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ON NEUMAN'S PROBLEM FOR A CLASS OF DEGENERATE QUASI-LINEAR PARABOLIC EQUATIONS

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In this paper the existence and uniqueness of the classical solution of Neuman's problem for a class of degenerate quasi-linear parabolic equations is proved. The method of parabolic regularization and successive approximations is used.

1. The aim of this paper is to investigate Neuman's problem for a class of degenerate quasi-parabolic equations

$$(1)\qquad Pu=\sum_{ij=1}^{n} a^{ij}(x, x_0, u)u_{x_ix_j}+\sum_{i=1}^{n} b^i(x, x_0, u)u_{x_i}-c(x, x_0, u)u_{x_0}$$
$$+d(x, x_0, u)u=0$$

in the cylinder $G_T=D_Tx(-M, M)=\Omega x(0, T)x(-M, M)$. Here Ω is a bounded domain in $\mathbb{R}^n$, which is $C^{2l+4+\lambda}$ smoothly diffeomorphic to a ball, $l\geqq 3$ is an integer and $0<\lambda<1$.

We consider the boundary value operator

$$(2)\qquad Bu=\sum_{k=1}^{n}\sigma^k(x, x_0)u_{x_k}+\sigma(x, x_0)u=\varphi(x, x_0)$$

on $\Gamma_T=\partial\Omega\times(0, T)$. If $(\nu^1, \nu^2, \dots\nu^n)$ is the inner unit normal to Γ_T, presuppose that

$$(3)\qquad \sum_{k=1}^{n}\sigma^k(x, x_0)\nu^k>0$$

and $\sigma(x, x_0)\leqq 0$ on Γ_T. Moreover, the coefficients of the boundary value operator B and φ and their derivatives $D_x^\alpha D_{x_0}^\beta$ of order $|\alpha|+2\beta\leqq 2l+3$ are Hölder continuous with exponent λ on Γ_T. Further we make the following assumptions regarding the operators P, B, φ and Ω:

(i) $\Sigma_{ij=1}^n a^{ij}(x, x_0, p)\xi^i\xi^j\geqq\mu(x, x_0, p)|\xi|^2\geqq 0$ in the domain $G'\supset\bar{G}_T$, $\xi\in\mathbb{R}^n$, $a^{ij}\in C^2(G')$, $c(x, x_0, p)\geqq 0$ in G', $c\in C^2(G')$, $c(x, x_0, p)+\mu(x, x_0, p)>0$, $d(x, x_0, p)\leqq 0$ in $\bar{G}$;

(ii) The coefficients of the operator P and their derivatives $D_{x,p}^\alpha D_{x_0}^\beta$ of order $|\alpha|+2\beta\leqq 2l+2$ are Hölder continuous with exponent λ in $\bar{G}_T$;

(iii) The boundary $S_T=\Gamma_T\times(-M, M)$ is non-characteristic i.e.

$$(4)\qquad \sum_{ij=1}^{n} a^{ij}(x, x_0, p)\nu^i\nu^j>0 \text{ on } \bar{S}_T;$$

(iv) $(\partial^k\varphi)/(\partial x_0^k)(x, 0)=0$ for $k=0, 1, \dots l+3$, $x\in\partial\Omega$.

For convenience we introduce the set $\Omega_0=\{x\in\overline{\Omega};\ c(x, 0, 0)=0\}$. Following Fichera [8], we consider for the equation (1) the boundary value condition

$$u(x, 0)=0 \text{ on } \Omega\setminus\Omega_0. \tag{5}$$

Remark 1. In case $\Omega_0\cap\partial\Omega=\emptyset$ the condition (iv) is the compatibility condition of the data up to the order $l+3$.

Under these assumptions we have the following results:

Theorem 1. *Suppose* (i)—(iv) *hold. If there exists a point* $Q_\tau\in D_T\cap\{x_0=\tau\}$, $0\le\tau\le T$ *and the operator* P *is strictly parabolic in* $Q_\tau\times[-M, M]$, *then the boundary value problem* (1), (2), (5) *has a unique classical solution* $u(x, x_0)\in C^l(\overline{D}_\delta)$, *where* δ *is sufficiently small.*

Some real processes can be described by means of the equation (1). For instance the equation describing the temperature distribution in the case of a quasi-stationary regime with a moving thermal source is (see [1, p. 178])

$$\operatorname{div}_{y,z}(\lambda \operatorname{grad} T)-c_v v\frac{\partial T}{\partial x}=0, \tag{6}$$

where v is the velocity of the source, λ is the heat conduction coefficient and $c_v=c_v(T)$ is the limiting volumetric thermal capacity. Here $v\geqq 0$ and $v=0$ only on the boundary of the tube.

In conclusion it should be mentioned that when $c(x, x_0, u)=1$ the existence and uniqueness of the classical solution of the second boundary value problem for the equation (1) has been proved by G. M. Fateeva [7] for $0<x_0<\delta$ (δ is sufficiently small).

Let us state that our results are not contained in [7], whose extension this paper is and unlike [7] the present work includes the equation (6).

2. We will use the following inequalities and identities (see [2, 5])

$$P(v_1v_2)-v_1Pv_2+v_2Pv_1+2\sum_{ij=1}^{n}a^{ij}(v_1)_{x_i}(v_2)_{x_j}-dv_1v_2$$

for any two functions $v_1,\ v_2\in C^2$;

$$\Big(\sum_{ij=1}^{n}a^{ij}\xi^i\eta^j\Big)^2\le\Big(\sum_{ij=1}^{n}a^{ij}\xi^i\xi^j\Big)\Big(\sum_{ij=1}^{n}a^{ij}\eta^i\eta^j\Big) \tag{7}$$

for any $\xi,\ \eta\in\mathbb{R}^n$;

$$\Big(\sum_{ij=1}^{n}a_z^{ij}u_{x_ix_j}\Big)^2\le M_1\sum_{kij=1}^{n}a^{ij}u_{x_kx_i}u_{x_kx_j}, \tag{8}$$

when $z=x_0, x_1, \dots x_n, p$, under the assumption (i) in 1. Here the constant M_1 depends on the maximum of the second derivatives of a^{ij}. The proof of (8) for $z=x_0, p$ follows with slight changes of Oleinik's proof for $z=x_1, x_2, \dots x_n$ (see [2, p. 71]).

Further we will use the short notation $u_k=u_{x_k}$, $b^i_{kl}=a^i_{x_kx_l}$ etc., and the summation convention is understood.

Without loss of generality, in order to prove Theorem 1, we assume that D_{T_0}, $T_0\le T$ is a cylinder with a base Ω, which is a ball, its centre and radius being respectively 0 and R. Besides, the operator P is strictly parabolic in the points $(0, x_0, p)$, $0\le x_0\le T_0$, $|p|\le M$, $T_0\le T$.

Let $u=v.w$, where $w=[2-\exp(-\alpha|x|^2)]\exp(\beta x_0)>0$ in $\bar{D}_{T_0}$ and let us consider the operators (see [4])

$$\tilde{P}v=(P(vw))/w=\sum_{ij=1}^{n} a^{ij}(x,\ x_0,\ vw)v_{ij}+\sum_{i=1}^{n}[b^i(x,\ x_0,\ vw)$$

$$+2\left(\sum_{j=1}^{n} a^{ij}w_j\right)/w]v_i-c(x,\ x_0,\ vw)v_{x_0}+[P(w)/w]v=0,$$

$$\tilde{B}v=(B(vw))/w=\sum_{k=1}^{n}\sigma^k v_k+\left[\sigma+\left(\sum_{k=1}^{n}\sigma^k w_k\right)/w\right]v=\varphi/w.$$

When α, β are sufficiently large the inequalities $Pw<0$, $\Sigma_{k=1}^{n}\sigma^k w_k<0$ hold in $\bar{G}_{T_0}$ and on $\bar{\Gamma}_{T_0}$. The operators $\tilde{P}$, $\tilde{B}$ satisfy the conditions (i)—(iv) in the domain $\Omega\times(0,T_1)\times(-M_1,\ M_1)$, where $T_1=\min(T_0,\ 1/\beta)$, $M_1=M/(2e)$. Consequently, if we preserve the previous notations without loss of generality we may assume that $d(x,\ x_0,\ p)<0$ in $\bar{G}_{T_1}$, $\sigma(x,\ x_0)<0$ on $\bar{\Gamma}_{T_1}$.

Of basic significance for the proof of Theorem 1 is the following regularized boundary value problem

$$(9)\qquad P^{\varepsilon,N}(u^{\varepsilon,N})=\sum_{ij=1}^{n} a^{\varepsilon,ij}(x,\ x_0,\ u^{\varepsilon,N-1})u_{ij}^{\varepsilon,N}+\sum_{i=1}^{n} b^i(x,\ x_0,\ u^{\varepsilon,N-1})u_i^{\varepsilon,N}$$

$$-[c(x,\ x_0,\ u^{\varepsilon,N-1})+\varepsilon]u_{x^0}^{\varepsilon,N}+d(x,\ x_0,u^{\varepsilon,N-1})u^{\varepsilon,N}=0 \text{ in } D_{T_1};$$

$$Bu^{\varepsilon,N}=\sum_{k=1}^{n}\sigma^k(x,\ x_0)u_k^{\varepsilon,N}+\sigma(x,\ x_0)u^{\varepsilon,N}=\varphi(x,\ x_0) \text{ on } \Gamma_{T_1};$$

$$u^{\varepsilon,N}(x,\ 0)=0 \text{ on } \Omega.$$

We choose $u^{\varepsilon,0}(x,\ x_0)\equiv 0$ for $\varepsilon>0$ and besides, we use the short notation $a^{\varepsilon,ij}(x,\ x_0,\ p)=a^{ij}(x,\ x_0,\ p)+\varepsilon\delta^{ij}$, where $\delta^{ij}=0$ for $i\neq j$ and $\delta^{ii}=1$ for $i=1$, $2,\ldots n$.

If $u^{\varepsilon,N-1}(x,\ x_0)$ has derivatives $D_x^\alpha D_{x_0}^\beta$ of order $|\alpha|+2\beta\leq 2l+2$, which are Hölder continuous with exponent λ in $\bar{D}_{T_1}$ and $|u^{\varepsilon,N-1}(x,\ x_0)|\leq M$, then (see [3]) the boundary value problem (9) has a unique solution $u^{\varepsilon,N}(x,\ x_0)$, which derivatives $D_x^\alpha D_{x_0}^\beta$ of order $|\alpha|+2\beta\leq 2l+2$ are Hölder continuous with exponent λ in $\bar{D}_{T_1}$.

We will show that the sequence of successive approximations $u^{\varepsilon,N}$ can be formed.

Lemma 1. *Under the assumptions of Theorem 1 the following estimates* $\sup_{\bar{D}_{T_2}}|u^{\varepsilon,N}(x,\ x_0)|\leq M$ hold, where $0<T_2\leq T_1$, $\varepsilon>0$, $N=1,\ 2,\ldots$

Proof. We will show something more, that the estimates

$$(10)\qquad \sup_{\bar{\Omega}\times[0,2/\zeta^3]}|u^{\varepsilon,N}(x,\ x_0)|\leq e/\zeta$$

hold. They will be necessary for us later on.

We consider the auxiliary function $v^0(x,\ x_0)=(u^{\varepsilon,N})^2+x_0^2\exp(\xi(R^2-|x|^2))-(1/\zeta^2)\exp(\zeta^3 x_0)$. A simple computation gives

$$Bv^0 = 2u^{\varepsilon,N}\varphi - \sigma(u^{\varepsilon,N})^2 - 2\xi x_0^2 \sum_{k=1}^{n} \sigma^k x_k + \sigma x_0^2 - (\sigma/\zeta^2)\exp(\zeta^3 x_0)$$

$$\geq \varphi^2/\sigma + 2\xi R x_0^2 \sum_{k=1}^{n} \sigma^k \nu^k + \sigma x_0^2 - \sigma/\zeta^2.$$

Since $\lim\limits_{\substack{x_0\to 0\\ x\in\partial\Omega}} \frac{\varphi(x,x_0)-\varphi(x,0)}{x_0-0} = \lim\limits_{\substack{x_0\to 0\\ x\in\partial\Omega}} \frac{\varphi(x,x_0)}{x_0} = \varphi_{x_0}(x,0)$, it follows that $|\varphi(x,x_0)/x_0|\leq C_0$ for $x\in\partial\Omega$, $0\leq x_0\leq T_1$. Analogously, using (iv) we have the estimates

(11) $$|(D_\tau^\alpha \varphi(x, x_0))/x_0| \leq C_m$$

for $|\alpha|=m$, $1\leq m\leq l+3$, $x\in\partial\Omega$, $0\leq x_0\leq T_1$, τ is a tangential direction to Γ_{T_1}, the estimates being necessary for us later on.

Consequently, from (3), when ξ is sufficiently large (depending on C^0 norm of the coefficients of B and C^1 norm of φ) we have

(12) $$Bv^0 \geq x_0^2 - \sigma/\varsigma^2 \quad \text{on } \Gamma_{T_1}.$$

The estimate

(13) $$P^{\varepsilon,N}v^0 \geq 2a^{ij}u_i^{\varepsilon,N}u_j^{\varepsilon,N} - d(u^{\varepsilon,N})^2 + x_0^2(4\xi^2\mu|x|^2 - 0(\xi))\exp(\xi(R^2-|x|^2))$$
$$-cx_0\exp(\xi(R^2-|x|^2)) + c\varsigma\exp(\varsigma^3 x_0) - (d/\varsigma^2)\exp(\zeta^3 x_0) \geq a^{ij}u_i^{\varepsilon,N}u_j^{\varepsilon,N} + x_0^2\mu + c - d/\varsigma^2$$

holds, when ξ, ς are sufficiently large (depending on C^0 norm of the coefficients of P).

Since $v^0(x,0) = -1/\varsigma^2 < 0$ from (12), (13) and from the maximum principle it follows that $v^0(x,x_0)$ can not attain a positive maximum in $\bar{D}_{T_1}$, i. e. $v^0(x,x_0)\leq 0$ in $\bar{D}_{T_1}$. Consequently $(u^{\varepsilon,N})^2 \leq (1/\varsigma^2)\exp(\varsigma^3 x_0) \leq e^2/\varsigma^2$, when $0\leq x_0\leq 2/\varsigma^3$. Lemma 1 follows immediatly from (10), when $0\leq x_0\leq T_2$, $T_2 = \min[T_1, (2/\varsigma^3)\, ln(M\varsigma)]$.

In our further calculations, for convenience, we omit the index ε and with M_i, K_i, C_i, $i=1, 2, \ldots$ we denote constants which depend on the coefficients of the equation, the boundary value operator and the domain Ω, but not on ε, N and ς.

In the following Lemmas 2—5 our aim will be to prove the uniformly boundedness of the derivatives up to the order $l+1$ of the solution $u^{\varepsilon,N}(x,x_0)$ with constants independent of ε and N.

Let r_0 be a small enough positive constant, so that the operator P is strictly parabolic for $\{(x, x_0, p);\ |x|\leq r_0,\ 0\leq x_0\leq T_2,\ |p|\leq M\}$ and $r_0 > r_1 > \ldots r_{2l+2} = r$. We choose the functions $\chi^m(x)\in C^{2l+2+\lambda}(\mathbb{R}^n)$ for $m=1, 2, \ldots 2l+2$ with the following properties: $0\leq\chi^m\leq 1$, $\chi^m\equiv 1$ for $|x|\leq r_m$ and $\chi^m(x)\equiv 0$ for $|x|\geq r_{m-1}$, $\chi^1\geq\chi^2\geq\ldots\geq\chi^{2l+2}$.

L e m m a 2. *Under the assumptions of Theorem 1 the following estimates* $\sup\limits_{\substack{|x|\leq r,\\ 0\leq x_0\leq T_3}} |D_x^\alpha D_{x_0}^\beta u^{\varepsilon,N}(x,x_0)| \leq k_m'/\varsigma$ *hold for* $|\alpha|+2\beta=m$, $1\leq m\leq 2l+2$, $\varepsilon>0$, $N=1, 2, \ldots$, *where the constants* k_m' *do not depend on* ε, N *and* ς.

P r o o f. Since the following estimates are similar to those in Lemmas 3—5, their proofs will only be sketched here.

Without loss of generality we assume that $c(x, x_0, p)\equiv 1$ for $|x|\leq r_0$, $0\leq x_0\leq T_2$, $|p|\leq M$ because $c(x, x_0, p)\neq 0$.

We introduce the functions

$$w^0(x,\ x_0)=(u^N)^2-(q_0/\varsigma^2)\exp(\varsigma^3 x_0),$$

$$w^1(x,\ x_0)=\chi^1(x)\sum_{k=1}^{n}(u_k^N)^2+q_1 w^0(x,\ x_0).$$

The positive constant q_0 is sufficiently large so that the inequalities

$$P^N w^0\geq\sum_{ij=1}^{n}a^{ij}u_i^N u_j^N+1,$$

$$w^0\leq 0 \tag{14}$$

for $|x|\leq r_0,\ 0\leq x_0\leq T_3$, where $T_3=\min(T_2,\ 2/\varsigma^3)$ hold.

We assume inductively that

$$\chi^1(x)\sum_{k=1}^{n}(u_k^{N-1})^2\leq(q_1 q_0 e^2)/\varsigma^2 \tag{15}$$

for $|x|\leq r_0,\ 0\leq x_0\leq T_3$.

A simple computation gives

$$P^N w^1\geq\chi^1\sum_{k=1}^{n}a^{ij}u_{ki}^N u_{kj}^N+[q_1\mu_0-(M_2 q_1 q_0 e^2)/\varsigma^2$$

$$-M_3]\sum_{k=1}^{n}(u_k^N)^2+q_1-M_4\geq\chi^1\sum_{k=1}^{n}a^{ij}u_{ki}^N u_{kj}^N+1,$$

when $\mu(x,\ x_0,\ p)\geq\mu_0>0$ for $|x|\leq r_0,\ 0<x_0\leq T_3,\ |p|\leq M$ and $\varsigma^2\geq\varsigma_1^2=(2M_2 q_0 e^2)/\mu_0$, $q_1\geq(2M_3/\mu_0)+M_4+1$.

From the maximum principle and (14) it follows that

$$w^1(x,\ x_0)\leq 0 \tag{16}$$

and (15) holds for N.

Let us consider the auxiliary function $w^2(x,\ x_0)=\chi^2\sum_{k,l=1}^n(u_{kl}^N)^2+q_2 w^1(x,\ x_0)$ and let us assume that

$$\chi^2\sum_{k,l=1}^{n}(u_{kl}^{N-1})^2\leq(q_2 q_1 q_0 e^2)/\varsigma^2 \tag{17}$$

for $|x|\leq r_1,\ 0\leq x_0\leq T_3$. Then we have

$$P^N w^2\geq\chi^2\sum_{k,l=1}^{n}a^{ij}u_{kli}^N u_{klj}^N+[q_2\mu_0-(M_5 q_2 q_1 q_0 e^2)/\varsigma^2$$

$$-M_6]\sum_{k,l=1}^{n}(u_{kl}^N)^2+q_2-M_7\geq\chi^2\sum_{k,l=1}^{n}a^{ij}u_{kli}^N u_{klj}^N+1,$$

when $q_2\geq\max[(2M_6)/\mu_0,\ M_7+1]$ and $\varsigma^2\geq\varsigma_2^2=(2q_1 q_0 M_5 e^2)/\mu_0$.

Consequently from (16) and the maximum principle it follows that $w^2(x,\ x_0)\leq 0$ and (17) holds for N. From (15), (17) and the equation we have the needed estimates for $u_{x_0}^N$.

By induction we prove $\chi^m \Sigma_{|\alpha|=m}(D_x^\alpha u^{N-1})^2 \leqq (q_m q_{m-1} \dots q_0 e^2)/\varsigma^2$ for $|x| \leqq r_{m-1}$, $0 \leqq x_0 \leqq T_3$ with the auxiliary functions $w^m(x,x_0) = \chi^m \Sigma_{|\alpha|=m}(D_x^\alpha u^N)^2 + q_m w^{m-1}(x, x_0)$.

From the equation and the derivatives of the equation up to the necessary order we estimate $D_x^\alpha D_{x_0}^\beta u^N$ for $|\alpha|+2\beta=m$, $\beta \neq 0$.

In order to prove the uniformly boundedness of the derivatives of $u^{\varepsilon,N}$ in $\bar{U}_{T_3}$, $U_{T_3}=\{(x, x_0);\ r<|x|<R,\ 0<x_0<T_3\}$ we make a polar change of the x-variables and for convenience, we preserve the previous notations considering, that $x_1, x_2, \dots, x_{n-1}$ are angular variables and x_n is a radial variable. In the new variables (3) denotes that $\sigma^n<0$ on Γ_{T_3}.

Lemma 3. *Under the assumptions of Theorem 1 the estimates* $\sup_{\bar{U}_{T_4}} |D^\alpha_{xx_0} u^{\varepsilon,N}(x, x_0)| \leqq K_1/\varsigma$ *hold for* $T_4 \leqq T_3$, $|\alpha|=1$, $\varepsilon>0$, $N=1, 2, \dots$, *where the constant* K_1 *does not depend on* ε, N *and* ς.

Proof. Let us introduce the auxiliary functions:

$$v^1(x, x_0) = n_1[m_1 \sum_{k=1}^{n-1} (u_k^N)^2 + 2(u_n^N)^2 + u_n^N T(u^N) + k_1 x_0^2$$

$$+ K_1^2/\varsigma^2] \exp(\xi_1(R-x_n) - \eta_1 x_0) + (u_{x_0}^N)^2 \exp(-\eta_1 x_0) + p_1 v^0(x, x_0),$$

$$T(u^N) = \sum_{k=1}^{n-1} \theta^k(x, x_0) u_k^N + \theta(x, x_0) u^N + \Phi(x, x_0),$$

where $\theta^k(x, x_0)$, $\theta(x, x_0)$ are smooth extensions into $\bar{U}_{T_3}$, respectively of the functions $(-4\sigma^k)/\sigma^n$, $(-4\sigma)/\sigma^n$ which are defined on Γ_{T_3} so, that their derivatives in $\bar{U}_{T_3}$, $D_x^\alpha D_{x_0}^\beta$ of order $|\alpha|+2\beta \leqq 2l+3$ are Hölder continuous with exponent λ. Analogously we introduce the function $\Phi(x, x_0)$, for example $\Phi(x, x_0) = (-4x_n\varphi)/(R\sigma^n)$. The positive constants m_1, k_1 are chosen as it follows $m_1 = 2+(4nH_1)^2$, where H_1 is the maximum of the coefficients before the first order derivatives of the operator T and k_1/ς, $k_1 x_0$ are upper bounds, respectively for the zero order operator in T and Φ (see (11)).

From the choice of m_1, k_1 we have

$$m_1 \sum_{k=1}^{n-1} (u_k^N)^2 + 2(u_n^N)^2 + u_n^N T(u^N) + k_1^2 x_0^2 + k_1^2/\varsigma^2 \geqq \sum_{k=1}^{n} (u_k^N)^2 + x_0^2 + 1/\varsigma^2,$$

$$2m_1 \sum_{k=1}^{n-1} a^{ij} u_{ki}^N u_{kj}^N + 4a^{ij} u_{ni}^N u_{nj}^N + a^{ij}(Tu^N)_i u_{nj}^N \tag{18}$$

$$\geqq \frac{3m_1}{2} \sum_{k=1}^{n-1} a^{ij} u_{ki}^N u_{kj}^N + 3a^{ij} u_{ni}^N u_{nj}^N - M_8 \sum_{k=1}^{n} (u_k^N)^2 - M_8 x_0^2 - M_8/\varsigma^2.$$

We will show that $v^1(x, x_0)$ can not attain a positive maximum on Γ_{T_3}. A simple computations gives

$$Bv^1 \geqq n_1\{-\sigma^n \xi_1(\sum_{k=1}^{n} (u_k^N)^2 + x_0^2 + 1/\varsigma^2) + m_1 \sum_{k=1}^{n-1} 2u_k^N(\varphi_k$$

$$+ [B, \partial/\partial x_k](u^N)) + 4u_n^N Bu_n^N + T(u^N) Bu_n^N + u_n^N(T(\varphi)$$

$$+ [B, T](u^N)) + k_1^2 \sigma x_0^2 + k_1^2 \sigma/\zeta^2\} \exp(-\eta_1 x_0) + \{2u_{x_0}^N(\varphi_{x_0}$$

$$+ [B, \partial/\partial x_0](u^N)) - \sigma(u_{x_0}^N)^2\} \exp(-\eta_1 x_0) + p_1 Bv^0.$$

Since $[B, \partial/\partial x_0](u^N)$ is an operator of the first order and does not depend on $\partial/\partial x_0$, when ξ_1 is sufficiently large (depending on the C^1 norm of the coefficients of B and C^2 norm of φ), we have the estimate

$$Bv^1>0 \text{ on } \Gamma_{T_3}. \tag{19}$$

Analogously for $P^N v^1$ we have

$$P^N v^1 = n_1(I_1+I_2)\exp(\xi_1(R-x_n)-\eta_1 x_0)+(I_3+I_4)\exp(-\eta_1 x_0)+p_1 P^N v^0$$

where

$$I_1=\sum_{k=1}^{n-1}(2m_1 u_k^N+\theta^k u_n^N)[-(a^{ij})_k u_{ij}^N-(b^i)_k u_i^N+(c)_k u_{x_0}^N$$

$$-(d)_k u^N]+[T(u^N)+4u_n^N][-(a^{ij})_n u_{ij}^N-(b^i)_n u_i^N+(c)_n u_{x_0}^N-(d)_n u^N].$$

We assume that

$$\sum_{k=1}^{n-1}(u_k^{N-1})^2\leq(p_1 e^2)/(n_1\varsigma^2), \tag{20}$$

$$(u_{x_0}^{N-1})^2\leq(p_1 e^2)/\varsigma^2.$$

Then from (8) and (20) we have

$$|I_1|\leqq\frac{m_1}{2}\sum_{k=1}^{n-1}a^{ij}u_{ki}^N u_{kj}^N+a^{ij}u_{ni}^N u_{nj}^N+[(M_9 p_1 e^2)/(n_1\varsigma^2)$$

$$+M_{10}]\sum_{k=1}^{n}(u_k^N)^2+M_{11}c(u_{x_0}^N)^2+[M_{12}/\varsigma^2+M_{13}x_0^2][1+(p_1 e^2)/(n_1\varsigma^2)]$$

Similarly for I_3 we obtain

$$|I_3|=|2u_{x_0}^N[-(a^{ij})_{x_0}u_{ij}^N-(b^i)_{x_0}u_i^N+(c)_{x_0}u_{x_0}^N-(d)_{x_0}u^N]$$

$$-d(u_{x_0}^N)^2\geqq-[(M_{14}p_1 e^2)/\varsigma^2+M_{15}]\sum_{k=1}^{n}a^{ij}u_{ki}^N u_{kj}^N-[(M_{16}p_1 e^2)/\varsigma^2$$

$$+M_{17}]\sum_{k=1}^{n}(u_k^N)^2-[d/2+(M_{18}p_1 c e^2)/\varsigma^2](u_{x_0}^N)^2-M_{18}/\varsigma^2-M_{18}p_1 e^2/\varsigma^4.$$

The estimates of I_2, I_4 follow from (7), (8), (11), (18) as in [13], so we have

$$P^N v^1\geqq e^{-2}[n_1-(M_{14}p_1 e^2)/\varsigma^2-M_{15}]\sum_{k=1}^{n}a^{ij}u_{ki}^N u_{kj}^N+a^{ij}u_{ix_0}^N u_{jx_0}^N$$

$$+n_1[c\eta_1+(p_1\mu)/n_1-(M_{19}p_1 e^2)/(n_1\varsigma^2)-M_{20}]\sum_{k=1}^{n}(u_k^N)^2+[\eta_1 c$$

$$-d/2-(M_{18}cp_1 e^2)/\varsigma^2-M_{21}n_1 c](u_{x_0}^N)^2+(-p_1 d-M_{22}n_1)/\varsigma^2$$

$$+p_1(\mu x_0^2+c)-M_{23}x_0^2-(M'_{22}p_1 e^2)/\varsigma^4-(M'_{23}x_0^2 p_1 e^2)/\varsigma^2.$$

Let $n_1>M_{15}$. Then if $p_1\geq\max[2(M_{22}n_1+1)/(-d),\ 2(M_{23}+1)/(\mu+c)]$; $\eta_1>M_{21}n_1$; $c\eta_1+(p_1\mu)/n_1\geq 2M_{20}\varsigma^2\geq\varsigma_3^2$, $\varsigma_3^2=\max[(M_{14}p_1 e^2)/(n_1-M_{15}),\ ((M_{19}p_1 e^2)/(n_1 M_{20})),\ (M_{18}p_1 e^2)/(\eta_1-M_{21}n_1),\ (2M'_{22}e^2)/(-d),\ (2M'_{23}e^2)/(\mu+c)]$

the estimate

$$P^N v^1 \geq e^{-2}\Big(\sum_{k=0}^{n} a^{ij} u^N_{ki} u^N_{kj}\Big) + x_0^2 \mu + c + 1/\varsigma^2$$

for $0 \leq x_0 \leq T_4$, $T_4 = \min(T_3, 2/(\eta_1 + \varsigma^3))$, holds.

From Lemma 2, when p_1 is sufficiently large (depending on k_1', n_1, m_1, ξ_1), $v^1(x, x_0) \leq 0$ for $|x| = r$, $0 \leq x_0 \leq T_4$.

Since $v^1(x, 0) = -p_1/\varsigma^2 < 0$ it follows that $v^1(x, x_0)$ can not attain a positive maximum in $\bar{U}_{T_4}$, i.e. $v^1(x, x_0) \leq 0$ and (20) hold for N.

Lemma 4. *Under the assumptions of Theorem 1 the estimates* $\sup_{\bar{U}_{T_5}} |D^\alpha_{xx_0} u^{\varepsilon,N}(x, x_0)| \leq K_2/\varsigma$ *hold for* $T_5 \leq T_4$, $|\alpha| = 2$, $\varepsilon > 0$, $N = 1, 2, \ldots$, *where the constant* K_2 *does not depend on* ε, N *and* ς.

Proof. We consider the auxiliary functions

$$v^2(x, x_0) = n_2\Big[m_2 \sum_{\substack{k=0\\l=1}}^{n-1} (u^N_{kl})^2 + 2 \sum_{k=0}^{n} (u^N_{kn})^2 + \sum_{k=0}^{n} u^N_{kn} T^k(u^N)$$

$$+ k_2^2 x_0^2 + k_2^2/\varsigma^2\Big] \exp(\xi_2(R - x_n) - \eta_2 x_0) + (u^N_{x_0x_0})^2 \exp(-\eta_2 x_0) + p_2 v^1,$$

$$T^k(u^N) = \Big(\sum_{i=1}^{n-1} \theta^i(x, x_0) u^N_i + \theta(x, x_0) u^N + \Phi\Big)_k, \quad k = 0, 1, \ldots, n-1,$$

$$T^n(u^N) = \sum_{i=1}^{n-1} A^{ni}\Big(\sum_{k=1}^{n-1} \theta^k u^N_k + \theta u^N + \Phi\Big)_i + \sum_{ij=1}^{n-1} A^{ij} u^N_{ij}$$

$$+ \sum_{i=1}^{n-1} B^i u^N_i + B^n\Big(\sum_{k=1}^{n-1} \theta^k u^N_k + \theta u^N + \Phi\Big) + C u^N_{x_0} + D u^N.$$

The functions θ^k, θ, Φ are introduced in Lemma 3. The functions A^{ij}, B^i C, D are smooth extensions respectively of $(-4a^{ij})/a^{nn}$, $(-4b^i)/a^{nn}$, $(4c)/a^{nn}$ $(-4d)/a^{nn}$ from $\bar{\Gamma}_{T_1} \times [-M, M]$ into $\bar{U}_{T_1} \times [-M, M]$, so that their derivatives $D^\alpha_x D^\beta_{x_0}$ of order $|\alpha| + 2\beta \leq 2l + 2$ are Hölder continuous with exponent λ.

The positive constants m_2, k_2 are chosen as follows $m_2 = 2 + (4n)^3 H_2^2$, where, H_2 is the maximum of the coefficients before the second order derivatives of the operators T^k, $k = 0, 1, \ldots, n$, and k_2/ς, $k_2 x_0$ are upper bounds respectively for the operators of order ≤ 1 in T^k, $k = 0, 1, \ldots, n$, and the loose members in T^k, $k = 0, 1, \ldots, n$.

From the choice of m_2, k_2 we have

$$m_2 \sum_{\substack{k=0\\l=1}}^{n-1} (u^N_{ke})^2 + 2 \sum_{k=0}^{n} (u^N_{kn})^2 + \sum_{k=0}^{n} u^N_{kn} T^k(u^N) + k_2^2/\zeta^2$$

$$+ k_2^2 x_0^2 \geq \sum_{\substack{k=0\\l=1}}^{n} (u^N_{ke})^2 + x_0^2 + 1/\varsigma^2,$$

$$2m_2 \sum_{\substack{k=0\\l=1}}^{n-1} a^{ij} u^N_{kli} u^N_{klj} + 4 \sum_{k=0}^{n} a^{ij} u^N_{kni} u^N_{knj} + 2 \sum_{k=0}^{n} a^{ij} u^N_{kni} (T^k(u^N))_j$$

$$\geq \frac{3m_2}{2} \sum_{\substack{k=0\\l=1}}^{n-1} a^{ij} u^N_{kli} u^N_{klj} + 3 \sum_{k=0}^{n} a^{ij} u^N_{kni} u^N_{knj} - M_{25} \sum_{\substack{k=0\\l=1}}^{n} (u^N_{kl})^2 - M_{26}/\zeta^2 - M_{27} x_0^2.$$

We will show that $v^2(x, x_0)$ can not attain a positive maximum on Γ_{T_1}. For Bv^2 we have

$$Bv^2 \geq n_2\{-\sigma^n\xi_2(\sum_{\substack{k=0\\l=1}}^{n}(u_{kl}^N)^2+1/\varsigma^2+x_0^2)+2m_2\sum_{\substack{k=0\\l=0}}^{n-1}u_{kl}^N(\varphi_{kl}$$

$$+[B, \partial^2/\partial x_k\partial x_l](u^N))+4\sum_{k=0}^{n}u_{kn}^N Bu_{kn}^N+\sum_{k=0}^{n}B(u_{kn}^N)T^k(u^N)$$

$$+\sum_{k=0}^{n}u_{kn}^N(T^k(\varphi)+[B, T^k](u^N))+k_2^2\sigma x_0^2+k_2^2\sigma/\varsigma^2\}\exp(-\eta_2 x_0)$$

$$+[2u_{x_0x_0}^N(\varphi_{x_0x_0}+[B, \partial^2/\partial x_0^2](u^N))-\sigma(u_{x_0x_0}^N)^2]\exp(-\eta_2 x_0)+p_2Bv^1.$$

Since $[B, \partial^2/\partial x_0^2](u^N)$ is an operator of second order, which does not depend on $u_{x_0x_0}^N$, from (11), (19), when ξ_2 is sufficiently large (depending on C^2 norm of the coefficients of B and C^3 norm of φ), we have the estimate $Bv^2>0$ on Γ_{T_1}.

Similarly we obtain for P^Nv^2

$$p^Nv^2=\{n_2(I_1+I_2)\exp(\xi_2(R-x_n))+I_3+I_4\}\exp(-\eta_2x_0)+p_2P^Nv^1$$

where

$$I_1=2m_2\sum_{\substack{k=0\\l=1}}^{n-1}u_{kl}^N[-(a^{ij})_k u_{lij}^N-(a^{ij})_l u_{kij}^N]+\sum_{k=0}^{n}(4u_{kn}^N$$

$$+T^k(u^N))[-(a^{ij})_k u_{nij}^N-(a^{ij})_n u_{kij}^N].$$

Let us assume that

$$\sum_{\substack{k=0\\l=1}}^{n}(u_{kl}^{N-1})^2\leq(p_2p_1e^2)/(n_2\varsigma^2), \tag{21}$$

$$(u_{x_0x_0}^{N-1})^2\leq(p_2p_1e^2)/\varsigma^2.$$

Then from (8), (21) we have

$$|I_1|\leq\frac{m_2}{2}\sum_{\substack{k=0\\l=1}}^{n-1}a^{ij}u_{kli}^N u_{klj}^N+\sum_{k=0}^{n}a^{ij}u_{kni}^N u_{knj}^N+[M_{29}$$

$$+(M_{28}p_1p_2e^2)/(n_2\varsigma^2)]\sum_{\substack{k=0\\l=1}}^{n}(u_{kl}^N)^2+[M_{30}/\varsigma^2+M_{31}x_0^2](1+(p_1p_2e^2)/(n_2\varsigma^2)).$$

Analogously we estimate I_3, $I_3=-4u_{x_0x_0}^N(a^{ij})_{x_0}u_{ijx_0}^N$. As for I_2, I_4 we estimate them in the same way as in [13]. Consequently

$$P^Nv^2\geq e^{-2}\{n_2-(M_{32}p_1p_2e^2)/\varsigma^2-M_{33}\}\sum_{\substack{k=0\\l=1}}^{n}a^{ij}u_{kli}^N u_{klj}^N$$

$$+a^{ij}u_{x_0x_0i}^N u_{x_0x_0j}^N+n_2[c\eta_2+(p_2\mu)/(e^2n_2)-(M_{34}p_1p_2e^2)/n_2\varsigma^2)$$

$$-M_{35}]\sum_{\substack{k=0\\l=1}}^{n}(u^N_{kl})^2+[\eta_2c-d/2-M_{36}n_2c-(M'_{36}p_2p_1e^2)/(n_2\varsigma^2)](u^N_{x_0x_0})^2$$

$$+(p_2-n_2M_{37})/\varsigma^2+p_2(\mu x_0^2+c)-M_{38}x_0^2-M'_{37}p_1p_2/\varsigma^4-M'_{38}x_0^2p_1p_2/\varsigma^2.$$

Let $n_2>M_{33}$. Then, when $p_2\geq\max[2n_2M_{37}+2,2(M_{38}+1)/(\mu+c)]\eta_2>n_2M_{36}$; $c\eta_2+(p_2\mu)/(e^2n_2)\geq 2M_{35}$; $\varsigma^2\geq\varsigma_4^2$, $\varsigma_4^2=\max[(M_{32}p_1p_2e^2)/(n_2-M_{33}),\ (M_{34}p_1p_2e^2)/(n_2M_{35}),\ (M'_{36}p_1p_2e^2)/(n_2(\eta_2-n_2M_{36})),\ 2M'_{37}p_1,\ (2M'_{38}p_1)/(\mu+c)]$, the estimate

$$P^Nv^2\geq e^{-2}\sum_{k,l=0}^{n}a^{ij}u^N_{kli}u^N_{klj}+x_0^2\mu+c+1/\varsigma^2$$

holds, for $0\leq x_0\leq T_5$, $T_5=\min(T_4,\ 2/(\eta_2+\varsigma^3))$.

From Lemma 2, when p_2 is sufficiently large (depending on k_2' n_2, m_2, ξ_2), $v^2(x,\ x_0)\leq 0$ for $|x|=r$, $0\leq x_0\leq T_5$. Since $v^2(x,\ 0)=-(p_1p_2)/\varsigma^2<0$ it follows that $v^2(x,\ x_0)$ can not attain a positive maximum in $\overline{U}_{T_5}$ i.e. $v^2(x,\ x_0)\leq 0$. Therefore (21) holds for N.

Lemma 5. *Under the assumptions of Theorem 1 the estimates* $\sup_{\overline{U}_{T_{\rho+3}}}|D^\alpha_{xx_0}u^N(x,\ x_0)|\leq K_\rho/\varsigma$ *hold, for* $T_{\rho+3}\leq T_{\rho+2}$, $|\alpha|=\rho$, $3\leq\rho\leq l+1$, $\varepsilon>0$, $N=1,\ 2,\ldots$, *where the constants* K_ρ *do not depend on* ε, N *and* ς.

Proof. We prove inductively the estimates

$$\sum_{\substack{|\alpha|+\beta=\rho\\ \beta\neq\rho}}(D^\alpha_xD^\beta_{x_0}u^{N-1}(x,\ x_0))^2\leq(p_\rho p_{\rho-1}\ldots p_1e^2)/(n_\rho\varsigma^2),$$

$$(D^\beta_{x_0}u^{N-1}(x,\ x_0))^2\leq(p_\rho p_{\rho-1}\ldots p_1e^2)/\varsigma^2$$

in $\overline{U}_{T_{\rho+3}}$, $T_{\rho+3}=\min[T_{\rho+2},\ 2/(\eta_\rho+\varsigma^2)]$ by means of the auxiliary functions

$$v^\rho(x,\ x_0)=\{n_\rho[m_\rho\sum_{\substack{|\alpha+\beta|=\rho\\ \beta\neq\rho}}(D^\alpha_{x'}D^\beta_{x_0}u^N)^2+2\sum_{\substack{|\alpha+\beta+\gamma|=\rho\\ \beta\neq\rho,\ \gamma\neq 0}}(D^\alpha_{x'}D^\beta_{x_0}D^\gamma_{x_n}u^N)^2$$

$$+\sum_{\substack{|\alpha+\gamma+\beta|=\rho\\ \beta\neq\rho,\ \gamma\neq 0}}D^\alpha_{x'}D^\beta_{x_0}D^\gamma_{x_n}u^N\cdot T^{\alpha\beta\gamma}(u^N)+k_\rho^2/\varsigma^2+k_\rho^2x_0^2]\exp(\xi_\rho(R-x_n)$$

$$+(D^\rho_{x_0}u^N)^2\}\exp(-\eta_\rho x_0)+p_\rho v^{\rho-1}(x,\ x_0).$$

The coefficients of $T^{\alpha\beta\gamma}(x,\ x_0,u,\ u_{x_0},\ldots,\ ux_{n-1})$ are determined on $\overline{\Gamma}_{T\rho+2}$ by means of the condition $T^{\alpha,\beta,\gamma}(u^N)=-4D^\alpha_{x'}D^\beta_{x_0}D^\gamma_{x_n}u^N$, where the derivatives $D^\gamma_{x_n}v^N$ are substituted for by their equivalent expressions on $\overline{\Gamma}_{T\rho+2}$ using the operators B, P^N and the derivatives of P^N up to the necessary order. In $\overline{U}_{T_{\rho+2}}\times[-M,\ M]$ the coefficients of $T^{\alpha\beta\gamma}$ are smoothly extended so that their derivatives $D^a_xD^b_{x_0}$ of order $|a|+2b\leq 2l+4-\rho$ are Hölder continuous with exponent λ. The positive constants m_ρ, k_ρ are chosen as follows $m_\rho=2+(4n)^{\rho+1}H_\rho^2$, where H_ρ is the maximum of the coefficients in front of the ρ-th derivatives of the operators $T^{\alpha\beta\gamma}$ and k_ρ/ς, $k_\rho x_0$ are upper bounds, respectively, for the operators of order $\leq\rho-1$ in $T^{\alpha\beta\gamma}$ and the loose members in $T^{\alpha\beta\gamma}$.

P r o o f of Theorem 1. By means of a priori estimates proved in Lemmas 1—5, we have the result that $u^{\varepsilon,N}(x, x_0)$ and their derivatives $D^{\alpha}_{xx_0}$ of order $|\alpha|\leq l+1$ are uniformly bounded by the constants which do not depend on ε and N. Using the Ascoli—Arzela theorem and a diagonalization argument we can find a subsequences $\varepsilon_k\to 0$, $N_k\to\infty$, such that $D^{\alpha}_{xx_0}u^{\varepsilon_k,N_k}(x, x_0)\rightrightarrows D^{\alpha}_{xx_0}u$ for $|\alpha|\leq l$.

Let us now consider

$$P^N(u^N-u^{N-1})=[a^{ij}(x,\ x_0,\ u^{N-2})-a^{ij}(u,\ x_0,\ u^{N-1})]u^{N-1}_{ij}$$
$$+[b^i(x,\ x_0,\ u^{N-2})-b^i(x,\ x_0,\ u^{N-1})]u^{N-1}_i-[c(x,\ x_0,\ u^{N-2})$$
$$-c(x,\ x_0,\ u^{N-1})]u^{N-1}_{x_0}+[d(x,\ x_0,\ u^{N-2})-d(x,x_0,u^{N-1})]u^{N-1}\equiv g(x,\ x_0,\ u^{N-1},\ u^{N-2})$$

From the mean value theorems, since (see also (11)) $|u^N_{ij}(x,\ x_0)|\leq A_1x_0$, $|u^N_i(x, x_0)|\leq A_1x_0$, $|u^N_{x_0}(x,\ x_0)|\leq A_1x_0$, $|u^N(x, x_0)|\leq A_1x_0$ for $N=1,\ 2,\ldots,$ where the constant A_1 depend on ε and N, we have

$$|g(x,\ x_0,\ u^{N-1},\quad u^{N-2})|\leq A_2x_0\sup|u^{N-1}-u^{N-2}|.$$

As in Lemma 1 we consider the auxiliary function

$$Z(X,\ X_0)=(u^N-u^{N-1})^2+\gamma[x_0^2\exp(\xi(R^2-|x|^2))-(1/\varsigma^2)\exp(\varsigma^3x_0)].$$

A simple computations gives

$$BZ=-\sigma(u^N-u^{N-1})^2+\gamma[-2\xi x_0^2\sum_{k=1}^{n}\sigma^kx_k+\sigma x_0^2-(\sigma/\varsigma^2)\exp(\varsigma^3x_0)]>0\ \text{ on }\ \Gamma_{\delta}',$$

$$P^NZ\geqq 2(u^N-u^{N-1})g-d(u^N-u^{N-1})^2+\gamma[4\xi^2x_0^2|x|^2\mu$$
$$-O(\xi)+c\varsigma]\geqq g^2/d+\gamma[4\xi^2x_0^2\mu\,|x|^2-O(\xi)+c\varsigma].$$

When ξ, ς are sufficiently large and $\gamma\geqq A_2^2\sup|u^{N-1}-u^{N-2}|^2/d_0$ $d\leqq -d_0<0$, the estimates $P^NZ>0$ hold. Since $Z(x,\ 0)=-1/\varsigma^2<0$ it follows that $Z\leqq 0$ in $\overline{U}_{\delta}'$, $\delta'=\min[T_{l+1},\ 2/\varsigma^3]$ i.e.

$$\sup_{\overline{U}_{\delta}'}|u^N-u^{N-1}|\leqq(A_3/\varsigma^2)\sup_{\overline{U}_{\delta}'}|u^{N-1}-u^{N-2}|.$$

When $\varsigma^2\geqq A_3/2$ it follows that $u^N(x,\ x_0)\rightrightarrows u(x,\ x_0)$. Therefore, when $\varepsilon_k\to 0$, $N_k\to\infty$ from (9), we have that $u(x,\ x_0)$ is a solution of (1), (2), (5).

U n i q u e n e s s. Let us assume that besides $u(x, x_0)$, $v(x,\ x_0)\in C^l(\overline{\Omega}\times[0,\delta'])$ is a solution of (1), (2), (5) and let us consider

$$a^{ij}(x,\ x_0,\ v)(v-u)_{ij}+b^i(x,\ x_0,\ v)(v-u)_i-c(x,\ x_0,\ v)(v-u)_{x_0}$$
$$+d(x,\ x_0,\ v)(v-u)=[a^{ij}(x,\ x_0,\ u)-a^{ij}(x,\ x_0,\ v)]u_{ij}+[b^i(x,\ x_0,\ u)$$
$$-b^i(x,\ x_0,\ v)]u_i-[c(x,\ x_0,\ u)-c(x,\ x_0,\ v)]u_{x_0}+[d(x,\ x_0,\ u)$$
$$-d(x,\ x_0,\ v)]u\equiv h(x,\ x_0,\ u,\ v).$$

As in the proof above the estimate

$$\sup_{\overline{D}_{\delta}}|u-v|\leq(A_4/\varsigma^2)\sup_{\overline{D}_{\delta}}|u-v|$$

hold, where $\delta=\min(\delta', 2/\varsigma^3)$. When $\varsigma^2\geq\frac{1}{2}A_4$ we have $u\equiv v$.

Remark 2. Theorem 1 holds also in case of non-homogenious equation (1) i.e. the boundary value problem

$$Pu=f(x, x_0) \text{ in } D_\delta,$$
$$Bu=\varphi(x, x_0) \text{ on } \Gamma_\delta,$$
$$u(x, 0)=0 \text{ on } \Omega\setminus\Omega_0$$

under the assumptions of Theorem 1 and the additional condition $(\partial^k f)/(\partial x_0^k)(x, 0)=0$ for $k=0, 1, \ldots, l+3$, $x\in\Omega$ has a unique classical solution $u(x, x_0)\in C^l(\bar{D}_\delta)$.

Remark 3. Since the constants K_δ in Lemmas 3—5 depend on C^{l+1} norm of the coefficients of B and P and C^{l+2} norm of φ, Theorem 1 holds, when the coefficients of B and P are of the class $C^{l+1}(\bar{G})$ and $\varphi\in C^{l+2}(\bar{D})$.

Remark 4. If

$$\sum_{ij=1}^{n} a^{ij}(x, x_0, p)\xi^i\xi^j\geq\mu|\xi|^2, \quad \mu>0$$

or $(x, x_0, p)\in\bar{G}$, $\xi\in\mathbb{R}^n$ and $c(x, x_0, p)=c_1(x, x_0)\cdot c_2(x, x_0, p)$, $c_2(x, x_0, p)>0$, it is not necessary for the condition $c(x, x_0, p)\geq 0$ in G', $c(x, x_0, p)\in C^2(G')$ to be fulfilled. It is enough for $c(x, x_0, p)\geq 0$ to be valid in $G''=\bar{\Omega}\times[0, T'']\times[-M, M]$, $c\in C^2(G'')$, $T''>T$.

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