Serdica

Bulgariacae mathematicae
publicationes

Сердика

Българско математическо
списание

APPLICATION OF INTERPOLATION FORMULAS TO THE NUMERICAL SOLUTION OF SINGULAR INTEGRAL EQUATIONS

N. I. IOAKIMIDIS

The Lagrange and Hermite polynomial interpolation formulas are applied to the direct numerical solution of one-dimensional singular integral equations with Cauchy-type kernels by using the quadrature method. The application of these formulas permits to use essentially arbitrary collocation points along the integration interval without decreasing the accuracy of the obtained numerical results. In this way, it is not necessary to evaluate special collocation points and, moreover, the distribution of these points along the integration interval is sufficiently arbitrary to be made optimal from various points of view. The method is also applicable to singular integrodifferential equations. A simple application of the method to a singular integral equation appearing in engineering problems is also made.

1. Introduction. Several methods were proposed during the last decade for th direct numerical solution of one-dimensional singular integral equations with Cauchy-type kernels (called from now on simply singular integral equations). Some of these methods are reported in the review paper [1]. Among these methods, the direct quadrature method has gained high popularity. A series of modifications and extensions of this method are due to Ioakimidis [2] and were reported in a series of papers by Ioakimidis and Theocaris (see, e. g. [1; 3; 4]). Moreover, in Refs. [2; 5] numerical integration rules for Cauchy-type principal value integrals were suggested. A major disadvantage of the direct quadrature method is that it requires the determination not only of the nodes and the weights of the quadrature rule used, but also of the collocation points used. These points depend both on the quadrature rule and on the singular integral equation to be solved and they are generally the roots of complicated transcendental functions. Moreover, the lack of any freedom in the selection of the collocation points is a disadvantage of the method since physical conditions or theoretical results of numerical analysis may put some restrictions on their distribution or may suggest optimal distributions for these points.

In this paper, we will apply the classical Lagrange and Hermite interpolation formulas [6] to the above-described direct quadrature method of numerical solution of singular integral equations. Then the selection of the collocation points will become in principle arbitrary and this will not affect the accuracy of the obtained numerical results. Furthermore, the cases of singular integral equations with generalized kernels and of singular integrodifferential equations will be considered. A numerical illustration of the method will also be presented.

2. The direct quadrature method. We consider the singular integral equation of the second kind

$$A(x)w(x)g(x)+\int_a^b w(t)K(t,x)g(t)\,dt=f(x),\ a<x<b, \tag{2.1}$$

along a finite or infinite interval (a, b). In this equation $A(x)$ and $f(x)$ are known functions, $K(t, x)$ a known kernel of the form

$$K(t,x)=B(x)/(t-x)+k(t,x), \tag{2.2}$$

(where $B(x)$ and $k(t,x)$ are known functions), $w(x)$ is a weight function and $g(x)$ the unknown function. Because of (2.2), (2.1) can also be written as

$$A(x)w(x)\,g(x)+B(x)\int_a^b w(t)\,\frac{g(t)}{t-x}\,dt+\int_a^b w(t)k(t,x)\,g(t)\,dt=f(x),\ a<x<b. \tag{2.3}$$

Now, we apply an appropriate quadrature rule of the form [6]

$$\int_a^b w(t)\,\varphi(t)dt=\sum_{i=1}^{n} A_i\,\varphi(t_i)+E_n, \tag{2.4}$$

where t_i are the nodes, A_i the weights and E_n the error term, to the approximation of the integrals in (2.3). For Cauchy-type principal value integrals, this rule is modified as [2;5]

$$\int_a^b w(t)\frac{\varphi(t)}{t-x}\,dt=\sum_{i=1}^{n} A_i\frac{\varphi(t_i)}{t_i-x}+M_n(x)\,\varphi(x)+E_n,\ x\neq t_m,\ m=1\,(1)\,n, \tag{2.5}$$

$$\int_a^b w(t)\,\frac{\varphi(t)}{t-x}\,dt=\sum_{\substack{i=1\\ i\neq m}}^{n} A_i\frac{\varphi(t_i)}{t_i-x}+A_m\,\varphi'(x)+\Lambda_n(x)\varphi(x)+E_n,\ x=t_m,\ m=1(1)n, \tag{2.6}$$

where $M_n(x)$ and $\Lambda_n(x)$ are generally transcendental functions dependent on the quadrature rule (2.4) [2;5]. Assuming for the moment that $x\neq t_m$ $(m=1\,(1)\,n)$ and applying (2.4) and (2.5) to (2.3), we obtain

$$[A(x)w(x)+B(x)M_n(x)]\,y(x)+\sum_{i=1}^{n} A_iK(t_i,x)\,y(t_i)=f(x),\ a<x<b,\ x\neq t_m,\quad m=1\,(1)\,n, \tag{2.7}$$

where $y(x)$ is an approximation to $g(x)$.

As was suggested in [1] for the general case of (2.3), we can select as collocation points x_k the roots (or some of the roots) of the function

$$F_n(x)=A(x)w(x)+B(x)M_n(x). \tag{2.8}$$

Generally, there exist n such collocation points, permitting the reduction of (2.7) to the system of linear equations [1]:

$$\sum_{i=1}^{n} A_i\,K(t_i,x_k)\,y(t_i)=f(x_k),\ k=1\,(1)p,\ x_k\neq t_m,\ m=1\,(1)\,n. \tag{2.9}$$

But if $p=n-1$, then a supplementary condition of the form

$$\int_a^b w(t)\,g(t)\,dt=C \tag{2.10}$$

holds generally true, which, by using (2.4), gives

$$\sum_{i=1}^{n} A_i y(t_i) = C. \tag{2.11}$$

After the numerical solution of (2.9), probably supplemented by (2.11), the approximate values $y(t_i)$ of $g(x)$ at the nodes t_i are determined. Then $y(x)$ can be considered to be the interpolating polynomial to these values.

3. Application of the Hermite interpolation formula. A new possibility for constructing the system of linear equations (2.9) will be suggested in this section. Instead of selecting the collocation points x_k as the roots of (2.8), we can select them in principle in an arbitrary way along (a, b). Then, because of (2.7), (2.9) will take the form

$$F_n(x_k) y(x_k) + \sum_{i=1}^{n} A_i K(t_i, x_k) y(t_i) = f(x_k),\ k = 1\,(1)\,p,\ x_k \neq t_m,\ m = 1\,(1)\,n. \tag{3.1}$$

We apply also (2. 3) at the nodes t_i, taking into account (2.4) and (2.6). Then we obtain

$$G_n(t_j) y(t_j) + \sum_{\substack{i=1 \\ i \neq j}}^{n} A_i K(t_i, t_j) y(t_i) + A_j B(t_j) y'(t_j) = f(t_j),\ j = 1\,(1)\,n, \tag{3.2}$$

where now

$$G_n(x) = A(x) w(x) + B(x) \Lambda_n(x) \tag{3.3}$$

and $y'(x)$ denotes an approximation to the derivative $g'(x)$ of $g(x)$.

We assume now that $y(x)$ is determined by the Hermite interpolation formula, based on the values $y(t_j)$ and $y'(t_j)$. This formula has the form [6]

$$y(x) = \sum_{j=1}^{n} h_j(x) y(t_j) + \sum_{j=1}^{n} h_j^*(x) y'(t_j), \tag{3.4}$$

where

$$h_j(x) = [1 - 2l'_j(t_j)(x - t_j)]\, l_j^2(x),\ j = 1\,(1)\,n, \tag{3.5}$$

$$h_j^*(x) = (x - t_j)\, l_j^2(x),\ j = 1\,(1)\,n, \tag{3.6}$$

and

$$l_j(x) = \sigma_n(x) / [(x - t_j)\, \sigma'_n(t_j)],\ j = 1\,(1)\,n, \tag{3.7}$$

with

$$\sigma_n(x) = \prod_{j=1}^{n} (x - t_j). \tag{3.8}$$

By applying (3.4) at the arbitrarily selected collocation points x_k, we find

$$y(x_k) = \sum_{j=1}^{n} h_j(x_k) y(t_j) + \sum_{j=1}^{n} h_j^*(x_k) y'(t_j),\ k = 1\,(1)\,p. \tag{3.9}$$

If we substitute in these equations the values of $y(x_k)$ and $y'(t_j)$ by their expressions resulting from (3.1) and (3.2), respectively, we obtain the following system of linear equations

$$(3.10)\qquad \sum_{i=1}^{n} D_{ik}\, y(t_i) = Z_k,\quad k = 1\,(1)\,p,$$

where

$$(3.11)\qquad D_{ik} = A_i \{ \sum_{\substack{j=1\\ j\neq i}}^{n} h_j^*(x_k)\, K(t_i, t_j)/[A_j\, B(t_j)] - K(t_i, x_k)/F_n(x_k)\}$$

$$- h_i(x_k) + h_i^*(x_k)\, G_n(t_i)/[A_i\, B(t_i)],\ i = 1\,(1)\,n,\ k = 1\,(1)\,p,$$

and

$$(3.12)\qquad Z_k = \sum_{j=1}^{n} h_j^*(x_k) f(t_j)/[A_j\, B(t_j)] - f(x_k)/F_n(x_k),\ k = 1\,(1)\,p.$$

The numerical solution of (3.10) with $p=n$ if the index $\varkappa$ of (2.3) is equal to 0, or with $p=n-1$ if $\varkappa=1$, but with (2.11) supplementing in this case (3.10), permits the determination of the approximate values $y(t_i)$ of the unknown function $g(x)$ at the nodes t_i. Moreover, it is possible from (3.1) to determine afterwards directly the values of $y(x_k)$ and from (3.2) the values of $y'(t_j)$. It is also possible to use (3.4) as an expression of $y(x)$ along the whole integration interval $[a, b]$.

4. Application of the Lagrange interpolation formula. It is simpler to use the Lagrange interpolation formula [6]

$$(4.1)\qquad y(x) = \sum_{k=1}^{q} l_k(x)\, y(x_k)$$

for the approximation $y(x)$ to the unknown function $g(x)$ in (2.3). In this way, we can use $q=n+p$ arbitrary collocation points x_k not coinciding with one or more of the nodes t_i in (2.4) and the polynomials $l_k(x)$ are given by

$$(4.2)\qquad l_k(x) = \sigma_q(x)/[(x-x_k)\,\sigma_q'(x_k)],\ k = 1\,(1)\,q,$$

where

$$(4.3)\qquad \sigma_q(x) = \prod_{k=1}^{q} (x - x_k).$$

Equations (4.2) and (4.3) are analogous to (3.7) and (3.8) of section 3. The Lagrange interpolation formula (4.1) permits to use a set of q arbitrary collocation points x_k along (a, b). Then we obtain a system of q linear equations of the form (3.1) and we further find

$$(4.4)\qquad y(x_k) = [\,f(x_k) - \sum_{i=1}^{n} A_i\, K(t_i, x_k)\, y(t_i)]/F_n(x_k),\ k = 1\,(1)\,q.$$

These equations can be inserted into (4.1) with x restricted to the values t_j to give

$$y(t_j)+\sum_{i=1}^{n} A_i\Big[\sum_{k=1}^{q} l_k(t_j)\,K(t_i,\ x_k)/F_n(x_k)\Big]\,y(t_i)=\sum_{k=1}^{q} l_k(t_j)\,f(x_k)/F_n(x_k), \tag{4.5}$$

$$j=1\,(1)\,n.$$

This is a system of n linear equations in n unknowns, the values $y(t_i)$ at the nodes used. After the determination of these values, the values of $y(x)$ at the collocation points x_k can be determined from (4.4) and $y(x)$ along the whole interval $[a, b]$ from (4.1).

We can also add that (4.5) were constructed for the case where the collocation points x_k do not coincide with the nodes t_i and the roots of $F_n(x)$. In the first case, we obtain in our equations derivatives of the unknown function $y(x)$ (because of (2.6)) and this should be avoided. In the latter case, we obtain (2.9). The same equations can also be seen to result from (4.4) or, further, from (4.5) if the collocation points x_k are the roots of $F_n(x)$. Finally, we can mention that, although when the index $\varkappa$ of (2.3) is equal to 0, (4.5) are sufficient for the determination of $y(t_i)$, yet, if $\varkappa=1$, we must use (2.11), which will replace one of (4.5).

5. A. numerical application. For the application of the numerical technique of the previous section, we consider the airfoil equation [7] (with $\varkappa=0$)

$$\int_{-1}^{1} w(t)\frac{g(t)}{t-x}\,dt=\pi f(x),\quad -1<x<1, \tag{5.1}$$

where now

$$w(t)=[(1-t)/(1+t)]^{1/2}. \tag{5.2}$$

For the numerical solution of (5.1), we will use the quadrature rule

$$\int_{-1}^{1} w(t)\,\varphi(t)\,dt=\sum_{i=1}^{n} A_i\,\varphi(t_i)+E_n, \tag{5.3}$$

where t_i are determined by

$$(1+t_i)\,U_{n-1}(t_i)=0\ \text{ or }\ t_i=\cos\theta_i,\ \theta_i=i\pi/n,\ i=1\,(1)\,n, \tag{5.4}$$

where $U_{n-1}(x)$ denotes the Chebyshev polynomial of the second kind and degree $n-1$, and A_i are determined by

$$A_i=\pi(1-t_i)/n,\ i=1\,(1)\,(n-1),\ A_n=\pi/n. \tag{5.5}$$

Moreover, (5.3) can be modified on the basis of the results of [5] to apply to Cauchy-type principal value integrals as

$$\int_{-1}^{1} w(t)\frac{\varphi(t)}{t-x}\,dt=\sum_{i=1}^{n} A_i\frac{\varphi(t_i)}{t_i-x}+M_n(x)\,\varphi(x)+E_n,\ x\neq t_i,\ i=1\,(1)\,n, \tag{5.6}$$

where now

$$M_n(x)=-\pi T_n(x)/[(1+x)\,U_{n-1}(x)] \tag{5.7}$$

with $T_n(x)$ denoting the Chebyshev polynomial of the first kind and degree n.

In accordance with the results of section 2, we select the collocation points z_k as

(5.8) $$T_n(z_k)=0 \text{ or } z_k=\cos\omega_k,\ \omega_k=(k-0.5)\pi/n,\ k=1\,(1)\,n,$$

as is clear from (5.7). Then (5.1) can by approximated by the linear equations

(5.9) $$\sum_{i=1}^{n} A_i \frac{y(t_i)}{t_i-z_k}=\pi f(z_k),\ k=1\,(1)\,n.$$

After the determination of $y(t_i)$ from (5.9), $y(x)$ can be determined almost along the whole interval $[-1, 1]$ by

(5.10) $$y(x)=[\pi f(x)-\sum_{i=1}^{n} A_i \frac{y(t_i)}{t_i-x}]/M_n(x),\ x\neq t_i,\ i=1\,(1)\,n,\ x\neq z_k,\ k=1\,(1)\,n,$$

as is clear from (5.1) and (5.6).

In this section we wish mainly to illustrate the application of the method of numerical solution of singular integral equations based on the Lagrange interpolation formula and described in section 4. For the application of this method, we select a set of $q=2n$ collocation points x_k by

(5.11) $$T_{2n}(x_k)=0 \text{ or } x_k=\cos\psi_k,\ \psi_k=(k-0.5)\pi/(2n),\ k=1\,(1)\,2n.$$

Now (4.5) take the form

(5.12) $$y(t_i)+\sum_{j=1}^{n} A_j[\sum_{k=1}^{2n}\frac{l_k(t_i)}{(t_j-x_k)M_n(x_k)}]\,y(t_j)=\pi\sum_{k=1}^{2n} l_k(t_i)f(x_k)/M_n(x_k),\ i=1\,(1)\,n.$$

By taking into account that

(5.13) $$U_{n-1}(x)=\sin n\theta/\sin\theta,\ T_n(x)=\cos n\theta \text{ with } x=\cos\theta,$$

as well as (5.7) and (5.11), we can see that

(5.14) $$M_n(x_k)=-\pi\sin\psi_k\cos n\psi_k/[(1+\cos\psi_k)\sin n\psi_k],\ k=1\,(1)\,2n.$$

Similarly, by taking into account (4.2), (5.4) and (5.11), we easily find that

(5.15) $$l_k(t_i)=\cos 2n\,\theta_i\sin\psi_k/[2n(\cos\theta_i-\cos\psi_k)\sin 2n\,\psi_k],$$
$$i=1\,(1)\,n,\ k=1\,(1)\,2n.$$

Equation (5.1) was at first solved for $f(x)=-1-x$. In this simple case, it can be directly verified that this equation possesses the closed-form solution: $g(x)=2+x$. The numerical results, even for $n=2$, by both above-described methods verified this closed-form solution of (5.1) in this special case, that is, $y(x)\equiv g(x)$. As a second application, we assumed that $f(x)=-\exp x$. In this case, no closed-form solution of (5.1) is available. In Table 1 we present the numerical results obtained for $y(x)$ at $x=\pm 1$ for $n=2\,(1)\,7$ both by the method of section 2 and by the method of section 4. Since $t_n=-1$, $y(-1)$ was directly determined from the solution of (5.9) or (5.12). As regards $y(1)$, it was determined from (5.10) for $x=1$. We observe from the results of Table 1 that the above-described methods gave in this application numerical results convergent and of almost comparable accuracies. Similar very good agreement between the numerical results obtained by the two aforementioned methods was also observed in the interior points of the interval $(-1, 1)$.

Table 1

Numerical results for the solution $g(x)$ of (5.1) with $f(x)=-\exp x$ for $x=\pm 1$ obtained by the direct quadrature method, $y_D(\pm 1)$, as well as by the method based on the Lagrange interpolation formula, $y_L(\pm 1)$

n	$y_D(-1)$	$y_L(-1)$	$y_D(1)$	$y_L(1)$
2	1.260591837	1.266065679	4.920168522	4.901888128
3	1.266020900	1.266065878	4.928456335	4.928205066
4	1.266065679	1.266065878	4.928515593	4.928514062
5	1.266065877	1.266065878	4.928515841	4.928515835
6	1.266065878	1.266065878	4.928515841	4.928515841
7	1.266065878	1.266065878	4.923515841	4.928515841

6. Application to singular integral equations with generalized kernels.

The Lagrange and Hermite interpolation formulas can also be used for the numerical solution of singular integral equations with kernels presenting poles outside but near the integration interval. Consider, for example, the singular integral equation

$$\int_0^1 w(t)\left(\frac{1}{t-x}+\frac{\lambda}{t+x+a}\right) g(t)\,dt=f(x),\ a\geq 0,\ 0<x<1. \tag{6.1}$$

If a takes small positive values or if $a=0$ (in which case we say that (6.1) is a singular integral equation with a generalized kernel), then the contribution of the pole $t=-(x+a)$ when evaluating the integral in (6.1) should be taken into account in (2.4). The contribution of the pole $t=u$ with $u=-(x+a)$ to the error term E_n in (2.4) will be a term of the form [8]

$$E_{n0}=\lambda N_n(u)\,g(u) \tag{6.2}$$

in our case (with $N_n(u)$ a transcendental function related to $M_n(x)$).

Several techniques permitting to obtain accurate results in the case of (6.1) and similar cases were suggested by Ioakimidis (see, e. g., [9]). Here we confine ourselves to show the application of the methods of section 4 to this class of singular integral equations. In the case of (6.1), we have simply to use (4.1) with nodes x_k including both the n nodes t_i of (2.4) to (26) and the p nodes $u_k=-(z_k+a)$ $(k=1(1)p)$, where z_k are the roots of (2.8). Then, by applying (6.1) at the collocation points z_k by using (2.4) and (2.5) and taking into account (6.2), we obtain

$$\sum_{i=1}^{n} A_i K(t_i,z_k)\,y(t_i)+\lambda N_n(u_k)\,y(u_k)=f(z_k),\ k=1(1)p, \tag{6.3}$$

where

$$K(t,x)=1/(t-x)+\lambda/(t+x+a). \tag{6.4}$$

The existence of a sufficient number of collocation points z_k is assured by the developments of [1]. Furthermore, we obtain from (6.3)

$$y(u_k)=\left[f(z_k)-\sum_{i=1}^{n} A_i K(t_i,z_k)\,y(t_i)\right]/[\lambda N_n(u_k)],\ k=1(1)p. \tag{6.5}$$

Since $\{x_k\}=\{t_i\}\cup\{u_k\}$, we can insert the values of $y(u_k)$ from (6.5) into (4.1). Then we obtain an interpolation formula of the form

$$y(x)=\sum_{i=1}^{n} G_i(x)\,y(t_i)+G_0(x) \tag{6.6}$$

where $G_j(x)\,(j=0(1)n)$ are easily determined functions from (4.1) and (6.5).

Now we can select an arbitrary set of collocation points $v_k\,(k=1\,(1)\,n)$, not coinciding with the nodes t_i, inside the integration interval (0, 1) of (6.1) and apply again this equation at these points. Then, because of (2.4), (2.5) and (6.2), we obtain

$$\sum_{i=1}^{n} A_i K(t_i, v_k)\,y(t_i)+M_n(v_k)\,y(v_k)+\lambda N_n(\omega_k)\,y(\omega_k)=f(v_k),\ k=1\,(1)\,n, \tag{6.7}$$

where $\omega_k=-(v_k+a)\,(k=1\,(1)\,n)$. By using (6.6) for $x=v_k$ and $x=\omega_k\,(k=1\,(1)\,n)$ and inserting the resulting values for $y(x)$ into (6.7), we obtain a system of n linear equations with unknowns the values of $y(t_i)$. Of course, if we have to take into account the condition (2.10), the number of collocation points v_k should be $(n-1)$ instead of n. After the determination of $y(t_i)$ from (6.7), $y(u_k)$ can be determined from (6.5) and $y(x)$ along the whole integration interval [0,1] can be determined from (4.1) or, better, (6.6).

7. Application to singular integrodifferential equations. In this section we will illustrate the application of the Hermite and the Lagrange interpolation formulas to the numerical solution of two classes of singular integrodifferential equations appearing in engineering problems. The Hermite interpolation formula is more appropriate than the Lagrange interpolation formula for the first class of singular integrodifferential equations, whereas the contrary happens for the second class of these equations.

We consider at first the singular integrodifferential equation

$$A(x)\,w(x)\,g'(x)+\int_a^b w(t)\,K(t, x)\,g(t)\,dt=f(x),\ a<x<b, \tag{7.1}$$

where $K(t, x)$ is given again by (2.2). This equation differs from (2.1) only in the replacement of $g(x)$ in the free term of (2.1) by its derivative $g'(x)$. For the numerical solution of this equation by application of the Hermite interpolation formula (3.4), we can apply this equation at the nodes t_i of the quadrature rules (2.4) to (2.6). Then, because of (2.6), we determine directly the approximate values $y'(t_i)$ of the derivative $g'(x)$ of the unknown function $g(x)$ on the basis of the approximate values $y(t_i)$. Then, the Hermite interpolation formula (3.4) can be written as

$$y(x)=\sum_{i=1}^{n} H_i(x)\,y(t_i)+H_0(x), \tag{7.2}$$

where $H_j(x)\,(j=0(1)\,n)$ are easily determined polynomials. By differentiating (7.2), we obtain

$$y'(x)=\sum_{i=1}^{n} H_i'(x)\,y(t_i)+H_0'(x), \tag{7.3}$$

Now, if (7.1) is applied at a set of n more collocation points x_k along the interval (a, b), by using (2.4) and (2.5), and (7.2) and (7.3) are taken into account, there results a system of n linear equations for the determination of $y(t_i)$. After the determination of these values, (7.2) can be used as the interpolation formula for the approximation $y(x)$ to $g(x)$ along $[a, b]$.

Second, we consider the singular integrodifferential equation

$$(7.4)\qquad A(x)h(x)+\int_a^b K(t, x)\,h'(t)\,dt=f(x),\ a<x<b,\ h(a)=h(b)=0.$$

Now the unknown function $h(x)$ appears in the free term of this equation and its derivative under the integral sign. For the special case of (7.4) known as Prandtl's integral equation, where $[a, b]\equiv[-1, 1]$ and $K(t, x)=1/(t-x)$, the well-known Multhopp's collocation method [10] is generally applicable.

Here we will apply the Lagrange interpolation formula (4.1) to the numerical solution of (7.4) without restrictions on the interval $[a, b]$ and the kernel $K(t, x)$. At first, by taking into account the expression (2.2) for the kernel $K(t, x)$, as well as the developments of Ref. [11], we rewrite (7.4) in the equivalent form

$$(7.5)\quad A(x)w(x)g(x)+\frac{d}{dx}\int_a^b w(t)\frac{g(t)}{t-x}\,dt+\int_a^b w(t)\frac{dk(t, x)}{dx}\,g(t)\,dt=f(x),\ a<x<b,$$

with $h(x)=w(x)\,g(x)$ and $w(x)$ an appropriate weight function. Next, we differentiate (2.5) with respect to x and we obtain

$$(7.6)\qquad \frac{d}{dx}\int_a^b w(t)\frac{\varphi(t)}{t-x}\,dt=\sum_{i=1}^{n} A_i\frac{\varphi(t_i)}{(t_i-x)^2}+M_n'(x)\,\varphi(x)+M_n(x)\,\varphi'(x)+E_n,$$

$$x\neq t_m,\ m=1(1)n.$$

Then we apply (7.5) at the collocation points z_k, determined as the roots of $M_n(x)$. By using (2.4) and (7.6), we obtain the values of $y(z_k)$, on the basis of the values of $y(t_i)$, and the Lagrange interpolation formula (4.1) with nodes x_k including these collocation points z_k $(k=1(1)p)$, as well as the nodes t_i $(i=1(1)n)$ of (2.4) and (7.6), takes the form (6.6). Finally, the application of (7.5) at an arbitrary set of collocation points v_k $(k=1(1)n)$ inside the interval (a, b) permits the construction of n linear equations with unknowns the values of $y(t_i)$ exactly as happened in section 6.

8. Discussion. The aim of the results of sections 2 to 7 was to show that it is possible to solve numerically (2.1) (or the other equations considered) by reducing it to a system of only n linear equations. The results of this paper can evidently be generalized to apply to a series of other classes of singular integral equations and related equations (or systems of such equations) not considered in this paper. Of course, there remains the question of establishing the conditions under which the proposed methods converge to the correct results for the unknown function, as well as the optimum (in some sense) selection of the collocation points in all cases where their location inside the integration interval (a, b) has been left arbitrary. These problems require much theoretical work. In practice, it is in several cases the computer which reveals the convergence as happened in the application of section 5 although theoretical results have begun to appear (see, e. g., [12]). It is hoped that these results will become complete in future.

Acknowledgements. The results reported here belong to a research project supported by the National Hellenic Research Foundation. The financial support of this Foundation is most gratefully acknowledged. The author wishes also to express his sincere thanks to his colleagues P. Yannopoulos and M. Eftaksias for their kind help during the preparation of the numerical results contained in this paper.

REFERENCES

1. P. S. Theocaris, N. I. Ioakimidis. Numerical solution of Cauchy type singular integral equations. *Pragm. Akad. Athēnōn (Trans. Acad. Athens)*, **40**, 1977, 1-39.
2. N. I. Ioakimidis. General methods for the solution of crack problems in the theory of plane elasticity. Doctoral thesis, Athens, 1976. [Available from University Microfilms, Ann Arbor, Michigan; Order No. 76-21, 056].
3. P. S. Theocaris, N. I. Ioakimidis. On the Gauss-Jacobi numerical integration method applied to the solution of singular integral equations. *Bull. Calcutta Math. Soc.*, **71**, 1979, 29-43.
4. N. I. Ioakimidis, P. S. Theocaris. Numerical solution of Cauchy type singular integral equations by use of the Lobatto-Jacobi numerical integration rule. *Appl. Math.*, **23**, 1978, 439-452.
5. N. I. Ioakimidis, P. S. Theocaris. On the numerical evaluation of Cauchy principal value integrals. *Rev. Roumaine Sci. Tech. Sér. Méc. Appl.*, **22**, 1977, 803-818.
6. F. B. Hildebrand. Introduction to numerical analysis. Tata McGraw-Hill, New Delhi, 1956.
7. F. G. Tricomi. Integral equations. Interscience Publishers, New York, 1957.
8. J. D. Donaldson, D. Elliott. A unified approach to quadrature rules with asymptotic estimates of their remainders. *SIAM J. Numer. Anal.*, **9**, 1972, 573-602.
9. N. I. Ioakimidis, P. S. Theocaris. On the selection of collocation points for the numerical solution of singular integral eguations with generalized kernels appearing in elasticity problems. *Comput. & Structures*, **11**, 1980, 289-295.
10. H. Multhopp. Die Berechnung der Auftriebsverteilung von Tragflügeln. *Luftfahrtforschung*, **15**, 1938, 153-166.
11. N. I. Ioakimidis, P. S. Theocaris. On the numerical solution of singular integro-differential equations. *Quart. Appl. Math.*, **37**, 1979, 325-331.
12. N. I. Ioakimidis, P. S. Theocaris. On convergence of two direct methods for solution of Cauchy type singular integral equations of the first kind. *BIT*, **20**, 1980, 83-87.

Chair of Mathematics B', School of Engineering,
University of Patras, P. O. Box 1120
GR-261.10 Patras, Greece

Received 7. 4. 1982
Revised 23. 1. 1984