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ON QUASI-INJECTIVITY AND QUASI-CONTINUITY

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A generalisation of quasi-injective modules, noted SQC modules, is introduced. Such modules are proved to be quasi-continuous. Left continuous modules need not be SQC but prime left non-singular SQC rings are primitive left self-injective regular. Let A denote a SQC ring:

(1) If A is left non-singular, then $A=B\oplus C$, where B is a left self-injective regular ring such that any non-zero ideal contains a non-zero nilpotent element and C is a reduced quasi-continuous ring whose injective hull is a strongly regular ring;

(2) If either A is left p-injective or every simple left A-module is flat, then $A=B\oplus C$, where B is the quasi-continuous left minimal direct summand containing all the nilpotent elements of A and C is a left and right self-injective strongly regular ring.

Characteristic properties of left self-injective regular rings and semi-simple Artinian rings are given in terms of SQC rings. For example, a left non-singular SQC ring is left self-injective iff either A is left p-injective or every principal left ideal is isomorphic to a complement left ideal. A SQC ring is semi-simple Artinian iff either every essential left ideal is isomorphic to ${}_AA$ or A is semi-prime with maximum condition on left annihilators.

Introduction. Throughout, A represents an associative ring with identity and A-modules are unitary. Z, J will denote respectively the left singular ideal and the Jacobson radical of A. Recall that

(1) A left A-module M is quasi-injective iff any left A-homomorphism of a left sumbodule of M into M may be extended to an endomorphism of ${}_AM$;

(2) M is quasi-continuous iff (a) every complement left submodule is a direct summand of ${}_AM$ and (b) for any direct summands N, P of ${}_AM$ such that $N\cap P=0$, $N\oplus P$ is also a direct summand of ${}_AM$;

(3) M is p-injective (resp. Up-injective) iff, for any principal (resp. complement) left ideal I of A, $a\in A$, any left A-homomorphism $g: Ia\to M$, there exists $y\in M$ such that $g(ba)=bay$ for all $b\in I$. Then A is von Neumann regular (resp. left continuous regular) iff every left A-module is p-injective (resp Up-injective);

(4) M is called semi-simple [8] iff $J(M)$, the intersection of all maximal left submodules of M, is zero. Then A is a left V-ring iff every left A-module is semi-simple [8, Theorem 2.1].

Since several years guasi-injective and related modules are extensively studied. Y. Utumi introduced continuous rings as a generalization of self-injective rings (cf. [9, 10]). This notion of continuity is, in turn, extended to quasi-continuity by many authors (cf. the bibliography of [1]). The purpose of this note is to study the following generalisation of quasi-injective modules.

Definition. *A left A-module M is called SQC (strongly quasi-continuous) if, for any left submodule N of M such that there exist a non-zero complement left submodule C of M which is isomorphic to a factor module of N, then any left A-homomorphism from N into M may be extended to an endomorphism of ${}_AM$.*

A is called a SQC ring if $_AA$ is SQC.

Let us first prove an interesting result on SQC modules.

Theorem 1. *Let M be a SQC left A-module and C a left complement submodule of M. If E is a submodule of $_AM$ and $f: {}_AE \to {}_AC$ a left epimorphism, then f may be extended to a left epimorphism $g: {}_AM \to {}_AC$.*

Proof. Let U denote the set of submodules N of $_AM$ containing E such that f may be extended to a left A-homomorphism from N into C. By Zorn's Lemma, U contains a maximal member V. Let $h: {}_AV \to {}_AC$ be the extension of f to V. Since $h(V)=C$, if $i: C \to M$ is the canonical injection, by hypothesis, $ih: {}_AV \to M$ may be extended to an endomorphism g of $_AM$. Suppose that $g(M) \not\subseteq C$. If K is a relative complement of C in $_AM$, $(g(M)+C)\cap K \neq 0$. Let $0 \neq k \in K \cap (g(M)+C)$, $k=g(m)+c$, $m\in M$, $c \in C$. Then $L=\{\mathfrak{z} \in M \mid g(\mathfrak{z}) \in K \oplus C\}$ is a submodule of $_AM$ which strictly contains V (because $g(m) \notin C$ and hence $m \notin V$ but $g(m)=k-c \in K \oplus C$). If p is the projection of $K\oplus C$ onto C, then $pg: {}_AL \to {}_AC$ is clearly an extension of h to L and hence pg is an extension of f to L. This contradicts the maximality of V. Thus $g(M) \subseteq C$ and g is an epimorphism of M onto C.

We are now in a position to prove that SQC modules are intermediate between quasi-injective and quasi-continuous modules.

Proposition 2. *Let M be a SQC left A-module.*

(1) *If C is a complement left submodule of M, then for any relative complement D of C, $M=C\oplus D$;*

(2) *M is quasi-continuous.*

Proof. (1) If C is a complement left submodule of M, D a relative complement of $_AC$ in $_AM$, $L=C\oplus D$ and $p: L \to C$ the canonical projection, then by Theorem 1, p may be extended to $g: {}_AM \to {}_AC$. Now $\ker g \cap C = 0$ and for any $y \in M$, $y=g(y)+(y-g(y))$, where $g(y) \in C$, $y-g(y) \in \ker g$, which proves that $M=C\oplus \ker g$. Since $D \subseteq \ker g$ and D is a relative complement of C, then $D=\ker g$. It follows that condition (a) for quasi-continuity is satisfied.

(2) Let P, Q be direct summands of $_AM$ such that $P\cap Q=0$. The set of left submodules N of M such that $P \cap N = 0$ and $Q \subseteq N$ has a maximal member K which is a relative complement of P. By (1), $M=P\oplus K$ and if $M=Q\oplus R$, then $M=(P\oplus Q)\oplus(K\cap R)$, where $K=Q\oplus(K\cap R)$. This proves that M is quasi-continuous.

As usual, an ideal of A means a two-sided ideal.

Corollary 2.1. *If A is semi-prime SQC ring, then the left annihilator of any ideal is generated by a central idempotent.*

(Apply [1, Proposition 10] to Proposition 2)

Corollary 2.2. *A SQC left A-module M is continuous iff any left submodule of M isomorphic to a complement left submodule of M is a complement left submodule of M is a complement submodule.*

It may be noted that left self-injective regular rings need not be left V-rings.

Corollary 2.3. *The following conditions are equivalent for a SQC ring A:*

(1) *A is a left V-ring;*

(2) *Every cyclic singular left A-module is semi-simple and every minimal left ideal of A is a complement left ideal.*

Proof. (1) implies (2) by [8, Theorem 2.1].

Assume (2). Let I be a minimal left ideal, K a relative complement of I in ${}_AA$. Then by Proposition 2, $A=I\oplus K$ which proves that K is a maximal left ideal of A. Then (2) implies (1) by [12, Theorem 3].

Corollary 2.4. *If A is a SQC ring such that any proper essential left ideal contains a non-zero complement left ideal of A, then A is left self-injective.*

At this point, we note an important property of SQC rings.

Remark 1. If A is a SQC ring, then any non-zero-divisor is right invertible and hence invertible. It follows that any left or right A-module is divisible.

As usual, $Z(M)$ denotes the singular submodule of the left A-module M. Then ${}_AM$ is non-singular iff $Z(M)=0$.

Theorem 3. *The following conditions are equivalent:*

(1) *A is left self-injective regular;*

(2) *A is a SQC ring whose cyclic left modules are either isomorphic to ${}_AA$ or p-injective;*

(3) *A is a left non-singular right p-injective SQC ring;*

(4) *A is a left non-singular SQC ring such that any principal left ideal is isomorphic to a complement left ideal;*

(5) *For any finitely generated left A-module M, $M=Z(M)\oplus P$, where P is a p-injective SQC left A-module;*

(6) *A is a left non-singular left p-injective SQC ring;*

(7) *A is a left non-singalar ring whose finitely generated faitful non-singular left modules are p-injective projective.*

Proof. Obviously, (1) implies (2).

Assume (2). Since every cyclic left A-module not isomorphic to ${}_AA$ is p-injective, then A is either regular or a simple domain. Thus (2) implies (3) by Remark 1.

Assume (3). Since A is right p-injective, then every principal left ideal of A is a left annihilator [7, Theorem 1] and is therefore a complement left ideal [4, Lemma 1]. This proves that (3) implies (4).

Assume (4). Then A is left Up-injective by Proposition 2. Since $Z=0$, then A is von Neumann regular SQC which yields A left self-injective and (4) implies (5) by [14, Corollary 10].

Assume (5). Then ${}_AZ$ is a direct summand of ${}_AA$ which implies $Z=0$ and hence (5) implies (6).

If A is left p-injective, then any left ideal isomorphic to a direct summand of ${}_AA$ is direct summand of ${}_AA$. Therefore (6) implies (7) by [10, Lemma 4.1], [14, Corollary 10] and Proposition 2.

Assume (7). If F is a finitely generated non-singular left A-module, then $M=A_A\oplus {}_AF$ is a finitely generated non-singular faithful left A-module which is therefore p-injective. This implies that ${}_AF$ is p-injective projective which proves, in particular, that A is regular. Then (7) implies (1) by [2, Theorem 2.1].

Corollary 3.1. *The following conditions are equivalent:*

(1) *A is left and right self-injective strongly regular;*

(2) *A is a reduced SQC ring such that every principal left ideal is isomorphic to a complement left ideal;*

(3) *A is a reduced SQC ring such that every principal right ideal is a complement right ideal.*

Proof. The equivalence of (1) and (2) follows from Theorem 3 (4).

(1) implies (3) evidently.

Assume (3). By Proposition 2, every complement left ideal is an ideal and since A is reduced, then every complement right ideal is also an ideal and is therefore a right annihilator by [11, Lemma 1]. Since A is a Baer ring, then every principal right ideal is a direct summand of A_A and hence (3) implies (2).

It is known that continuous regular rings need not be self-injective [10 P. 1972]. It follows from Theorem 3 that left continuous rings need not be SQC

We now consider two decompositions of SQC rings.

Proposition 4. *If A is a left non-singular SQC ring, then $A=B\oplus C$, where B is a left self-injective regular ring such that any non-zero ideal contains a non-zero nilpotent element and C is a reduced quasi-continuous ring such that the injective hull of ${}_AC$ is a strongly regular ring.*

Proof. If H denotes the injective hull of ${}^AA$, then H is a left self-injective regular ring and $H=B\oplus K$, where B is a left self-injective regular ring such that any non-zero ideal contains a non-zero nilpotent element and K is a strongly regular ring [9, P. 604]. By Proposition 2, $A=(B\cap A)\oplus(K\cap A)$ and since $B\subsetneqq A$, then $A=B\oplus C$, where $C=K\cap A$ and K is the injective hull of ${}_AC$.

Corollary 4.1. *The following conditions are equivalent:*

(1) *A is a primitive left self-injective regular ring;*

(2) *A is a prime left non-singular SQC ring.*

We here make a remark which will contribute to another important decomposition result.

Remark 2. If A is a SQC ring such that $A=B\oplus D$, where B is an ideal of A and D is regular ring, then D is a left self-injective ring.

Applying [1, Corollary 13], Proposition 2 and Remark 2, we get

Theorem 5. *Let A be a SQC ring satisfying any one of the following conditions:* (a) *A is left p-injective or* (b) *Every simple left A-module is flat. Then $A=B\oplus C$, where B is the quasi-continuous left minimal direct summand containing all the nilpotent elements of A and C is a left and right self-injective strongly regular ring.*

Let us mention a result on SQC modules analogous to that of C. Faith and Y. Utumi for quasi-injective modules.

Proposition 6. *Let M be a SQC left A-module. If $E=\mathrm{End}({}_AM)$, then $E/J(E)$ is von Neumann regular, where*

$$J(E)=\{f\in E\}\mid \ker f \text{ is essential in } {}_AM\}$$

is the Jacobson radical of E.

Applying Corollary 2.3 to Proposition 6, we get

Corollary 6.1. *A is a left self-injective regular left V-ring iff A is a semi-prime SQC ring whose cyclic singular left modules are semi-simple.*

Our last theorem characterises semi-simple Artinian rings in terms of SQC rings.

Theorem 7. *The following conditions are equivalent for a SQC ring A:*

(1) *A is semi-simple Artinian;*

(2) *A is semi-prime with maximum condition on left annihilators;*

(3) *Every essential left ideal of A is isomorphic to ${}_AA$;*

(4) *$Z=0$ and for any essential left ideal L, every proper complement left subideal of L is a complement left ideal of A;*

(5) *Every cyclic left A-module not isomorphic to ${}_AA$ is injective.*

Proof. (1) implies (2) evidently.

Assume (2). A is then a left Goldie ring by Proposition 2. If L is an essential left ideal, then L contains a non-zero divisor c by a well-known theorem of A. W. Goldie and by Remark 1, $L=A$. Thus (2) implies (3).

Assume (3). If L is an essential left ideal, since ${}_AL\approx{}_AA$, by hypothesis, any left A-homomorphism from L to A may be extended to an endomorphism of ${}_AA$. This implies that A is left self-injective whence A is the only essential left ideal. Therefore (3) implies (4).

Assume (4). Suppose there exists an essential left ideal E such that any non-zero left subideal of E is essential in E. Then any non-zero left ideal of A (having non-zero intersection with E) is an essential left ideal which proves that A is left uniform. Since $Z=0$, A becomes a left Ore domain which yields A a division ring by Remark 1.

Otherwise, for any essential left ideal L, there exists a non-zero left subideal which is not essential in ${}_AL$. Then L contains a non-zero complement left subideal which $C\neq L$ and hence another non-trivial complement left subideal K such that $C\oplus K$ is essential in ${}_AL$. By hypothesis, C and K are complement left ideals of A and by Proposition 2(2), $C\oplus K$ is a direct summand of ${}_AA$ which implies $C\oplus K=L=A$. Thus (4) implies (5) in any case.

Finally, (5) implies (1) by [5, Theorem 1] and Remark 1.

We conclude with a few more remarks.

A theorem of Cateforis-Sandomierski [3, Theorem 2.1] yields

Remark 3. If A is commutative, then A is semi-simple Artinian iff A is a SQC ring such that any A-module contains its singular submodule as a direct summand. (This improves [3, Theorem 2.7]).

ALD (almost left duo) rings are considered in [13].

Remark 4. A is simple Artinian iff A is a prime ALD SQC ring. (Apply [13, Lemma 1.1]).

Remark 5. Suppose that every cyclic left A-module D is SQC and that every submodule of D isomorphic to a complement submodule is a complement submodule. Then A/J is semi-simple Artinian.

Remark 6. The following conditions are equivalent:

(1) Every factor ring of A is left self-injective regular;

(2) Every factor ring of A is semi-prime left non-singular SQC ring. (Apply [6, Corollary 1.18] to Corollary 4.1)

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