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ON CONVEXIES AND RELATED CONGRUENCES ON GRAPHS

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A closure operation on graphs is defined. The properties of congruences having the substitution property with respect to this operation are considered. A special class of graphs is found, where the congruences have properties like congruences in distributive lattices.

1. Introduction. Let $G=(X, E)$ be a finite, connected and undirected graph without loops and multiple lines. X is the set of points in G and E its set of lines. A geodesic (shortest path) closed pointset, called here briefly a convex. of G can be defined as follows: Let $x, y \in X$. The notation $SP(x, y)$ is a brief expression for the set $\{w \mid w \in X$ and w is on an $x-y$ geodesic in $G\}$. In general, if A and B are two non-empty subsets of X, $SP(A, B)=\{w \mid w \in X$, w is on an $a-b$ geodesic, where $a \in A$ and $b \in B\}$. Moreover, we denote by $SP^{i+1}(x, y)=SP(SP^i(x, y), SP^i(x, y))$, $i=1, 2, \ldots,$ and $SP^1(x, y)=SP(x, y)$. Because G is finite, there is certainly a value n of i such that $SP^n(x, y)=SP^{n+1}(x, y)=SP^{n+2}(x, y)=\cdots$ and this set $SP^n(x, y)$ is briefly denoted by $\langle x, y\rangle$. $\langle x, y\rangle$ is the least convex containing x and y in G. The pointset $\langle A, B\rangle$ is defined analogously. A non-empty pointset $A \subset X$ is called convex (or geodesic closed) if $\langle A, A\rangle=A$, and according to the definition of $\langle x, y\rangle$, $\langle\langle x, y\rangle, \langle x, y\rangle\rangle=\langle x, y\rangle$. $\langle\ ,\ \rangle$ is a setvalued operation on X and on G as well.

The purpose of this paper is to consider $\langle\ ,\ \rangle$-compatible congruences on G and $\langle\ ,\ \rangle$-compatible homomorphisms of G. In particular we will consider the lattice $C(G)$ of $\langle\ ,\ \rangle$-compatible congruences on G. It turns out that a class of graphs has properties that are analogous to the characteristic properties of distributive lattices and thus this class of graphs is a natural generalization of distributive lattices.

As easily seen, $\langle x, x\rangle=\{x\}$ for every $x \in X$, as well as $\langle X, X\rangle=X$. Moreover, if A is the pointset inducing the complete subgraph of G, then $\langle A, A\rangle=A$, and if $xy \in E$, then $\langle x, y\rangle=\{x, y\}$. One can easily prove that $\langle\langle A, B\rangle, \langle A, B\rangle\rangle=\langle A, B\rangle$ when A and B are non-empty subsets of X. A convex $S \neq X$ of G is called prime, if $X \setminus S$ is also a convex of G.

If A and B are two convexies of G such that $A \cap B \neq \emptyset$, then $A \cap B$ is a convex of G, too. By $A \vee B$ we mean the least convex of G containing A and B. Thus the convexies of G constitute a join-semilattice $\mathcal{J}$, where $J \wedge I$ exists whenever $J \cap I \neq \emptyset$. A special class of graphs are the graphs where every convex $J \neq X$ is the intersection of prime convexies containing J; we will denote this class by $\mathcal{G}_p$. As easily seen, every complete graph belongs to this class as well as every tree and the Hasse diagram graphs of finite distributive lattices. It will be shown that $\mathcal{G}_p$ is a natural generalization of finite distributive lattices.

As a general reference of graph theory we will use the book [2] of Harary and of lattice theory the book [1] of Grätzer. Convexies of graphs are considered in [3], [5—10], and related results are given also by Melter and Tomescu in [4].

In all what follows a congruence means a $\langle\, ,\rangle$-compatible congruence and a homomorphism a $\langle\, ,\rangle$-compatible homomorphism.

2. Congruences on arbitrary graphs. In this section we present properties of homomorphisms and congruences on arbitrary graphs.

A binary, reflexive, symmetric and transitive relation R is called a $\langle\, ,\rangle$-compatible congruence on G, if $(a, b), (x, y) \in R$ imply that $(\langle a, x\rangle, \langle b, y\rangle) \in R$, which means that for every $z \in \langle a, x\rangle$ there is a $v \in \langle b, y\rangle$ such that $(z, v) \in R$ and vice versa. Because R is transitive, R induces a partition $\mathscr{C} = \{C_1, \dots, C_n\}$ of X as follows: $C_1 \cup \dots \cup C_n = X$, $C_i \cap C_j = \emptyset$ whenever $i \neq j$, and

(1) if a, $b \in C_i$ and $x, y \in C_j$, then
$$\langle a, x\rangle \cap C_k = \emptyset \Leftrightarrow \langle b, y - \cap C_k = \emptyset \text{ for every } k, k = 1, \dots, n$$

(cf [9, Thm. 1]). The congruence classes C_i are convexies of G.

As it is well known, every congruence determines a homomorphism and conversely. In the homomorphism φ induced by congruence R, every congruence class C_i is mapped onto a point c_i of the homomorphic image $\varphi(G)$ of G. Moreover, the condition (1) shows that c_i and c_j are adjacent in $\varphi(G)$ whenever C_i and C_j have two adjacent points in G [9].

Theorem 1. *Let φ be a homomorphism of G and $H = (X_H, E_H)$ its homomorphic image under φ, where $X_H = \{c_1, \dots, c_n\}$. If A_H is a convex of H, then $\varphi^{-1}(A_h)$ is a convex of G, and if A_H is a prime convex, then $\varphi^{-1}(A_H)$ is prime, too. Moreover, if A is a convex of G, then $\varphi(A) \subset X_H$ is a convex of H.*

Proof. Let $A_H = \{c_{i1}, \dots, c_{im}\}$. Then $\varphi^{-1}(A_H) = C_{i1} \cup \dots \cup C_{im}$. Because A_H is a convex of H, $\langle c_{ij}, c_{ih}\rangle \subset A_H$ for every two points $c_{ij}, c_{ih} \in A_H$. But this means that if $x \in C_{ij}$ and $y \in C_{ih}$, then $\langle x, y\rangle \cap C_k = \emptyset$ for all $C_k \not\subset \varphi^{-1}(A_H)$, according to (1). Thus $\langle \varphi^{-1}(A_H), \varphi^{-1}(A_H)\rangle \subset \varphi^{-1}(A_H)$ in G, whence $\varphi^{-1}(A_H)$ is a convex of G. When A_H is a prime convex, then $X_H \setminus A_H$ is a convex of H, and we can show that $\varphi^{-1}(X_H \setminus A_H)$ is a convex of G. But this shows that $\varphi^{-1}(A_H)$ is a prime convex of G. The last assertion follows directly from (1). This completes the proof.

Theorem 2. *Let φ be a homomorphism of G and $H = (X_H, E_H)$ its homomorphic image under φ. Then the join-semilattice $\mathscr{I}_H$ of all convexies of H is a join-homomorphic image of $\mathscr{I}_G$.*

Proof. As proved in Theorem 1, every convex of G is mapped onto a convex of H under φ, and conversely every convex of H is the image of a convex of G under φ. Thus $\varphi: \mathscr{I}_G \to \mathscr{I}_H$ is onto and it remains to show that $\varphi(I \vee J) = \varphi(I) \vee \varphi(J)$ for every two convexies I and J of G. On the other hand, $I \vee J = \langle I, J\rangle$ and $\varphi(I) \vee \varphi(J) = \langle \varphi(I), \varphi(J)\rangle$. According to (1) we know that in G $\langle I, J\rangle \cap C_k \neq \emptyset$ if and only if $\langle \varphi(I), \varphi(J)\rangle \cap c_k \neq \emptyset$ in H, whence $\varphi(I \vee J) = \varphi(I) \vee \varphi(J)$. This completes the proof.

Melter and Tomescu introduced in [4] the concept of a base in a graph. Following them we say that a non-empty set $S \subset X$ is a base of G, if for every two distinct points $x, y \in X \setminus S$ there is a point $z \in S$ such that $\langle x, z\rangle \neq \langle y, z\rangle$. The least cardinality of a base in G is called the dimension of G and denoted by $\dim(G)$.

Theorem 3. *Let φ be a homomorphism of G and $H = (X_H, E_H)$ its homomorphic image under φ. If* $\dim(G) = 1$ *then also* $\dim(H) = 1$.

P r o o f. Let $\{z\}$ be a base of G. We will show that $\{\varphi(z)\}$ is a base of H. whence $\dim(H)=1$. Assume that there are two disjoint points $\varphi(x)$ and $\varphi(y)$ in H such that $\langle\varphi(x), \varphi(z)\rangle=\langle\varphi(y), \varphi(z)\rangle$. Let $\mathscr{C}=\{C_1, \ldots, C_n\}$ be the congruence partition of X in the congruence related to φ, and assume that $x\in C_x$ and $y\in C_y$ in C. Because $\langle\varphi(x), \varphi(z)\rangle=\langle\varphi(y), \varphi(z)\rangle$, $\langle x, z\rangle\cap C_y\neq\emptyset$ and $\langle y, z\rangle\cap C_x\neq\emptyset$ in G; this holds for every x from C_x and for every y from C_y. Let us denote x by x_1. Then $\langle x_1, z\rangle$ contains a point y_1 from $\mathscr{C}_y$. If $\langle x_1, z\rangle=\langle y_1, z\rangle$ we have obtained a contradiction and hence we assume that $\langle y_1, z\rangle$ is contained in $\langle x_1, z\rangle$ properly (because $y_1, z\in\langle x_1, z\rangle$, then $\langle y_1, z\rangle\subset\langle x_1, z\rangle$). Because $\langle y_1, z\rangle\cap C_x\neq\emptyset$, $\langle y_1, z\rangle$ contains a point x_2 ($\neq x_1$, since otherwise $\langle x_1, z\rangle=\langle y_1, z\rangle$) from C_x. If $\langle y_1, z\rangle=\langle x_2, z\rangle$, we have a contradiction, and thus we assume that $\langle x_2, z\rangle$ is contained in $\langle y, z\rangle$ properly. Because $\langle x_2, z\rangle\cap C_y\neq\emptyset$, $\langle x_2, z\rangle$ contains a point $y_2(\neq y_1)$ from C_y. Certainly $\langle y_2, z\rangle\subset\langle x_2, z\rangle$, and if the equality holds, we have a contradiction, and if not, we can continue the process. Because G is finite and because we have constructed a sequence $\langle x_1, z\rangle\supset\langle y_1, z\rangle\supset\langle x_2, z\rangle\supset\langle y_2, z\rangle\supset\cdots$, we will find after a finite number of steps two points y_s and x_s such that $\langle x_s, z\rangle=\langle y_s, z\rangle$ in contradiction with the base property of $\{z\}$. Hence the Theorem.

Let $C(G)$ be the set of all congruences on a graph G. If $\Phi, \theta\in C(G)$, the meet $\Phi\wedge\theta$ is usually defined as follows: $(a, b)\in\Phi\wedge\theta\Leftrightarrow(u, b)\in\Phi$ and $(a, b)\in\theta$. This is not valid in every graph, not even in every graph of the class $\mathscr{G}^d$ As an example one can consider the graph $G\in\mathscr{G}_p$ of Figure 2, where Φ consists of classes $\{u, w, v\}$, $\{z, y, x\}$ and θ of classes $\{z, u, w\}$, $\{y, x, v\}$; then $\Phi\wedge\theta$ is not any more a $\langle\ ,\ \rangle$-compatible congruence on G. On the other hand, there are many graphs, where $C(G)$ is a lattice with respect to the meet $\wedge$ defined above, namely complete graphs, trees (and thus also graphs, where every block is a complete graph), the graphs isomorphic to the Hasse diagrams of distributive lattices (and thus also graphs, where every block is a graph isomorphic to the Hasse diagram of a distributive lattice) and the graphs isomorphic to the Hasse diagrams of modular lattices. *In the following, when we consider the lattice $C(G)$, we assume that the meet can be defined in $C(G)$ as done above.*

T h e o r e m 4. *Let θ and Φ be two arbitrary congruences on G and the lattice $C(G)$ exist. Then $C(G)$ is distributive, if*

$(x, y)\in\theta\vee\Phi$ implies the existence of a path $u_0, u_1, \ldots, u_n$,

(2) *u_{n+1}, contained in $\langle x, y\rangle$ such that $x=u_0$, $y=u_{n+1}$, and*

$(u_i, u_{i+1})\in\theta$ or $(u_i, u_{i+1})\in\Phi$ or both for every $i=0, \ldots, n$.

P r o o f. Let the condition hold and $(x, y)\in\psi\wedge(\theta\vee\Phi)$. Thus $(x, y)\in\psi$ whence $(u_i, u_{i+1})\in\psi$ for every $i=0, \ldots, n$. Accordingly, $(u_i, u_{i+1})\in\psi\wedge\theta$ or $(u_i, u_{i+1})\in\psi\wedge\Phi$ for every i, which implies that $(u_i, u_{i+1})\in(\psi\wedge\theta)\vee(\psi\wedge\Phi)$ for every i. Hence also $(x, y)\in(\psi\wedge\theta)\vee(\psi\wedge\Phi)$ and thus $\psi\wedge(\theta\vee\Phi)\leq(\psi\wedge\theta)\vee(\psi\wedge\Phi)$ from which the distributivity of $C(G)$ follows.

Note that there are graphs G for which $C(G)$ exists and is non-distributive also in the class $\mathscr{G}_p$. For example, consider the complete graph of Figure 1, where θ has the classes $\{y, x\}$, $\{u, w\}$, ψ the classes $\{x\}$, $\{w\}$, $\{y, u\}$ and Φ the classes $\{y\}$, $\{u\}$, $\{x, w\}$. Thus $\theta\vee\psi=\theta\vee\Phi=1$ in $C(G)$, whence $\psi\wedge(\theta\vee\Phi)=\psi$ but $(\psi\wedge\theta)\vee(\psi\wedge\Phi)=0\vee 0=0$ in $C(G)$, from which the non-distributivity follows.

3. Congruences on graphs of $\mathcal{G}_P$. At first we like to show that the homomorphic image of a graph from $\mathcal{G}_P$ belongs to $\mathcal{G}_P$.

Theorem 5. *Let $G \in \mathcal{G}_P$ and φ be a homomorphism of G onto $H=(X_H, E_H)$. Then also $H \in \mathcal{G}_P$.*

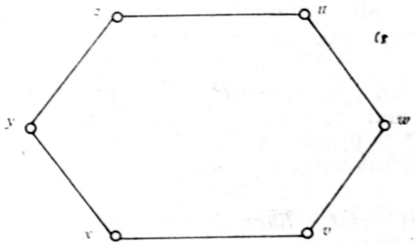

Fig. 1

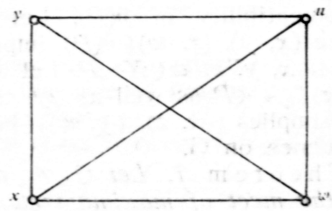

Fig. 2

Proof. Let $I_H, J_H \subset X_H$ be two non-empty convexies of H such that $I_H \cap J_H = \emptyset$. Because H is finite, we can find a maximal convex K_H which contains I_H and for which $K_H \cap J_H = \emptyset$. As shown in Theorem 1, $\varphi^{-1}(I_H)=I$ is a convex of G as well as $\varphi^{-1}(J_H)=J$ and $\varphi^{-1}(K_H)=K$. Moreover, $I \subset K$ and $J \cap K = \emptyset$. The maximality of K_H implies that for every $x_H \in X_H \setminus K_H$, $(K_H \vee \{x_H\}) \cap J_H \neq \emptyset$. Because φ is a homomorphism, K has the same maximality property in G. If K is not a prime convex in G, then there is a prime convex P containing properly K and $J \cap X \setminus P$. But this is absurd according to the maximality of K, and hence K is a prime convex of G.

Because $K=\varphi^{-1}(K_h)$, then $\varphi(X \setminus K) \cap K_H = \emptyset$ in H and $\varphi(X \setminus K) \cup K_H = X_H$. Moreover, because $X \setminus K$ is a convex of G, $\varphi(X \setminus K) = X_H \setminus K_h$ is a convex of H, whence K_H is a prime convex of H. Thus we have shown that every two convexies I_H and J_H, for which $I_H \cap J_H = \emptyset$, can be separated by a prime convex K_H in H. Hence every convex I_H of H is the intersection of prime convexies containing I_H, and consequently $H \in \mathcal{G}_P$. This completes the proof.

Let $x, y \in X$ in a graph G. Then $x/y = \{w \mid x \in \langle w, y \rangle\}$, see [6]. The following theorem shows a property of the sets x/y in the graphs of the class $\mathcal{G}_P$.

Theorem 6. *If $G \in G_P$, then x/y is a convex of G for every pair $x, y \in X$.*

Proof. Because $x \in \langle x, y \rangle$, $x \in x/y$, too. If every point of X belongs to x/y, then x/y is a convex. Usually, y is a point such that $y \notin x/y$, because $\langle y, y \rangle = y$ and if $x \in \langle y, y \rangle$, then $x=y$. So we assume that $x \neq y$, $x/y \neq X$ and consequently obtain the result $y \notin x/y$. According to the property of G, there is a prime convex P such that $x \in P$ and $y \notin P$, and thus we can form $W = \bigcap \{P \mid P$ is a prime convex in G, $x \in P$ and $y \notin P\}$. Trivially, W is a convex of G, and we will show that $W = x/y$, from which the assertion follows.

Let $w \in W$ but $x \notin \langle w, y \rangle$. Because every convex of G can be obtained as an intersection of prime convexies containing the convex under question, there is a prime convex P of G such that $x \in P$ but $\langle w, y \rangle \subset X \setminus P$, whence $w \notin W$, which is a contradiction. Thus $W \subset x/y$. On the other hand, if there is a point $z \in X \setminus W$ such that $x \in \langle z, y \rangle$, we obtain a contradiction, too. Indeed, when

$z \in X \setminus W$, there is a prime convex P containing x but not z, whence $\langle z, y \rangle \subset X \setminus P \subset X \setminus W$. Thus $x/y \subset W$, and consequently, $W = x/y$. This completes the proof.

Next we like to consider congruences on a graph $G \in \mathcal{G}_P$.

Lemma 1. *Let P be a prime convex of a graph G, $P \neq X$. Then $\theta(P)$, where $(x, y) \in \theta(P) \Leftrightarrow x, y \in P$ or $x, y \in X \setminus P$, is congruence on G.*

Proof. Obviously $\theta(P)$ is reflexive and symmetric. Because it has only two classes P and $X \setminus P$, it is transitive, too. So it remains to prove the $\langle\ ,\ \rangle$-compatibility of $\theta(P)$. Let $x, y, z, w \in P$. Then also $\langle x, z \rangle$, $\langle y, w \rangle \subset P$, whence (x, y), $(z, w) \in \theta(P)$ implies $(\langle x, z \rangle, \langle y, w \rangle) \in \theta(P)$. The proof is analogous, if $x, y, z, w \in X \setminus P$. Let $x, y \in P$ and $z, w \in X \setminus P$. Then $\langle x, z \rangle \cap P \neq \emptyset \neq \langle x, z \rangle \cap X \setminus P$ as well as $\langle y, w \rangle \cap P \neq \emptyset \neq \langle y, w \rangle \cap X \setminus P$, and so (x, y), $(z, w) \in \theta(P)$ implies $(\langle x, z \rangle, \langle y, w \rangle) \in \theta(P)$. Thus $\theta(P)$ is $\langle\ ,\ \rangle$-compatible and hence a congruence on G.

Theorem 7. *Let $G \in \mathcal{G}_P$ and $C(G)$ exist. Then every congruence θ on G is the meet of maximum elements of $C(G)$.*

Proof. Let P be a prime convex of G. Because $\theta(P)$ has at most two congruence classes, it is a maximum element in $C(G)$ (a maximum element θ has the property that if $\Phi \geq \theta$ then $\Phi = \theta$ or $\Phi = 1$).

Suppose now that $\psi, \Phi \in C(G)$ and $\psi \not\leq \Phi$. We will show that there is a prime convex P of G such that $\theta(P) \geq \Phi$ and $\theta(P) \not\geq \psi$. This property will be used as follows: Let Φ be a fixed element of $C(G)$ and $\psi = \{\theta(P) \mid \theta(P) \geq \Phi$ in $C(G)\}$. If $\Phi \neq \psi$, then $\psi \not\leq \Phi$, because $\Phi \leq \psi$ according to the definition of ψ. Then the property we will prove below determines a $\theta(P) \geq \Phi$ and $\theta(P) \not\geq \psi$, which contradicts the definition of ψ. Hence $\Phi = \psi = \bigwedge\{\theta(P) \mid \theta(P) \geq \Phi\}$.

Because $\psi \not\leq \Phi$, there is a pair x, y of points such that $(x, y) \in \psi$ and $(x, y) \notin \Phi$. Every congruence class of a congruence on X is a convex of G. Let C_{xy} be the congruence class of Φ containing x, and because $(x, y) \notin \Phi$, $y \notin C_{xy}$. Let φ be the homomorphism related to Φ. Because $y \notin C_{xy}$, $\varphi(x) \neq \varphi(y)$, and as shown in Theorem 5, $\varphi(G) \in \mathcal{G}_P$. Hence there is a prime ideal P' in $\varphi(G)$ containing $\varphi(x)$ but not $\varphi(y)$. $\varphi^{-1}(P')$ is a prime convex of G with the property: $z, u \in \varphi^{-1}(P')$ or $z, u \in X \setminus \varphi^{-1}(P')$ when $(z, u) \in \Phi$. But then $\theta(\varphi^{-1}(P')) \geq \Phi$ and $\theta(\varphi^{-1}(P')) \not\geq \psi$ because $(x, y) \notin \theta(\varphi^{-1}(P'))$. This completes the proof.

Now we can prove a theorem characterizing the least congruence having a given convex I as congruence class.

Theorem 8. *Let $G \in \mathcal{G}_P$, $C(G)$ exist, and I be a convex of G, $I \neq X$. Then* (3) *is a congruence on G having I as a congruence class. Moreover,* (4) *characterizes $\theta[I]$ and it is the least congruence having I as a congruence class.*

(3) $\theta[I] = \bigwedge\{\theta(P) \mid P$ *is a prime convex of G and $I \subset P\}$.*

(4) $(x, y) \in \theta[I] \Leftrightarrow$ *there are two points $i_1, i_2 \in I$* such *that $y \in \langle x, i_1 \rangle$ and $x \in \langle y, i_2 \rangle$.*

Proof. Clearly $\theta[I]$ is a congruence on X. Because every P in (3) contains I, $(x, y) \in \theta(P)$ for every P in (2) and for every two points $x, y \in I$, whence $(x, y) \in \theta[I]$. On the other hand, $I = \wedge\{P \mid P$ is a prime convex and $I \subset P\}$, whence $(w, y) \notin \theta(P)$ for at least one $\theta(P)$ in (3) and for every two points w, y, $w \in X \setminus I$ and $y \in I$. Hence I is a congruence class of $\theta[I]$. Because of Theorem 7 $\theta[I]$ is the least congruence having I as a congruence class.

Let x and y be two points such that $y \in \langle x, i_1 \rangle$ and $x \in \langle y, i_2 \rangle$ for some points $i_1, i_2 \in I$, and let P be a prime convex from (3). If $x \in P$, then also $y \in P$, because $i_1 \in I \subset P$ and thus $y \in \langle x, i_1 \rangle \subset P$. Similarly one sees that if $x \in X \setminus P$ then also $y \in X \setminus P$. Hence $(x, y) \in \theta(P)$, and the result is valid for every P in (3), whence $(x, y) \in \theta[I]$.

Conversely, let $(x, y) \in \theta[I]$ and assume that $\langle x, i \rangle$ does not contain y for any $i \in I$. Thus $I \vee \{x\}$ is a convex of G not containing y. Because $G \in \mathcal{G}_P$, $I \vee \{x\} = \wedge \{P' \mid P'$ is a prime convex and $I \vee \{x\} \subset P'\}$, and since $y \notin I \vee \{x\}$, there is a P' such that $(x, y) \notin \theta(P')$. Further, $I \subset I \vee \{x\} \subset P'$, whence every $\theta(P')$ is in (3). But this implies that $(x, y) \notin \theta[I]$, which is a contradiction. Hence there is an $i_1 \in I$ such that $y \in \langle x, i_1 \rangle$ and from the same reason there is an $i_2 \in I$ such that $x \in \langle y, i_2 \rangle$. This completes the proof.

In the case of lattices the lattice congruences $\theta[I]$ characterize the distributivity. A partial converse can be obtained in the case of graphs.

Theorem 9. *Let G be a graph and I a convex of G, $I \neq X$. If the least congruence $\theta[I]$ having I as a congruence class is characterized by* (4), *then any point $x \in X$ is a base of G.*

Proof. The least congruence having $I = \{x\}$ as a congruence class is the identity relation U: $(u, w) \in \Leftrightarrow u = w$, which is trivially a congruence on G. Because $U = \theta[\{x\}]$, we obtain $(u, w) \in \theta[\{x\}] \Leftrightarrow w \in \langle u, x \rangle$ and $u \in \langle w, x \rangle \Leftrightarrow \langle u, x \rangle = \langle w, x \rangle$, and this is possible only when $u = w$. Hence $\langle u, x \rangle \neq \langle w, x \rangle$ for every two disjoint points $u, w \in X \setminus \{x\}$. This shows that an arbitrary point x is a base of G, and the theorem follows.

Although in $G \in \mathcal{G}_P$ any point $x \in X$ is a base of G, the converse does not hold. A counterexample is given in [5].

Because of the defining property of the class $\mathcal{G}_P$ it is possible to prove the congruence extension property.

Theorem 10. *Let $G \in \mathcal{G}_P$, $C(G)$ exist, and K be a convex of G. If Φ is a congruence on K, it can be extended to G, i. e. there is a congruence θ on G such that $(x, y) \in \Phi \Leftrightarrow (x, y) \in \theta$ when $(x, y) \in K$.*

Proof. The proof follows from the corresponding proof for distributive lattices given in [1, Thm. II.3.6].

Because K is a convex in a graph $G \in \mathcal{G}_P$, it induces a subgraph G_K of G and $G_K \in \mathcal{G}_P$. Let φ be the homomorphism related to Φ in K, $\varphi\colon K \to \varphi(K)$. Because $G_K \in \mathcal{G}_P$, $\varphi(G_K) \in \mathcal{G}_P$ according to Theorem 5, and as shown in Theorem 1, the pointset $\varphi^{-1}(P')$ is a prime convex of G_K for every prime convex P' of $\varphi(G_K)$. Because K is a convex of G, $\varphi^{-1}(P') \subset K \subset X$ is a convex of G as well as $K \setminus \varphi^{-1}(P')$. Thus there is a prime convex P of G containing $\varphi^{-1}(P')$ such that $P \cap (K \setminus \varphi^{-1}(P')) = \emptyset$. Let A be the collection of all prime convexies P of G such that $\varphi^{-1}(P') \subset P$ and $K \setminus \varphi^{-1}(P') \subset X \setminus P$ for some prime convex P' of $\varphi(G_K)$. Let us now consider the congruence relation $\theta = \wedge \{\theta(P) \mid P \in A\}$. For $x, y \in K$ the condition $(x, y) \in \Phi$ is equivalent with $\varphi(x) = \varphi(y)$ and so for every $P \in A$ either $x, y \in P$ or $x, y \in X \setminus P$ and thus $(x, y) \in \theta$. Conversely, if $(x, y) \in \theta$, then for every $P \subset A$ either $x, y \in P$ or $x, y \in X \setminus P$ and so either $\varphi(x), \varphi(y) \in P'$ or $\varphi(x), \varphi(y) \notin P'$. Since every pair of distinct points of $\varphi(G_K)$ is separated by a prime convex, we conclude that $\varphi(x) = \varphi(y)$, whence $(x, y) \in \Phi$, and the theorem follows.

Theorem 11. *Let $G \in \mathcal{G}_P$, the lattice $C(G)$ exist, and θ be a congruence on G with classes $C_1, \ldots, C_n$. Then $\theta = \vee \{\theta[C_i] \mid i = 1, \ldots, n\}$.*

Proof. Every congruence class of θ is a convex of G. Trivially $\theta \leq \vee \{\theta[C_i] \mid i = 1, \ldots, n\}$. On the other hand, $\theta[C_i]$ is the least congruence having C_i as a congruence class, whence $\theta[C_i] \leq \theta$ for every i. Hence $\theta \geq \bigvee \{\theta[C_i] \mid i=1, \ldots, n\}$, and the theorem follows.

Theorem 12. *Let $G \in \mathcal{G}_P$ and the lattice $C(G)$ exist. $C(G)$ is distributive if and only if (2) holds.*

Proof. According to Theorem 4 it remains to show that the distributivity of $C(G)$ implies (2). Let $(x, y) \in \theta \vee \Phi$ and assume that there does not exist an $x-y$ path of (2). Thus every path giving the result $(x, y) \in \theta \vee \Phi$ contains at least one point u_i outside from $\langle x, y \rangle$. Because $G \in \mathcal{G}_P$, there is a prime convex P such that $\langle x, y \rangle \subset P$ and $u_i \subset X \setminus P$, whence $(x, y) \in \theta(P)$. Now we can construct a congruence ψ as follows: $\psi = \bigwedge \{\theta(P) \mid P$ is a prime convex, $\langle x, y \rangle \subset P$ and $u_i \notin P\}$. Trivially $(x, y) \in \psi$, whence also $(x, y) \in \psi \wedge (\theta \vee \Phi) = (\psi \wedge \theta) \vee (\psi \wedge \Phi)$ which is impossible because of the definition of ψ.

We call a graph complemented, if for every $x \in X$ there is a point z such that $\langle x, z \rangle = X$. Moreover, G is strongly complemented if its every subgraph G_I induced by a convex I of G is complemented. Every strongly complemented graph $\mathcal{G}$ is complemented, because $G_X = G$ for the convex X of G. The graph G of Figure 2 is from the class $\mathcal{G}_P$ and complemented, but it is not strongly complemented because in the induced subgraph of points v, x, y the point x does not have a complement.

Theorem 13. *If $G \in \mathcal{G}_P$ then the complement x of y is unique.*

Proof. It has been proved in [7, Thm. 4] that when $G \in \mathcal{G}_P$ then any point y of G is a base of G. If there are two different complements x and x' for y, then $\langle x, y \rangle = X = \langle x', y \rangle$, and y is not any more a base of G, which is a contradiction. Hence the theorem.

Theorem 14. *Let $G \in \mathcal{G}_P$, the lattice $C(G)$ exist, and I be a convex of G, $I \neq X$. If G is strongly complemented, then $(x, y) \in \theta[I] \Leftrightarrow$ there are two points $i_1, i_2 \in I$ such that $\langle x, i_1 \rangle = \langle y, i_2 \rangle$.*

Proof. Let $(x, y) \in \theta[I]$. Because $G \in \mathcal{G}_P$, there are $i_1, i_2 \in I$ such that $y \in \langle x, i_1 \rangle$ and $x \in \langle y, i_2 \rangle$. Let us consider the pointset of all classes of $\theta[I]$ that intersect $\langle x, i_1 \rangle$ and let this pointset be $C_{1x} \cup \cdots \cup C_{mx}$, where $C_{1x} = I$ and C_{mx} contains x and y. $C_{1x} \cup \cdots \cup C_{mx}$ is a convex of G because it is the preimage of the convex $\langle c_{1x}, c_{mx} \rangle$ in the homomorphic image of G under the homomorphism related to $\theta[I]$. Because $C_{1x} \cup \cdots \cup C_{mx}$ is a convex, it induces a complemented subgraph G_{xy} of G. In this graph x has a complement i_x and because $G_{xy} \in \mathcal{G}_P$, the complement is unique. $\langle x, i_x \rangle$ and $\langle x, i \rangle$ intersect both all classes $C_{1x}, \ldots, C_{mx}$ and if i_x is from another class than $C_{1x} = I$, say from C_{kx}, then $\langle c_{1x}, c_{kx} \rangle = \langle c_{ix}, c_{mx} \rangle$, which is a contradiction according to Theorem 3. Similarly, y has a complement $i_y \in i$ in G_{xv}, and hence there are points $i_x, i_y \in I$ such that $\langle x, i_x \rangle = \langle y, i_y \rangle$. This completes the proof.

In the following we give a graphtheoretic characterization of Boolean lattices.

Theorem 15. *A graph $G \in \mathcal{G}_P$ is strongly complemented if and only if G is isomorphic to the Hasse diagram of a finite Boolean lattice.*

Proof. In the Hasse diagram of a finite modular lattice the pointset A is a convex if and only if A is a convex sublattice (an interval) of L (see [9]). As well known, every convex sublattice of a Boolean lattice is also Boolean and hence every convex of the Hasse diagram H of a Boolean lattice is complemented. Thus the graph H is strongly complemented and obviously $H \in \mathcal{G}_P$.

Conversely, let G be from $\mathcal{G}_P$, strongly complemented and let x be an arbitrary point of G. Then x has a unique complement x' in G such that

$\langle x, x'\rangle = X$. When proving the isomorphism of G to the Hasse diagram H of a Boolean lattice, we will use the induction over the number of points in G. If $|X|$ is 1 or 2, G is obviously isomorphic to the Hasse diagram of a Boolean lattice, and so we assume that the isomorphism exists always when $|X| \leq k$. Let $|X| = k+1$ and let x be adjacent to $y_1, \ldots, y_r$ in G. Every $\langle y_i, x'\rangle$ is a convex of G and $\langle y_1, x'\rangle \cup \cdots \cup \langle y_r, x'\rangle = X \setminus \{x\}$. According to the induction assumption, the subgraph $G(y_i, x')$ induced by $\langle y_i, x'\rangle$ is isomorphic to the Hasse diagram of a Boolean lattice, and as well known, we can choose the greatest element freely and thus in every $G(y_i, x')$ x' is chosen as the greatest element, and consequently, y_i is the least element in $G(y_i, x')$, $i = 1, \ldots, r$. Let $y'_1, \ldots, y'_s$ be the points adjacent to x'. The induction assumption ensures that every $\langle y'_j, x\rangle$ induces a subgraph $G(x, y'_j)$ isomorphic to the Boolean lattice, and in every such lattice we can choose y'_j the greatest and x the least element. Obviously G is then isomorphic to the Hasse diagram of a lattice L_G (and this can be also proved by means of [6, Thm. 8]). In this lattice L_G every proper convex sublattice is complemented according to the induction assumption, and according to the strong complementarity, also L_G is a complemented lattice. Because $G \in \mathcal{G}_p$, all complements are unique. As well known, a lattice, where the complements in every convex sublattice are unique, is distributive, and hence L_G is distributive. Because L_G is distributive and complemented, it is Boolean. This completes the proof.

At last we like to mention that in the class $\mathcal{G}_p$ complete graphs can be characterized by means of congruence relations. The characterizations we have found are all based on the property that every convex of a complete lattice is a prime convex. Moreover, trees and, in general, graphs where every block is a complete graph (those graphs include complete graphs and trees as special cases) can also be characterized by means of congruences. In these graphs the congruences $\theta[I]$ have namely a sharpened form: $(x, y) \in \theta[I] \Leftrightarrow x, y \in I$ or $x = y$.

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