Serdica
Bulgariacae mathematicae publicationes

Сердика
Българско математическо списание

ON THE STABILITY OF SOLUTIONS OF FIRST ORDER PARTIAL DIFFERENTIAL-FUNCTIONAL EQUATIONS

ZDZISŁAW KAMONT

Let $C(E_0 \cup E, R)$ be a class of continuous functions from $E_0 \cup E$ into R, where $R=(-\infty, +\infty)$ and

$$E_0=\{(x, y): x_0-\tau_0 \leq x \leq x_0, \tau_0 \geq 0, \quad y=(y_1, \dots, y_n),$$

$$\bar{r}_i(x) \leq y_i \leq \bar{s}_i(x), \quad i=1, \dots, n\},$$

$$E=\{(x, y): x \in [x_0, +\infty), \tilde{r}_i(x) \leq y_i \leq \tilde{s}_i(x), \quad i=1, \dots, n\}.$$

Let $C(E_0 \cup E, R^m)$ be a set of all vector functions $u(\cdot)=(u^{(1)}(\cdot), \dots, u^{(m)}(\cdot))$, where $u^{(i)}(\cdot) \in C(E_0 \cup E, R)$.

Assume that

(i) $\bar{r}_i$ and $\bar{s}_i$, $i=1, \dots, n$, are continuous functions on $[x_0-\tau_0, x_0]$ and $\bar{r}_i(x) \leq \bar{s}_i(x)$, $x \in [x_0-\tau_0, x_0]$,

(ii) $\tilde{r}_i$ and $\tilde{s}_i$, $i=1, \dots, n$, are of class C^1 on $[x_0, +\infty)$ and $\tilde{r}_i(x) < \tilde{s}_i(x)$ for $x \in [x_0, +\infty)$, $\tilde{r}_i(x_0)=\bar{r}_i(x_0)$, $\tilde{s}_i(x_0) = \bar{s}_i(x_0)$.

Suppose that functions $f^{(i)}$, $i=1, \dots, m$, of the variables $(x, y, z, u(\cdot), q)$, where $z=(z^{(1)}, \dots, z^{(m)}), u(\cdot)=(u^{(1)}(\cdot), \dots, u^{(m)}(\cdot))$, $q=(q_1, \dots, q_n)$ are defined on $E \times R^m \times C(E_0 \cup E, R^m) \times R^n$. In this paper we consider the initial differential-functional problem

$$(1) \quad z_x^{(i)}(x, y)=f^{(i)}(x, y, z(x, y), z(\cdot), z_y^{(i)}(x, y)), \ (x, y) \in E, \quad i=1, \dots, m,$$

$$z^{(i)}(x, y)=\varphi^{(i)}(x, y), \ (x, y) \in E_0,$$

where $\varphi=(\varphi^{(1)}, \dots, \varphi^{(m)})$ is a given function defined on E_0 and $z(\cdot)=(z^{(1)}(\cdot), \dots, z^{(m)}(\cdot))$, $z(x, y)=(z^{(1)}(x, y), \dots, z^{(m)}(x, y))$, $z_y^{(i)}(x, y)=(z_{y_1}^{(i)}(x, y), \dots, z_{y_n}^{(i)}(x,y))$. We assume that the function $f=(f^{(1)}, \dots, f^{(m)})$ satisfies the following Volterra condition: if $(x, y, z, u(\cdot), q)$, $(x, y, z, v(\cdot), q) \in E \times R^m \times C(E_0 \cup E, R^m) \times R^n$ and $u(\xi, \eta)=v(\xi, \eta)$ for $(\xi, \eta) \in H_x$, where $H_x=\{(\xi, \eta) \in E_0 \cup E: \xi \leq x\}$, then $f(x, y, z, u(\cdot), q)=f(x, y, z, v(\cdot), q)$. We assume that the problem (1) for $\varphi(x, y)=0$ on E_0 possesses a solution $z(x, y)=0$ on $E_0 \cup E$.

We give here some theorems on stability and asymptotic stability of the trivial solution of (1). This will be a generalization of the results published in [6, Chapter 9].

In this paper, we study the stability and asymptotic stability of solutions of (1) by means of Lapunov functions and the theory of differential-functional inequalities.

We assume the existence of solutions of (1) on $E_0 \cup E$. Some results concerning the existence and uniqueness of solutions can be found in [1—5; 7; 9]. For a more detailed information and references see [5].

SERDICA Bulgaricae mathematicae publicationes. Vol. 9, 1983, p. 335—342.

We adopt the following definitions of stability and asymptotic stability of the trivial solution of (1).

The trivial solution $z(x, y)=0$, $(x, y)\in E_0\cup E$, of (1) is said to be stable if for every $\varepsilon>0$ there exists a $\delta>0$ such that $\|z(x, y)\|<\delta$ on E_0 implies $\|z(x, y)\|<\varepsilon$ on E. (We denote by $\|\cdot\|$ the norm in R^m.)

The trivial solution $z(x, y)=0$, $(x, y)\in E_0\cup E$, of (1) is said to be asymptotically stable if

(i) it is stable,

(ii) there exists a positive number δ_0 such, that for every $\varepsilon>0$ there corresponds $T(\varepsilon)$ such that $\|z(x, y)\|<\delta_0$ on E_0 implies $\|z(x, y)\|<\varepsilon$ for $x\geq x_0+T(\varepsilon)$, $\tilde{r}(x)\leq y\leq\tilde{s}(x)$.

We introduce the following class of solutions of (1). Let $S=\{(x, y)\in E:$ there exists j, $1\leq j\leq n$, such that $y_j=\tilde{r}_j(x)$ or $y_j=\tilde{s}_j(x)\}$.

A real function u of the variables (x, y) will be called the function of class D in $E_0\cup E$ if u is continuous on $E_0\cup E$, possesses partial derivatives u_x, $u_y=(u_{y_1},\ldots,u_{y_n})$, in an interior of E and total derivative on S.

In this paper we consider solutions $z(\cdot)=(z^{(1)}(\cdot),\ldots,z^{(m)}(\cdot))$ of (1) such that $z^{(i)}(\cdot)$ are of class D in $E_0\cup E$.

We introduce

Assumption H_1. Suppose that

1° σ is a continuous function of the variables $(t, z, w(\cdot))$ defined on $R_+\times R_+\times C_0([-\tau_0, +\infty), R_+)$, where $C_0([-\tau_0, +\infty), R_+)$ is a set of all continuous, bounded and non-negative functions on $[-\tau_0, +\infty)$.

2° σ is non-decreasing with respect to the functional argument and satisfies the following Volterra condition: if $w(\cdot)$, $\bar{w}(\cdot)\in C_0([-\tau_0, +\infty), R_+)$ and $w(\tau)=\bar{w}(\tau)$ for $\tau\in[-\tau_0, t]$, then $\sigma(t, z, w(\cdot))=\sigma(t, z, \bar{w}(\cdot))$. We define $S_t=\{y:(x_0+t, y)\in E_0\cup E\}$, $t\in[-\tau_0, +\infty)$. We shall denote a function w of the variable t for t belonging to some interval (α, β) by the symbol $w(\cdot)$ or $(w(\tau))_{(\alpha,\beta)}$.

If $u(\cdot)\in C(E_0\cup E, R)$ and $u_y(x, y)=(u_{y_1}(x, y),\ldots,u_{y_n}(x, y))$ exists, then we define $|u_y(x, y)|=(|u_{y_1}(x, y)|,\ldots,|u_{y_n}(x, y)|)$.

Assumption H_2. Suppose that

1° g is a function of the variables $(x, y, z, u(\cdot), q)$ defined on $E\times R_+\times C(E_0\cup E, R_+)\times R^n$, where $C(E_0\cup E, R_+)$ is a set of all continuous and non-negative functions on $E_0\cup E$,

2° g satisfies the Volterra condition, i. e. if $u(\cdot)$, $v(\cdot)\in C(E_0\cup E, R_+)$ and $u(\xi, \eta)=v(\xi, \eta)$ for $(\xi, \eta)\in H_x$, then $g(x, y, z, u(\cdot), q)=g(x, y, z, v(\cdot), q)$,

3° for each point $(x, y)\in E$ there exist sets of integers I_1, I_2, I_3 such that $I_1\cup I_2\cup I_3=\{1,\ldots,n\}$ and $y_j=\tilde{r}_j(x)$ for $j\in I_1$, $y_j=\tilde{s}_j(x)$ for $j\in I_2$, $\tilde{r}_j(x)<y_j<\tilde{s}_j(x)$ for $j\in I_3$, we assume that

$$g(x, y, z, u(\cdot), q)+\sum_{j\in I_1}\tilde{r}_j'(x)q_j+\sum_{j\in I_2}\tilde{s}_j'(x)q_j$$

$$\leq\sigma(x-x_0, z, (\max_{y\in S_t}u(x_0+t, y))_{[-\tau_0, x-x_0]})$$

where $q=(q_1,\ldots,q_n)$, $q_j\leq0$ for $j\in I_1$, $q_j\geq0$ for $j\in I_2$, $q_j=0$ for $j\in I_3$ and $u(\cdot)\in C(E_0\cup E, R_+)$.

We start with the following comparison lemma. This lemma is a some modification of Theorem 1.1 of [5].

Lemma 1. *Suppose, that*

1^0 *Assumptions* H_1 *and* H_2 *are satisfied,*

2^0 *the function* $\bar{u}(\cdot) \in C(E_0 \cup E, R_+)$ *of the variables* (x, y) *is of class* D *in* $E_0 \cup E$,

3^0 *the differential-functional inequality*

$$\bar{u}_x(x, y) \leq g(x, y, \bar{u}(x, y), \bar{u}(\cdot), u_y(x, y)), \ (x, y) \in E, \tag{2}$$

and the initial inequality

$$\bar{u}(x, y) \leq \eta(x - x_0), \quad (x, y) \in E_0, \tag{3}$$

are satisfied.

4^0 the function η is continuous on $[-\tau_0, 0]$ and the maximum solution $\omega(t;\eta)$ of the initial problem

$$\begin{aligned} w'(t) &= \sigma(t, w(t), w(\cdot)) \\ w(t) &= \eta(t) \ \text{for} \ t \in [-\tau_0, 0] \end{aligned} \tag{4}$$

is defined on $[-\tau_0, +\infty)$.

Under these assumptions

$$\bar{u}(x, y) \leq \omega(x - x_0; \eta) \tag{5}$$

for $(x, y) \in E$.

P r o o f. Let us define

$$\bar{w}(t) = \max_{y \in S_t} \bar{u}(x_0 + t, y), \quad t \in [-\tau_0, +\infty).$$

The function $\bar{w}(\cdot)$ is continuous on $[-\tau_0, +\infty)$ (see [8, Chapter 6]) and the estimation (5) on E is equivalent with

$$\bar{w}(t) \leq \omega(t;\eta) \tag{6}$$

for $t \in [0, +\infty)$.

We prove (6) for $t \in [0, a)$, where $a > 0$. For $\varepsilon > 0$, denote by $\omega(t; \eta, \varepsilon)$, the maximum solution of the problem

$$w'(t) = \sigma(t, w(t), w(\cdot)) + \varepsilon, \ w(t) = \eta(t) + \varepsilon \ \text{for} \ t \in [-\tau_0, 0].$$

For $\varepsilon > 0$ sufficiently small, $\omega(t; \eta, \varepsilon)$ is defined on $[-\tau_0, a)$ and

$$\lim_{\varepsilon \to \infty} \omega(t; \eta, \varepsilon) = \omega(t; \eta) \ \text{on} \ [-\tau_0, a).$$

To prove (6), it is sufficient to show that

$$\bar{w}(t) < \omega(t; \eta, \varepsilon) \tag{7}$$

for $t \in [0, a)$. Now we will prove (7) using Lemma 1.1 of [5].

It follows from (3) that $\bar{w}(t) < \omega(t; \eta, \varepsilon)$ for $t \in [-\tau_0, 0]$. Suppose, that for some $\tilde{t} \in (0, a)$ we have $\bar{w}(\tilde{t}) = \omega(\tilde{t}; \eta, \varepsilon)$ and $\bar{w}(t) \leq \omega(t; \eta, \varepsilon)$ for $t \in [0, \tilde{t}]$. It follows from the definition of $\bar{w}(\cdot)$ that $\bar{w}(\tilde{t}) \geq 0$ and that there is an $\tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_n)$ such that $\bar{w}(\tilde{t}) = \bar{u}(\tilde{x}, \tilde{y})$, where $\tilde{x} = x_0 + \tilde{t}$. Suppose that I_1, I_2, I_3 are sets of integers such that $I_1 \cup I_2 \cup I_3 = \{1, \ldots, n\}$ and $\tilde{y}_j = \tilde{r}_j(\tilde{x})$ for $j \in I_1$, $\tilde{y}_j = \tilde{s}_j(\tilde{x})$ for $j \in I_2$, $\tilde{r}_j(\tilde{x}) < \tilde{y}_j < \tilde{s}_j(\tilde{x})$ for $j \in I_3$. Then we have

(8) $\bar{u}_{y_i}(\tilde{x}, \tilde{y}) \leq 0$ for $i \in I_1$, $\bar{u}_{y_i}(\tilde{x}, \tilde{y}) \geq 0$ for $i \in I_2$, $\bar{u}_{y_i}(\tilde{x}, \tilde{y}) = 0$ for $i \in I_3$.

Let $\bar{y}(x) = (\bar{y}_1(x), \ldots, \bar{y}_n(x))$, where

$$\bar{y}_i(x) = \begin{cases} \tilde{r}_i(x) & \text{for } i \in I_1, \\ \tilde{s}_i(x) & \text{for } i \in I_2, \\ \tilde{y}_i & \text{for } i \in I_3, \end{cases}$$

and consider the composite function $\bar{u}(x, \bar{y}(x))$, $x \in [x_0, x_0 + a)$. We have $\bar{u}(x_0 + \tilde{t}, \bar{y}(x_0 + \tilde{t})) = \bar{w}(\tilde{t})$ and $\bar{u}(x_0 + \tau, \bar{y}(x_0 + \tau)) \leq \bar{w}(\tau)$ for $\tau \in [0, \tilde{t}]$ and, therefore $D_- \bar{w}(\tilde{t}) \leq D_- [\bar{u}(x_0 + t, \bar{y}(x_0 + t))]\,|_{t=\tilde{t}}$. ($D_- \varphi(t)$ is the left-hand lower Dini's derivative of φ at the point t).

From Assumption H_2 it follows that

$$D_- \bar{w}(\tilde{t}) \leq \bar{u}_x(\tilde{x}, \tilde{y}) + \sum_{i \in I_1} \bar{u}_{y_i}(\tilde{x}, \tilde{y})\, \tilde{r}_i'(\tilde{x}) + \sum_{i \in I_2} \bar{u}_{y_i}(\tilde{x}, \tilde{y})\, \tilde{s}_i'(\tilde{x})$$

$$\leq g(\tilde{x}, \tilde{y}, \bar{u}(\tilde{x}, \tilde{y}), \bar{u}(\cdot), \bar{u}_y(\tilde{x}, \tilde{y})) + \sum_{i \in I_1} \bar{u}_{y_i}(\tilde{x}, \tilde{y})\, \tilde{r}_i'(\tilde{x}) + \sum_{i \in I_2} \bar{u}_{y_i}(\tilde{x}, \tilde{y})\, \tilde{s}_i'(\tilde{x})$$

$$< \sigma(\tilde{t}, \bar{w}(\tilde{t}), \bar{w}(\cdot)) + \varepsilon.$$

It therefore follows that the assumption $\bar{w}(\tilde{t}) = \omega(\tilde{t}; \eta, \varepsilon)$ and $\bar{w}(t) \leq \omega(t; \eta, \varepsilon)$ for $t \in [0, \tilde{t}]$ implies the differential inequality $D_- \bar{w}(\tilde{t}) < \sigma(\tilde{t}, \bar{w}(\tilde{t}), \bar{w}(\cdot)) + \varepsilon$. Since $\bar{w}(t) < \omega(t; \eta, \varepsilon)$ for $t \in [-\tau_0, 0]$, it follows that all the assumptions of Lemma 1.1 [5] are satisfied. From this Lemma we obtain (7) in $[0, a]$. Since a is arbitrary, inequality (7) holds true in $[0, +\infty)$. From (7) we obtain in the limit (letting ε tend to 0) inequality (6). This completes the proof.

Theorem 1. *Suppose that*

1° *Assumptions* H_1 *and* H_2 *are satisfied,*

2° V *is a function of the variables* (x, u), $u = (u_1, \ldots, u_m)$, *defined on* $[x_0, +\infty) \times R^m$,

3° V *possesses continuous partial derivatives with respect to* (x, u) *and for* $u(\cdot) = (u^{(1)}(\cdot), \ldots, u^{(m)}(\cdot))$ *such that* $u^{(i)}(\cdot)$ *are of class D in* $E_0 \cup E$ *we have*

$$\frac{\partial V(x, u(x, y))}{\partial x} + \frac{\partial V(x, u(x, y))}{\partial u} \cdot f(x, y, u(x, y), u(\cdot), u_y(x, y))$$

$$\leq g\Big(x, y, V^{(u)}(x, y), V^{(u)}(\cdot), \frac{\partial V^{(u)}(x, y)}{\partial y}\Big), \quad (x, y) \in E,$$

where

$$\frac{\partial V(x, u(x, y))}{\partial u} \cdot f(x, y, u(x, y), u(\cdot), u_y(x, y))$$

$$= \sum_{j=1}^{m} \frac{\partial V(x, u(x, y))}{\partial u_j} f^{(j)}(x, y, u(x, y), u(\cdot), u_y^{(j)}(x, y)),$$

$V^{(u)}$ *is a function defined by*

(9) $$V^{(u)}(x, y) = V(x, u(x, y))$$

and

$$\frac{\partial V^{(u)}(x, y)}{\partial y} = \left(\frac{\partial V^{(u)}(x, y)}{\partial y_1}, \ldots, \frac{\partial V^{(u)}(x, y)}{\partial y_n}\right),$$

4^0 *the maximum solution* $\omega(\cdot\,; \eta)$ *of the initial problem* (4) *exists for* $t \geq 0$,

5^0 $\bar{v}(\cdot) = (\bar{v}^{(1)}(\cdot), \ldots, \bar{v}^{(m)}(\cdot))$ *is a solution of* (1) *defined in* $E_0 \cup E$ such that $\bar{v}^{(i)}(\cdot)$ *are of class D on* $E_0 \cup E$ *and* $V(x, \bar{v}(x, y)) \leq \eta(x - x_0)$ *for* $(x, y) \in E_0$.

Under these assumptions $V(x, \bar{v}(x, y)) \leq \omega(x - x_0\,; \eta)$ for $(x, y) \in E$.

Proof. Consider the function $\bar{z}(x, y) = V(x, \bar{v}(x, y))$ for $(x, y) \in E_0 \cup E$. By assumption 3^0 we have $\bar{z}_x(x, y) \leq g(x, y, \bar{z}(x, y), \bar{z}(\cdot), \bar{z}_y(x, y))$, $(x, y) \in E$, and $\bar{z}(x, y) \leq \eta(x - x_0)$ for $(x, y) \in E_0$. It is evident that the hypotheses of Lemma 1 are fulfilled and, as a result $\bar{z}(x, y) \leq \omega(x - x_0\,; \eta)$ on E. The proof is therefore completed.

Now we give theorems on stability of the trivial solution of (1).

Theorem 2. *Suppose that*

1^0 *Assumptions* H_1 *and* H_2 *are satisfied,*

2^0 *the conditions* 2^0 *and* 3^0 *of Theorem* 1 *hold and the differential-functional equation*

$$w'(t) = \sigma(t, w(t), w(\cdot)), \quad t \in [0, +\infty), \tag{10}$$

has the trivial solution on $[-\tau_0, +\infty)$,

3^0 α *and* β *are continuous, strictly increasing and non-negative functions on* $[0, +\infty)$,

4^0 $\alpha(0) = \beta(0) = 0$ *and for* $u \in R^m$ *we have* $\beta(\|u\|) \leq V(t, u) \leq \alpha(\|u\|)$. *Under these assumptions the stability or asymptotic stability of the trivial solution of* (10) *implies the stability or asymptotic stability of the trivial solution of the system* (1).

Proof. Suppose that the trivial solution of (10) is stable. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that inequality $0 < w(t) < \delta$ on $[-\tau_0, 0]$ implies $w(t) < \beta(\varepsilon)$ for $t \geq 0$, where $w(\cdot)$ is any solution of (10). Choose a positive number δ_1 such that $\alpha(\delta_1) = \delta$ and assume that $\|u(x, y)\| < \delta_1$ on E_0, where $u(\cdot)$ is a solution of (1). This implies that $V(x, u(x, y)) \leq \alpha(\|u(x, y)\|) \leq \alpha(\delta_1) = \delta$ for $(x, y) \in E_0$. Choose

$$\eta(t) = \max_{y \in S_t} V(x_0 + t, u(x_0 + t, y)), \quad t \in [-\tau_0, 0].$$

It then follows by Theorem 1 that $V(x, u(x, y)) \leq \omega(x - x_0\,; \eta)$ for $(x, y) \in {}'E$, where $\omega(\cdot\,; \eta)$ is the maximum solution of the initial problem (4). Since $\eta(t) < \delta$, then $\omega(x - x_0\,; \eta) < \beta(\varepsilon)$ for $x \geq x_0$ and $\beta(\|u(x, y)\|) \leq V(x, u(x, y)) \leq \omega(x - x_0\,; \eta) < \beta(\varepsilon)$ on E, which leads to the inequality $\|u(x, y)\| < \varepsilon$ on E, provided that $\|u(x, y)\| \leq \delta$, on E_0. This proves the stability of the trivial solution of (1).

Now suppose that the trivial solution of (10) is asymptotically stable. Then there exists δ_0 such that for each $\varepsilon > 0$ there corresponds $T(\varepsilon)$ such that $|w(t)| < \delta_0$ for $t \in [-\tau_0, 0]$ implies $|w(t)| < \beta(\varepsilon)$ for $t \in [T(\varepsilon), +\infty)$]. We choose $\eta(t) = \max_{y \in S_t} V(x_0 + t, u(x_0 + t, y))$, $t \in [-\tau_0, 0]$. Let $\alpha(\tilde{\delta}_0) = \delta_0$ and assume that $\|u(x, y)\| \leq \tilde{\delta}_0$ for $(x, y) \in E_0$. These considerations show that, as previously,

$$\beta(\|u(x, y)\|) \leq V(x, u(x, y)) \leq \omega(x - x_0\,; \eta) < \beta(\varepsilon) \quad \text{for } x \geq x_0 + T,$$

$\tilde{r}(x) \leq y \leq \tilde{s}(x)$, provided that $\|u(x, y)\| \leq \tilde{\delta}_0$ for $(x, y) \in E_0$. This assures the asymptotic stability of the trivial solution of (1).

In the above theorem we obtain the stability (or asymptotic stability) of the solution $z(x, y) = 0$ on $E_0 \cup E$ of (1) as a consequence of the stability (or asymptotic stability) of the trivial solution of some ordinary differential-functional equation. In the next theorem the stability of solutions of (1) will be a consequence of the stability of some partial differential-functional equations.

We introduce

Assumption H_3. Suppose that

1^0 g is a function of the variables $(x, y, z, u(\cdot), q)$ defined on $E \times R_+ \times C(E_0 \cup E, R_+) \times R^n$,

2^0 g satisfies the Volterra condition and is non decreasing with respect to the functional argument,

3^0 for each point $(x, y) \in S$ there exist sets of integers I_1, I_2, I_3 such that $I_1 \cup I_2 \cup I_3 = \{1, \ldots, n\}$ and $y_i = \tilde{r}_i(x)$ for $i \in I_1$, $y_i = \tilde{s}_i(x)$ for $i \in I_2$, $\tilde{r}_i(x) < y_i < \tilde{s}_i(x)$ for $i \in I_3$, we assume that

$$g(x, y, z, u(\cdot), q) - g(x, y, z, u(\cdot), \bar{q}) \leq - \sum_{i \in I_1} \tilde{r}'_i(x)(q_i - \bar{q}_i) - \sum_{i \in I_2} \tilde{s}'_i(x)(q_i - \bar{q}_i),$$

where $q = (q_1, \ldots, q_n)$ $\bar{q} = (\bar{q}_1, \ldots, \bar{q}_n)$ and $q_i \leq \bar{q}_i$ for $i \in I_1$, $q_i \geq \bar{q}_i$ for $i \in I_2$, $q_i = \bar{q}_i$ for $i \in I_3$,

4^0 if $0 \leq z \leq \bar{z}$ and $0 \leq u(\xi, \eta) \leq v(\xi, \eta)$ on H_x, then

$$g(x, y, z, u(\cdot), q) - g(x, y, \bar{z}, v(\cdot), q) \geq - \sigma(x - x_0, \bar{z} - z, (\max_{y \in S_t} [v(x_0 + t, y) - u(x_0 + t, y)])_{[-\tau_0, x - x_0]}).$$

The following lemma is a simple generalization of Theorem 3.2 of [5]

Lemma 2. *Suppose that*

1^0 *Assumption* H_3 *is satisfied,*

2^0 *the function* σ *satisfies* Assumption H_1 *and the right-hand maximum solution of the initial problem*

$$w'(t) = \sigma(t, w(t), w(\cdot)), \quad t \in [0, +\infty), \ w(t) = 0 \text{ for } t \in [-\tau_0, 0], \tag{11}$$

is $w(t) \equiv 0$,

3^0 $\bar{u}(\cdot), \bar{v}(\cdot) \in C(E_0 \cup E, R_+)$ *are of class D in* $E_0 \cup E$ *and for* $(x, y) \in E$

$$\bar{u}_x(x, y) \leq g(x, y, \bar{u}(x, y), \bar{u}(\cdot), \bar{u}_y(x, y)),$$

$$\bar{v}_x(x, y) \geq g(x, y, \bar{v}(x, y), \bar{v}(\cdot), \bar{v}_y(x, y)),$$

4^0 *for* $(x, y) \in E_0$ *we* have $\bar{u}(x, y) \leq \bar{v}(x, y)$. *Under these assumptions*

$$\bar{u}(x, y) \leq \bar{v}(x, y) \quad \textit{for} \quad (x, y) \in E. \tag{12}$$

Proof. At first we prove (12) for $(x, y) \in \{(x, y) \in E : x_0 \leq x < x_0 + a, \ a > 0\}$ For $\varepsilon > 0$ denote by $\omega(t; \varepsilon)$ the right-hand side maximum solution of the problem

$$w'(t) = \sigma(t, w(t), w(\cdot)) + \varepsilon, \ w(t) = \varepsilon \text{ for } t \in [-\tau_0, 0].$$

For $\varepsilon > 0$ sufficiently small $\omega(t; \varepsilon)$ is defined on $[0, a)$ and

$$\lim_{\varepsilon\to 0} \omega(t;\,\varepsilon)=0 \text{ on } [0,\ a]. \tag{13}$$

Consider now the function

$$\tilde{v}(x,\ y)=\bar{v}(x,\ y)+\omega(x-x_0;\ \varepsilon). \tag{14}$$

We shall prove that

$$\bar{u}(x,\ y)<\tilde{v}(x,\ y) \text{ on } \{(x,\ y)\in E:\ x_0\leq x<x_0+a\}. \tag{15}$$

By assumption 3^0 and (14) we have

$$\tilde{v}_x(x,\ y)\geq g(x,\ y,\ \bar{v}(x,\ y),\ \bar{v}(\cdot),\ \tilde{v}_y(x,\ y))+\omega'(x-x_0;\ \varepsilon)$$

$$+[g(x,\ y,\ \bar{v}(x,\ y),\ \bar{v}(\cdot),\tilde{v}_y(x,\ y))-g(x,\ y,\ \tilde{v}(x,\ y),\ \tilde{v}(\cdot),\ \tilde{v}_y(x,\ y))]$$

$$\geq g(x,\ y,\ \tilde{v}(x,\ y),\ \tilde{v}(\cdot),\ \tilde{v}_y(x,\ y))-\sigma(x-x_0,\ \tilde{v}(x,\ y)-\bar{v}(x,y),(\max_{y\in S_t}[\tilde{v}(x_0+t,\ y),$$

$$-\bar{v}(x_0+t,\ y)])_{[-\tau_0,\ x-x_0]})$$

$$+\sigma(x-x_0,\ \omega(x-x_0;\ \varepsilon),\ (\omega(\tau,\ \varepsilon))_{[-\tau_0,\ x-x_0]})+\varepsilon$$

$$=g(x,\ y,\ \tilde{v}(x,\ y),\ \tilde{v}(\cdot),\ \tilde{v}_y(x,\ y))+\varepsilon.$$

It follows from the above estimations that

$$\tilde{v}_x(x,\ y)>g(x,\ y,\ \tilde{v}(x,\ y),\ \tilde{v}(\cdot),\ \tilde{v}_y(x,\ y))$$

on $\{(x,\ y)\in E:\ x\in[x_0,\ x_0+a)\}$. By the definition of $\tilde{v}(\cdot)$ we have $\bar{u}(x,\ y)<\tilde{v}(x,\ y)$ for $(x,\ y)\in E_0$. From the strong partial differential-functional inequalities theorem (see [5, Th. 1.2]) we obtain (15). By (13) we obtain in the limit (letting ε tend to 0 in (15)) inequality (12) on $\{(x,\ y)\in E:\ x\in[x_0,\ x_0+a)\}$. Since $a>0$ is arbitrary, inequality (12) holds true in E.

Theorem 3. *Suppose that*

1^0 *Assumptions* H_1 *and* H_3 *are satisfied,*

2^0 *the right-hand maximum solution of the initial problem* (11) *is* $w(t)=0$, $t\in[0,\ +\infty)$, *and the initial problem*

$$\begin{aligned} &z_x(x,\ y)=g(x,\ y,\ z(x,\ y),\ z(\cdot),\ z_y(x,\ y)), \quad (x,\ y)\in E,\\ &z(x,\ y)=0 \quad \text{for} \quad (x,\ y)\in E_0, \end{aligned} \tag{16}$$

possesses the trivial solution.

3^0 *conditions* 2^0 *and* 3^0 *of Theorem* 1 *are satisfied,*

4^0 α *and* β *are continuous, strictly increasing and non-negative functions on* $[0,\ +\infty)$,

5^0 $\alpha(0)=\beta(0)=0$ *and for* $u\in R^m$ *we have* $\beta(\|u\|)\leq V(t,\ u)\leq\alpha(\|u\|)$.

Under these assumptions stability or asymptotic stability of the trivial solution of the comparison problem (16) *implies the stability or asymptotic stability of the trivial solution of the system* (1).

Proof. Suppose that the trivial solution of (16) is stable. Then for $\varepsilon>0$ there exists a $\delta>0$ such that inequality $0\leq w(x,\ y)<\beta(\varepsilon)$ on E_0 implies $0\leq w(x,y)<\beta(\varepsilon)$ on E, where $w(\cdot)$ is a solution of (16). Choose $\delta_1>0$ such that $\alpha(\delta_1)=\delta$ and assume that $\|u(x,\ y)\|<\delta_1$ on E_0, where u is a solution of (1). This implies that for $w(x,\ y)=V(x,\ u(x,\ y))$ we have $w(x,\ y)\leq\alpha(\|u(x,\ y)\|)\leq\alpha(\delta_1)=\delta$. From the assumption 3^0 of our theorem we have on E

$$\frac{\partial V^{(u)}(x, y)}{\partial x} \leq g(x, y, V^{(u)}(x, y), V^{(u)}(\cdot), \frac{\partial V^{(u)}(x, y)}{\partial y}),$$

where $V^{(u)}(\cdot)$ is defined by (9). It follows from Lemma 2 that $V(x, u(x, y)) \leq w(x, y)$ for $(x, y) \in E$. Then

$$\beta(\| u(x, y) \|) \leq V(x, u(x, y)) \leq w(x, y) < \beta(\varepsilon) \text{ on } E,$$

which leads to the inequality $\| u(x, y) \| < \varepsilon$ on E, provided that $\| u(x, y) \| < \delta_1$ on E_0. This proves the stability of the trivial solution of (1).

We omit the simple proof of the second part of Theorem 3.

REFERENCES

1. W. E. Abolina, A. D. Myshkis. A mixed problem for almost linear hyperbolic systems on a plane. *Mat. Sb.*, **50**, 1960, 423-442.
2. Z. Kamont. On the existence and uniqueness of solutions of the Cauchy problem for linear partial differential-functional equations of the first order. *Math. Nachr.*, **80**, 1977, 183—200.
3. Z. Kamont. On the Cauchy problem for non-linear partial differential-functional equations of the first order. *Math. Nachr.*, **88**, 1979, 13-29.
4. Z. Kamont. On the estimation of the existence domain for solutions of a non-linear partial differential-functional equation of the first order. *Glasnik Matem.*, **13**, 1978, 277-291.
5. Z. Kamont. On first order partial differential-functional equations. *Uniw. Gd., Rozprawy i Monografie*, **10**, 1978, 1-156.
6. V. Lakshmikantham, S. Leela. Differential and integral inequalities, New York and London, 1969.
7. A. D. Myshkis, A. S. Slopak. A mixed problem for systems of differential-functional equations with partial derivatives and with operators of type Volterra. *Math. Sb.*, **41** 1957, 239-256.
8. J. Szàrski. Differential inequalities, Warszawa, 1967.
9. J. Szàrski. Generalized Cauchy problem for differential-functional equations with first order partial derivatives. *Bull. Acad. Polon. Sci., Ser. Math. Astr. Phys.*, **24**, 1976, 575-580.
10. K. Zima. Sur le équations aux dérivées partielles du premier ordre à argument fonctionel. *Ann. Polon. Math.*, **22**, 1969, 49-59.

Received 2. 7. 1981

Institute of Mathematics, University of Gdansk
08-952 Gdansk Poland