppose the contrary. Then,

(24) $$\mu(D_{n,\lambda})=f_n(\lambda)\geqq 2\varepsilon.$$

Choose A with $\mu(A)=f_n(\lambda)$ so that

(25) $$\int_A (x_n+y_n)\,d\mu>\overline{x_n+y_n}(f_n(\lambda))-\gamma/2.$$

By (16), (18) and (24), we obtain $\delta<\mu(A)\leqq\eta$. But $\xi<\delta$, hence $\xi\leqq\mu(A)\leqq\eta$. Therefore, by (23) and (25), the assumptions of Lemma 4.8 are satisfied, whence $\mu(A\cap K_{n,\lambda-\alpha})<\alpha$. Thus, (19) implies that

(26) $$\mu(A\cap K_{n,\lambda-\alpha})<\delta/2.$$

The following representation holds:

$$A=(A\cap D_{n,\lambda})\cup(A\cap K_{n,\lambda}\cap D_{n,\lambda-\alpha})\cup(A\cap K_{n,\lambda-\alpha}).$$

By (20), we get that $\mu(A\cap K_{n,\lambda}\cap D_{n,\lambda-\alpha})<\delta/2$. The above formula, the equality $\mu(A)=\mu(D_{n,\lambda})$, (26) and the representation of A imply

(27) $$\mu(A\cap E_{n,\lambda})>\mu(D_{n,\lambda})-\delta.$$

Consider now the set $C=A\cap E_{n,\lambda-\beta}$. We shall prove that $\xi\leq\mu(C)$. Assume the contrary, i.e. $\mu(C)<\xi$. Then, by (17), we get

$$\int_C y_n\,d\mu\leq\int_C x_n\,d\mu+\beta\varepsilon. \tag{28}$$

By (23) and (25), we obtain $2\tilde{x}_n(\mu(A))-\gamma<\int_A(x_n+y_n)\,d\mu$, whence

$$2\tilde{x}_n(\mu(A))-\gamma<\tilde{x}_n(\mu(A))+\int_C y_n\,d\mu+\int_{A\setminus C}y_n\,d\mu.$$

Since $y_n<\lambda-\beta$ on $A\setminus C$, the last inequality and (28) imply

$$\tilde{x}_n(\mu(A))-\gamma<\int_C x_n\,d\mu+\beta\varepsilon+(\lambda-\beta)\,\mu(A\setminus C). \tag{29}$$

It follows from (27) that

$$\mu((A\cap D_{n,\lambda})\setminus C)\geq\mu(A\cap D_{n,\lambda})-\mu(C)>\mu(A)-\delta-\xi>\mu(A)-2\delta.$$

In particular, $\mu(A\setminus C)\leq\mu(A)<\mu(A\cap D_{n,\lambda})+2\delta$. Putting these together with (29), we obtain

$$\tilde{x}_n(\mu(A))-\gamma<\int_C x_n\,d\mu+\beta\varepsilon+\lambda\mu(A\cap D_{n,\lambda}\setminus C)+2\delta\lambda-\beta(\mu(A\cap D_{n,\lambda}\setminus C)+2\delta)$$

$$\leq\int_C x_n\,d\mu+\int_{A\cap D_{n,\lambda}\setminus C}x_n\,d\mu+\beta\varepsilon+2\delta\lambda-\beta\mu(A)$$

$$\leq\int_A x_n\,d\mu+\beta\varepsilon+2\delta\lambda-\beta\mu(A)\leq\tilde{x}_n(\mu(A))+\beta\varepsilon+2\delta\lambda-\beta\mu(A).$$

Hence, $\gamma+\delta\varepsilon+2\delta\lambda-\beta\mu(A)>0$. Therefore, by (16), (19), (21) and (24),

$$0<\gamma+\beta\varepsilon+2\delta\lambda-2\beta\varepsilon<\delta^2/4+\beta\varepsilon+2\lambda\beta\varepsilon/4\lambda-2\beta\varepsilon<0.$$

The contradiction implies $\mu(C)\geq\xi$. Since the inequality $\mu(C)\leq\eta$ is also valid, (23) holds for $t=\mu(C)$. Thus, by (16), (19) and (21), the assumptions of Lemma 4.8 are satisfied, whence

$$\mu(A\cap M_{n,\lambda-\beta})<\beta. \tag{30}$$

It follows from (27) that $\mu(D_{n,\lambda}\setminus A)<\delta$. Hence, by (30), we obtain

$$\mu(D_{n,\lambda}\cap M_{n,\lambda-\beta})<\delta+\beta<2\varepsilon.$$

This completes the proof of Lemma 4.11.

4.12. Proof of Theorem 3.1. Let $x_n, y_n\in L_1(S,\Sigma,\mu)$, $n=1,2,\ldots$ with $|||x_n|||\to1$, $|||y_n|||\to1$ and $|||x_n+y_n|||\to2$. According to Lemma 4.1, let $x_n-y_n=z$, $n=1,2,\ldots$.

Put $\Omega_1=\bigcup_{n=1}^{\infty}(\operatorname{supp}x_n\cup\operatorname{supp}y_n)$. Since Ω_1 is σ-finite, it can be represented by $\Omega_1=(\bigcup_{i=1}^{\infty}A_i)\cup\Omega$, where A_i are atoms and Ω is free of atoms.

Since the sequences $\{\|x_n\|\}_{n=1}^{\infty}$, $\{p(x_n)\}_{n=1}^{\infty}$, $\{q(x_n)\}_{n=1}^{\infty}$, $\{r(x_n)\}_{n=1}^{\infty}$ and the corresponding sequences to $\{y_n\}_{n=1}^{\infty}$ and $\{x_n+y_n\}_{n=1}^{\infty}$ are bounded, without loss of generality (if necessary passing to a subsequence) we may assume that they are convergent. By the triangle inequality, we get

$$|||x_n+y_n|||\leqq[(\|x_n\|+\|y_n\|)^2+(p(x_n)+p(y_n))^2+(q(x_n)+q(y_n))^2$$
$$+(r(x_n)+r(y_n))^2]^{1/2}\leqq|||x_n|||+|||y_n|||.$$

Thus, it follows from $|||x_n+y_n|||\to 2$ and $|||x_n|||+|||y_n|||\to 2$ that

$$[(\|x_n\|+\|y_n\|)^2+(p(x_n)+p(y_n))^2+(q(x_n)+q(y_n))^2+(r(x_n)+r(y_n))^2]^{1/2}\to 2.$$

The uniform convexity of l_2 gives $\|x_n\|-\|y_n\|\to 0$, $p(x_n)-p(y_n)\to 0$, $q(x_n)-q(y_n)\to 0$ and $r(x_n)-r(y_n)\to 0$. Put $\lim_{n\to\infty}\|x_n\|=u$, $\lim_{n\to\infty}p(x_n)=p$, $\lim_{n\to\infty}q(x_n)=q$, $\lim_{n\to\infty}r(x_n)=r$. Next, by the triangle inequality,

$$|||x_n+y_n|||\leqq[\|x_n+y_n\|^2+(p(x_n)+p(y_n))^2+(q(x_n)+q(y_n))^2+(r(x_n)+r(y_n))^2]^{1/2}$$
$$\leqq|||x_n|||+|||y_n|||.$$

Letting $n\to\infty$, we get that $(\lim_{n\to\infty}\|x_n+y_n\|^2+4p^2+4q^2+4r^2)^{1/2}=2$. Since $(4u^2+4p^2+4q^2+4r^2)^{1/2}=2$, then $\lim_{n\to\infty}\|x_n+y_n\|=2u$. Similarly, $\lim_{n\to\infty}p(x_n+y_n)=2p$, $\lim_{n\to\infty}q(x_n+y_n)=2q$ and $\lim_{n\to\infty}r(x_n+y_n)=2r$.

We have that for each $x\in L_1(S,\Sigma,\mu)$ with supp $x\subset\Omega_1$

$$r(x)=(\sum_{i=1}^{\infty}x^2(a_i)\mu^2(A_i))^{1/2},\text{ where, } a_i\in A_i.$$

Since $r(x_n)\to r$, $r(y_n)\to r$ and $r(x_n+y_n)\to 2r$, the uniform convexity of l_2 implies $\lim_{n\to\infty}(x_n(a_i)-y_n(a_i))=0$, $i=1,2,\ldots$. On the other hand, $x_n-y_n=z$, $n=1,2,\ldots$ Hence,

(31) $$z\chi_{A_i}=0,\ i=1,2,\ldots$$

Next, since $x_n-y_n=z$, $n=1,2,\ldots$ implies

$$\lim_{\mu(C)\to 0}\sup_n\{|\int_C(|x_n|-|y_n|)\,d\mu|\}=0,$$

the assumptions of Lemma 4.11 are satisfied. Hence, there exists a subsequence so that $(x_n-y_n)\chi_\Omega$ tends to zero in measure. Then, we deduce that $z\chi_\Omega=0$. Putting this together with (31), we conclude that $z=0$, which completes the proof of Theorem 3.1.

5. In this section we present the proof of Theorem 3.5.

5.1. Let (S,Σ,μ) be a probability space and X be a Köthe function space on (S,Σ,μ) with a σ-order continuous norm. Since μ is funite, S can be represented by $S=(\bigcup_{i=1}^{\infty}A_i)\cup\Omega$, where A_i, $i=1,2,\ldots$ are atoms and Ω is free of atoms.

There exists (cf. [6, p. 29]) an order isometry $T:X^*\to Y$, where Y is a Köthe function space on (S,Σ,μ) consisting of all measurable functions g on (S,Σ,μ) such that $fg\in L_1(S,\Sigma,\mu)$ for every $f\in X$, moreover $x^*(f)=\int_S fg\,d\mu$, where $g=Tx^*$.

Let $x^*\in X^*$, $Tx^*=g$, Put

$$\tilde{g}(t)=\sup_{B\subset\Omega,\mu(B)\leqq t}\int_B|g(s)|\,d\mu,\ t\in[0,1],$$
$$\|g\|_1=(\int_0^1\tilde{g}^2(t)\,dt)^{1/2}.$$

Since $\tilde{g}(t)\leqq\int_\Omega|g|\,d\mu=|x^*|(X_\Omega)\leqq\|x^*\|\cdot\|\chi_\Omega\|$, then $\|g\|_1\leqq\|x^*\|\cdot\|\chi_\Omega\|$.

We introduce in X^* an equivalent lattice norm by the formula:

$$|||x^*||| = (\|x^*\|^2 + \|Tx^*\|_1^2 + \sum_{i=1}^{\infty} |x^*(\chi_{A_i})|/2^i \|\chi_{A_i}\|_X^2)^{1/2}.$$

Lemma 5.2. *The norm* $|||\cdot|||$ *is* w^**-lower semi-continuous.*

Proof. Let $\{x_\alpha^*\} \subset X^*$, $x_\alpha^*(x) \to x^*(x)$ for each $x \in X$. Put $g_\alpha = Tx_\alpha^*$, $g = Tx^*$. Let $B \in \Sigma$, then $f_B = \chi_B \operatorname{sign} g \in X$. Thus, we obtain that

$$(32) \qquad \int_B |g|\, d\mu = \int_S f_B\, g\, d\mu = x^*(f_B) = \lim_\alpha x_\alpha^*(f_B) \leq \lim_\alpha \inf \int_B |g_\alpha|\, d\mu.$$

Let $t \in [0, 1]$ and $\eta > 0$. Select $B \subset \Omega$ such that $\mu(B) = t$ and $\tilde{g}(t) \leq \int_B |g| d\mu + \eta$. It follows from (32) that $\int_B |g|\, d\mu \leq \lim_\alpha \inf \tilde{g}_\alpha(t)$. Therefore, $\inf_\alpha \tilde{g}_\alpha(t) + \eta$. Since η is chosen arbitrarily, then

$$\tilde{g}(t) \leq \lim_\alpha \inf \tilde{g}_\alpha(t) + \eta.$$

Since η is chosen arbitrarily, then (33) $\tilde{g}(t) \leq \lim_\alpha \inf \tilde{g}_\alpha(t)$.

Let $\varepsilon > 0$. Since $\tilde{g}$ is concave and bounded, then $\tilde{g}^2$ is Riemann integrable. Hence, there exists n such that

$$\int_0^1 \tilde{g}^2(t)\, dt \leq \sum_{j=0}^{n-1} \tilde{g}^2(j/n)/n + \varepsilon.$$

By (33), there is α_0 such that $\alpha > \alpha_0$ implies

$$\tilde{g}_\alpha^2(j/n) \geq \tilde{g}^2(j/n) - \varepsilon, \quad j = 0, 1, 2, \ldots, n-1.$$

Therefore, we obtain for $\alpha > \alpha_0$

$$\int_0^1 \tilde{g}^2(t)\, dt \leq \sum_{j=0}^{n-1} \tilde{g}_\alpha^2(j/n)/n + 2\varepsilon.$$

Since $\tilde{g}_\alpha^2$ are increasing, then $\Sigma_{j=0}^{n-1} \tilde{g}_\alpha^2(j/n)/n \leq \int_0^1 \tilde{g}_\alpha^2(t)\, dt$. Hence, for each $\alpha > \alpha_0$

$$\int_0^1 \tilde{g}^2(t)\, dt \leq \int_0^1 \tilde{g}_\alpha^2(t)\, dt + 2\varepsilon.$$

Thus, we get that $\|g\|_1 \leq \lim_\alpha \inf \|g_\alpha\|_1$.

Lemma 5.3. *The dual space* X^**, equipted with the norm* $|||\cdot|||$*, is* w^**-uniformly convex.*

Proof. Let $x_n^*, y_n^* \in X^*$, $n = 1, 2, \ldots$ with $|||x_n^*||| \to 1$, $|||y_n^*||| \to 1$, $|||x_n^* + y_n^*||| \to 2$. Let T be the order isometry between X and the Köthe function space on (S, Σ, μ), defined in 5.1. Let $Tx_n^* = f_n$, $Ty_n^* = g_n$. In order to prove that

$$\lim_{n\to\infty} \int_S (f_n - g_n)\, h\, d\mu = 0$$

holds for every $h \in X$, it suffices to obtain it for a dense set in X. Hence (cf. e.g. [4. p. 142]), it remains to show that $\lim_{n\to\infty} \int_S (f_n - g_n)\, \chi_A\, d\mu = 0$ for each $A \in \Sigma$.

As in 4.12, we may select a subsequence $\{n\}$ of indices so that $\lim_{n\to\infty} |f_n(a_i)| = u_i$, $\lim_{n\to\infty} |g_n(a_i)| = u_i$, $\lim_{n\to\infty} |f_n(a_i) + g_n(a_i)| = 2u_i$, where $a_i \in A_i$,

$i=1, 2, \ldots,$ and $\lim_{n\to\infty}\|f_n\|_1=u,$ $\lim_{n\to\infty}\|g_n\|_1=u,$ $\lim_{n\to\infty}\|f_n+g_n\|_1=2u$. Evidently,

(34) $$\lim_{n\to\infty}(f_n(a_i)-g_n(a_i))=0,\ i=1, 2, \ldots$$

By the same argument as in Lemma 4.10, we get that there is a subsequence so that $\tilde{f}_n(t)\to v(t)$, $\tilde{g}_n(t)\to v(t)$, $\overline{f_n+g_n}(t)\to 2v(t)$ for $t\in[0, 1]$. Putting $t_0=\mu(\Omega)$, we deduce that $\int_\Omega|f_n|\,d\mu\to v(t_0)$, $\int_\Omega|g_n|\,d\mu\to v(t_0)$ and $\int_\Omega|f_n+g_n|\,d\mu\to 2v(t_0)$. For each $A\in\Sigma$ we have

$$\int_A(|f_n|-|g_n|)\,d\mu\leq\int_S|f_n-g_n|\chi_A\,d\mu\leq 2\|\chi_A\|.$$

Since the norm is σ-order continuous, it follows that

$$\lim_{\mu(A)\to 0}\sup_n\int_A(|f_n|-|g_n|)\,g\mu=0.$$

Therefore, it is clear that the assumptions of Lemma 4,11 are satisfied, whence there is a subsequence {n} of indices so that $(f_n-g_n)\chi_\Omega\to 0$ in measure. Thus, by the theorem of Riesz, we may choose a subsequence such that

$$(f_n-g_n)\chi_\Omega\to 0 \text{ a.e.}$$

Putting this together with (34), we conclude that

(35) $$f_n-g_n\to 0 \text{ a.e.}$$

Next, we have as above that for every $A\in\Sigma$

(36) $$\lim_{\mu(A)\to 0}\sup_n\int_A(f_n-g_n)\,d\mu=0.$$

By (35), (36) and Vitali's theorem, we obtain that

$$\lim_{n\to\infty}\int_S|f_n-g_n|\,d\mu=0.$$

5. 4. Proof of Theorem 3.5. It follows from Lemma 5.2 that the new lattice norm $|||\cdot|||$ in X^* is induced by an equivalent lattice norm $|||\cdot|||$ in X. By Lemma 5.3 and [8], we obtain that the norm $|||\cdot|||$ in X is uniformly differentiable in every direction.

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