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COMPACT SUBSPACE OF R^n AND FIXED POINTS

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In this note we study the compact subspaces K of the Euclidean space R^n, in relation with the fixed point property. We state some conditions in order to, for every continuous map f of K into K with $\Lambda_f \neq 0$, either f has a fixed point or some "spiral" continuum is mapped by f into itself. It is also introduced the concept of the so-called isotopic pseudo-retracts. These compact retracts, as also the Sieclucki's deformation quasi-retracts, have an "aspiral" structure and the fixed point property.

Let $J=[0, 1)$ and $J_t=[t, 1)$, for every $t \in J$. We denote by d the metric of the Euclidean space R^n.

In the space R^n let K be a compact subspace and U an open neighbourhood of K. The frontier of a set $M \subseteq R^n$ is denoted by $\mathrm{Er}\, M$.

A homeomorphism ψ of $\mathrm{Fr}\, U \times J$ onto $\bar{U} \setminus K$ is called an isotopic contraction of $\mathrm{Fr}\, U$ to K in U if 1) $\psi(a, 0)=a$, for every $a \in \mathrm{Er}\, U$ and 2) for every open neighbourhood V of K, there exists $t \in J$ such that $\psi(\mathrm{Fr}\, U, J_t) \subseteq V$.

For an arbitrary subset A of J we set $D_A=\{\psi(a, t): a \in \mathrm{Fr}\, U, t \in A\}$. We also set $D_{A \cup \{1\}}=D_A \cup K$. Obviously $D_{A \cup B}=D_A \cup D_B$. Let $p \in \bar{U} \setminus K$ and $\psi(a, t_0)=p$. We set $l_p=\{\psi(a, t): t \in [0, t_0]\}$, $l^p=\{\psi(a, t): t \in [t_0, 1)\}$, and $l(p)=l_p \cup l^p$. For an arbitrary subset T of $\bar{U} \setminus K$ we set $\mathrm{tr}\, T=\cup_{p \in T} l_p$. Every set l of the form $l(p)$ is called a fibre of $\bar{U} \setminus K$ with respect to the homeomorphism ψ. Obviously by every point of $\bar{U} \setminus K$ passes one and only one fibre. It is also obvious that every fibre l intersects every set $D_{\{t\}}=D_t$, $t \in J$, in a unique point.

We observe that: 1) if $p_n \in \bar{U} \setminus K$, $n=1, 2, \ldots, p \in \bar{U} \setminus K$ and $\lim_{n\to\infty} p_n=p$ then $\overline{\lim}_{n\to\infty} l p_n=l_p$ and $\overline{\lim}_{n\to\infty} l^{p_n}=l^p$, 2) if $p_n=\psi(a_n, t_n)$, $n=1, 2, \ldots$, $\lim_{n\to\infty} a_n=a$ and $\lim_{n\to\infty} t_n=1$ then $\overline{\lim}\, l_{p_n}=l(a)$ (Here, as also in what follows, the upper limit is considered in the space $\bar{U} \setminus K$).

A subset T of $U \setminus K$ is called a cut with respect to ψ if 1) the set T is a closed of $\bar{U} \setminus K$, 2) for every fibre l the set $l \cap T$ either is empty or is consisted of a unique point, 3) the set $(U \setminus K) \setminus T$ is union of two non-empty open subsets G_1 and G_2 with empty intersection and for which $\mathrm{Fr}\, G_1=\mathrm{Fr}\, G_2=T$ (the frontier is considered with respect to $U \setminus K$).

The cut T is called complete if every fibre has non-empty intersection with the set T.

Obviously for every $t \in J$, the set D_t is a complete cut.

We say that the compact K is of finite type if 1) for all $n=0, 1, 2, \ldots$ but a finite number, the homology groups $H_n(K)$ (Alexandroff-Cech homology) are null and 2) if for some n the group $H_n(K)$ is non-null then $H_n(K)$ has a finite number of generators.

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We say that a compact K of finite type has the fixed point property if for every continuous map $f\colon K\to K$ for which the Lefschetz's number Λ_f is different than zero, there exists $x\in K$ such that $f(x)=x$.

Lemma 1. *Let T be a cut and G_1, G_2 the open sets mentioned in the definition of a cut. Then, for every $p\in T$ either $l_p\subseteq\overline{G}_1$ and $l^p\subseteq\overline{G}_2$ or $l_p\subseteq\overline{G}_2$ and $l^p\subseteq\overline{G}_1$.*

Proof. If $q\in G_1$ then either $l_q\subseteq\overline{G}_1$ or $l^q\subseteq\overline{G}_1$ for if not, then the set $l(q)\cap T$ should contain more than one point. Similarly, for the points of the set G_2.

Since $\operatorname{Fr}G_1=\operatorname{Fr}G_2=T$ (the frontier is considered with respect to $U\setminus K$) we have that every point $p\in T$ must be the limit of a sequence of points of G_1 and G_2. Hence either $l_p\subseteq\overline{G}_1$ and $l^p\subseteq\overline{G}_2$ or $l_p\subseteq\overline{G}_2$ and $l^p\subseteq\overline{G}_1$.

Let T be a cut. We define on the set $\operatorname{Fr}U$ a function φ_T as follows: if $a\in\operatorname{Fr}U$ and $l(a)\cap T=\emptyset$, we set $\varphi_T(a)=1$, and if $l(a)\cap T\neq\emptyset$ then $l(a)\cap T=\{\psi(a,t)\}$, $t\in J$. We set $\varphi_T(a)=t$.

Obviously, for every cut T, we have $0<\varphi_T(a)\leq 1$ for every $a\in\operatorname{Fr}U$.

Lemma 2. *For every cut T the function φ_T is continuous.*

Proof. *Let* $p_n=\psi(a_n,t_n)\in T$, $n=1,2,\ldots$, $\lim_{n\to\infty}a_n=a$ and $\lim_{n\to\infty}t_n=\lim_{n\to\infty}\varphi_T(a_n)=t$. In order to prove the lemma, it suffices to prove that $\varphi_T(a)=t$.

If $t<1$, then, since T is closed in $\overline{U}\setminus K$, we have that $\psi(a,t)\in T$ and hence $\varphi_T(a)=t$.

If $t=1$, then $\overline{\lim}_{n\to\infty}l_{p_n}=l(a)$. By lemma 1, for every $n=1,2,\ldots$, we have either $l_{p_n}\subseteq\overline{G}_1$ or $l_{p_n}\subseteq\overline{G}_2$. Consequently either $l(a)\subseteq\overline{G}_1$ or $l(a)\subseteq\overline{G}_2$. But, by Lemma 1, this means that $l(a)\cap T=\emptyset$ hence $\varphi_T(a)=1$.

Lemma 3. *If T is a complete cut, then T is compact.*

Proof. In order to prove that T is compact it suffices to prove that $\overline{T}^{R_n}\cap K=\emptyset$. For if this is not true then there exists a sequence $p_1,p_2,\ldots,p_k,\ldots$ of points of T such that if $p_k=\psi(a_k,t_k)$ then $\lim_{k\to\infty}a_k=a\in\operatorname{Fr}U$ and $\lim_{k\to\infty}t_k=1$. We can suppose that for all $k=1,2,\ldots$, $l_{p_k}\subseteq\overline{G}_1$ (or $l_{p_k}\subseteq\overline{G}_2$). Let $l=\{\psi(a,t)\colon 0\leq t<1\}$ and $p\in l\cap T$. Then, obviously, $l=l(p)\subseteq\overline{G}_1$ (or $l=l(p)\subseteq\overline{G}_2$) which, because of Lemma 1, is impossible.

Obviously, the cut T is complete if and only if $0<\varphi_T(a)<1$ for every $a\in\operatorname{Fr}U$.

Lemma 4. *If φ is a continuons function of $\operatorname{Fr}U$, $\varphi\neq 1$, such that for every $a\in\operatorname{Fr}U$, $0<\varphi_T(a)\leq 1$, then there exists one and only one cut $T=T_\varphi$ such that $\varphi_T=\varphi$.*

Proof. We set $T_\varphi=\{\psi(a,\varphi(a))\colon a\in\operatorname{Fr}U,\ \varphi(a)<1\}$. Obviously, $T_\varphi\subseteq U\setminus K$. We prove that this set is a cut with respect to ψ.

Let $p\in U\setminus K$, $p_i\in T_\varphi$, $i=1,2,\ldots$ and $\lim_{i\to\infty}p_i=p$. If $p=\psi(a,t)$ and $p_i=\psi(a_i,t_i)$ then, obviously, $0<t<1$, $\lim_{i\to\infty}a_i=a$, $\varphi(a_i)=t_i$ and $\lim_{i\to\infty}t_i=t$. Since φ is a continous function, we have that $\varphi(a)=t$. Hence $p\in T_\varphi$. Thus, the set T_φ is a closed subset of $\overline{U}\setminus K$. Obviously, for every fibre l the set $l\cap T_\varphi$ either is empty or is consisted of a unique point.

We set $G^T=G_1=(\bigcup_{p\in T_\varphi}l^p)\setminus T_\varphi$. Obviously, $G^T=\{\psi(a,t)\in U\setminus K\colon t>\varphi(a)\}$. Hence the set G^T is an open subset of $U\setminus K$. Similarly, the set $G_T=G_2=\{\psi(a,t)$

$\in U \setminus K$: $t<\varphi(a)\}$ is an open subset of $U \setminus K$. We have $G_1 \cup G_2=(U \setminus K)\ T$, $\overline{G}_1=G_1 \cup T$ and $\overline{G}_2=G_2 \cup T$ (the closure is considered with respect to $U \setminus K$).

All the above mean that the set T_φ is a cut with respect to ψ. Obviously, if for some cut T we have $\varphi_T=\varphi$, then $T=T_\varphi$.

Obviously, for every complete cut T, the set $Q=(U \setminus \operatorname{tr} T) \cup T$ is a compact set and the set U is an open neighbourhood of Q in the space R^n. If $T=D_t$, $t \in J$, then $Q=(U \setminus \operatorname{tr} T) \cup T=D_{[t,1]}$.

Lemma 5. *If T is a complete cut then the set $Q=(U \setminus \operatorname{tr} T) \cup T$ is a deformation retract of the set U.*

Proof. We define the map F^Q: $U \times [0, 1] \to U$ by 1) if $p \in U \setminus Q$, $p=\psi(a, t)$ and $s \in [0, 1]$ then $F^Q(p, s)=\psi(a, t+(\varphi_T(a)-t) . s)$, 2) if $p \in Q$ and $s \in [0, 1]$ then $F^Q(p, s)=p$. Obviously the map F^Q is a deformation retraction of U onto Q.

Corollary 1. *The compact set Q is an absolute neighbourhood retract. Therefore, the set Q is compact of finite type and has the fixed point property.*

Obviously, if $T \subseteq D_{[t,1]}$, $0<t<1$, then $F^Q(D_{[t,1]} \times [0, 1]) \subseteq D_{[t,1]}$. The map from $D_{[t,1]} \times [0, 1]$ to $D_{[t,1]}$ which on every point of $D_{[t,1]} \times [0, 1]$ coincides with the map F^Q we, also, denote by F^Q. Hence, F^Q is a deformation retraction of $D_{[t,1]}$ onto Q.

Corollary 2. *If $t, t \in J$ and $t<t'$ then the set $D_{[t',1]}$ is a deformation retract of the set $D_{[t,1]}$.*

Lemma 6. *Let K be a compact subspace of R^n and U an open neighbourhood of K such that there exists an isotopic contraction ψ of $\operatorname{Fr} U$ to K in U. Then K is compact of finite type.*

Proof. It suffices to prove that for every $k=1, 2, \ldots$ there exists an open neighbourhood V_k of K such that: 1) $\overline{V}_{k+1} \subseteq V_k \subseteq U$. 2) $\bigcap_{k=1}^{\infty} \overline{V}_k=K$. 3) the compact $\overline{V}_{k+1}$ is compact of finite type. 4) the homomorphism $(i_k^{k+1})_*$ is an isomorphism onto, where i_k^{k+1} is the embedding of $\overline{V}_{k+1}$ in the set $\overline{V}_k$.

We set $V_k=D_{(k/k+1,1]}$, $k=1, 2, \ldots$. Obviously, $\overline{V}_{k+1} \subseteq V_k \subseteq U$ and $\bigcap_{k=1}^{\infty} \overline{V}_k=K$.

By Corollary 1, of Lemma 5, the subspace $\overline{V}_{k+1}=D_{[k/k+1,1]}$ is compact of finite type. By Corollary 2, of Lema 5, the compact $\overline{V}_{k+1}$ is a deformation retract of the set $\overline{V}_k$. Therefore, the homomorphism $(i_k^{k+1})_*$ is an isomorphism onto.

Theorem. *Let K be a compact subspace of R^n and U an open neighbourhood of K such that there exists an isotopic contraction ψ of $\operatorname{Fr} U$ to K in U. Let, further, that for every $\varepsilon>0$ there exists $\delta>0$ such that for every $k \in U \setminus K$ with $d(p, K)<\delta$ there exists a cut T of $U \setminus K$ with respect to homeomorphism ψ and having the following properties:* 1) $T \cap l(p) \neq \varnothing$, 2) $\operatorname{diam} T<\varepsilon$, 3) $d(p, T)<\varepsilon$ *and* 4) *if* $q \in \operatorname{tr} T$ *then* $d(q, l_p)<\varepsilon$. *Then for every continuous map f: $K \to K$ whose Lefschetz's number is different than zero either there exists a fixed point or there exists a fibre l such that $f(\overline{l}^{R^n} \setminus l) \subseteq \overline{l}^{R^n} \setminus l$.*

Proof. Let f be a continuous map of K in K, whose $\Lambda_f \neq 0$ and for which does not exist a fixed point. We prove that there exists a fibre l such that $f(\overline{l}^{R^n} \setminus l) \subseteq \overline{l}^{R^n} \setminus l$.

Let V_k, $k=1, 2, \ldots$ be neighbourhoods of K constructed as in Lemma 6. There exists $k_0>1$ such that the map f can be extended in a map $\bar{f}$ of $\bar{V}_{k_0}$ into $\bar{V}_1$ and for which, also does not exist a fixed point.

A set Φ of $\bar{V}_{k_0}\setminus K$ is said to be marked if it has the following properties: 1) if $x\in\Phi$ then $\bar{f}(x)\in l_x$, 2) the set Φ is closed in the space $\bar{V}_{k_0}\setminus K$, 3) for every complete cut T which is contained in $\bar{V}_{k_0}$, the set $T\cap\Phi$ is non-empty.

The set Φ of all the points $x\in V_{k_0}\setminus K$ for which $\bar{f}(x)\in l_x$, is marked. In fact, if $a_i\in\Phi$, $i=1, 2, \ldots$ and $\lim_{i\to\infty} a_i=a\in\bar{V}_{k_0}\setminus K$ then obviously $\bar{f}(a)\in l_a$ hence $a\in\Phi$ and the set Φ is a closed subspace of $\bar{V}_{k_0}\setminus K$.

Let T be a complete cut and $T\subseteq\bar{V}_{k_0}$. The map F^Q is a deformation retraction of $\bar{V}_1$ onto the set $Q=(U\setminus \mathrm{tr}\, T)\cup T$. By Corollary 1 of Lemma 5, the compact Q is a compact of finite type and has the fixed point property. Let F_1^Q be the map from $\bar{V}_1$ to Q for which $F_1^Q(p)=F^Q(p, 1)$, for every point $p\in\bar{V}_1$. We observe that the Lefschetz's number Λ_g of the map $g=F_1^Q\circ\bar{f}|_Q$ is equal to the Lefschetz's number Λ_f of the map f. Therefore, the map g has a fixed point. Since $F_1^Q(\bar{V}_1\setminus Q)\subseteq T$, if $(F_1^Q\circ\bar{f}|_Q)(p)=p$, then it holds that $p\in T$ and $\bar{f}(p)\in l_p$. Hence $p\in\Phi$ and $\Phi\cap T\neq\emptyset$. All the above mean that the set Φ is marked set.

A marked set Φ is called minimal marked if every marked subset Φ' of Φ coincides with Φ. By the definition of a marked set and Lemma 3, it follows that the intersection of a transfinite decreasing sequence of marked sets, is a marked set. Therefore the existence of a marked set implies the existence of a minimal marked set.

Let Φ_0 be a minimal marked set and $\bar{p}_1, \bar{p}_2, \ldots$ a sequence of Φ_0 such that $\lim_{i\to\infty}\bar{p}_i=\bar{p}\in K$. We can suppose that if $\bar{p}_i=\psi(a_i, t_i)$ then the sequence $a_1, a_2, \ldots$ converges to a point $a\in \mathrm{Fr}\, U$ and $\lim_{i\to\infty} t_i=1$.

We prove that the fibre $l=\{\psi(a, t): 0\leq t<1\}$ is the required one, that is $f(\bar{l}^{R^n}\setminus l)\subseteq\bar{l}^{R^n}\setminus l$.

Let $p\in\bar{l}^{R^n}\setminus l$ and $p_1, p_2, \ldots$ be a sequence of points of l such that $\lim_{i\to\infty} p_i=p$.

We prove that $f(p)\in\bar{l}^{R^n}\setminus l$. Let $\varepsilon>0$ be an arbitrary number and $\delta>0$ the number corresponding to the hypothesis of the theorem. There exists an integer i_1 such that $d(p_{i_1}, K)<\delta$. For the point p_{i_1}, there exists a cut T satisfying all the conditions mentioned in the theorem. We will prove that $T\cap\Phi_0\neq\emptyset$.

Suppose the contrary, that is, $T\cap\Phi_0=\emptyset$. Let D be a complete cut such that $D\subseteq V_{k_0}$. Consider the complete cut $\min(T, D)$. By the definition of the set Φ_0 we have that $\min(T, D)\cap\Phi_0\neq\emptyset$. Since $T\cap\Phi_0=\emptyset$ and $\min(T, D)\subseteq T\cup D\setminus G^T)$ we will have $(D\setminus G^T)\cap\Phi_0=D\cap(\Phi_0\setminus C^T)\neq\emptyset$. Let $\Phi'=\Phi_0\setminus G^T$.

By the construction of l and T there exists a number i such that $\bar{p}_i\in G^T$. Hence the set Φ' is a proper subset of Φ_0, closed in $\bar{U}\setminus K$ and as we, above, proved $D\cap\Phi'\neq\emptyset$ for every complete cut D. This means that Φ' is a marked set. But this is impossible because we supposed that Φ_0 is a minimal marked set.

Therefore, there exists a point $q_1' \in T \cap \Phi_0$. Let $q_1'' = \bar{f}(q_1')$. Obviously $q_1'' \in \mathrm{tr} T$. By the suppositions of the theorem there exists $q_1 \in l$ such that $d(q_1, q_1'') < \varepsilon$.

Thus, we are able to construct: 1) a subsequence $p_{i_1}, p_{i_2}, \ldots, p_{i_k}, \ldots$ of the sequence $p_1, p_2, \ldots$, 2) a sequence of points $q_1', q_2', \ldots, q_k', \ldots$ with the property $d(p_{i_k}, q_k') < 1/k$, $k = 1, 2, \ldots$ and 3) a sequence $q_1, q_2, \ldots, q_k$ of points of the fibre l such that $d(q_k, \bar{f}(q_k')) < 1/k$, $k = 1, 2, \ldots$. Obviously $\lim_{i\to\infty} p_{i_k} = p$, $\lim_{k\to\infty} p_{i_k} = \lim_{k\to\infty} q_k' = p$, $\lim_{k\to\infty} \bar{f}(q_k') = \lim_{k\to\infty} q_k = \lim_{k\to\infty} \bar{f}(p_{i_k}) = \bar{f}(p)$. Since $q_k \in l$ we have $f(p) \in \bar{l}^{R^n} \setminus l$. This proves the theorem.

A compact K is called an isotopic neighbourhood pseudo-retract if there exist 1) an embedding of K in R^n, for some n, 2) an open neighbourhood U of K for which there exists an isotopic contraction ψ of $\mathrm{Fr}\, U$ to K in U such that ψ satisfies α) all the conditions mentioned in the theorem and β) for every fibre l we have that the set $\bar{l}^{R^n} \setminus l$ consists of a single point.

Corollary. *Every isotopic neighbourhood pseudo-retract K has the fixed point property.*

Remark. We can prove that:

1. In the plane R^2, every compact K of finite type has a neighbourhood U for which all the conditions of the theorem are satisfied.

2. Every compact $K \subseteq R^2$ of finite type is an isotopic neighbourhood pseudo-retract if every prime end (see, for example [1]) is of first or second type.

3. The examples 3, 4, 5 and 7 of [2] are isotopic neighbourhood pseudo-retracts.

Problem. Which is the relation between the acyclic isotopic neighbourhood pseudo-retracts and the deformation quasi-retracts? (see [2])

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