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ERROR ESTIMATES FOR NUMERICAL SOLUTIONS OF CAUCHY-PROBLEM

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The error estimations are obtained for Runge-Kutta's and Adams' methods for the numerical solution of the Cauchy-problem. The estimations are expressed by a new averaged modulus of smoothness, without additional conditions of smoothness of the solution of the equation. From the properties of the modulus series of estimates can be obtained under the various assumption for smoothness of the solution.

In this paper we shall consider the most commonly used methods of Runge-Kutta and Adams for numerical solution of the Cauchy-problem [1; 2]

$$(1)\qquad \begin{aligned} &y' = f(x, y), \quad x \in [x_0, x_0 + A], \quad A > 0, \\ &y(x_0) = y_0, \end{aligned}$$

where the function f satisfies the condition

$$(2)\qquad |f(x, y) - f(x, z)| \leq K|y - z|, \quad K \text{ — absolute constant.}$$

Our purpose is to find estimates by means of the so-called averaged modulus of smoothness $\tau_k(f; \delta)_{L_p}$, $1 \leq p < \infty$, [3; 4; 5]. For a given function f which is defined in the interval $[a, b]$ the definition of the modulus $\tau_k(f; \delta]_{L_p}$ is

$$\tau_k(f; \delta)_{L_p} = \|\omega_k(f, \cdot\,; \delta)\|_{L_p}, \text{ where } \omega_k(f, x; \delta) = \sup\{|\Delta_h^k f(t)|,$$
$$t, t + kh \in [x + k\delta/2, x - k\delta/2] \cap [a, b]\}$$

and $\Delta_h^k f(t)$ is the finite difference of the k-th order of the functlon f.

In [3; 5: 6] the following properties of the modulus $\tau_k(f; \delta)_{L_p}$ are proved:

a. $\tau_k(f; \delta)_L \xrightarrow[\delta\to 0]{} 0$ iff f is integrable in the sense of Riemann;

b. $\tau_k(f; \delta')_{L_p} \leq \tau_k(f; \delta'')_{L_p}$ for $\delta' \leqq \delta''$;

c. $\tau(f; \delta)_L \leq 2\delta V(f)$, where $V(f)$ is a variation of the function f;

d. $\tau(f; \delta)_{L_p} \leq \delta \|f'\|_{L_p}$;

e. $\tau_k(f; \lambda\delta)_{L_p} \leqq (2\lambda + 2)^{k+1} \tau_k(f; \delta)_{L_p}$, in the case when n is an integer number $\tau(f; n\delta)_L \leq n\tau(f; \delta)_L$;

f. $\tau_k(f; \delta)_{L_p} \leq c(k)\, \omega_{k-1}(f'; \delta)_{L_p}$, where $\omega_k(f; \delta)_{L_p}$ is the usual modulus of f in L_p, i. e.

$$\omega_k(f; \delta)_{L_p} = (\sup_{0 \leq h \leqq \delta} \int_a^{b-kh} |\Delta_h^k f(t)|^p\, dt)^{1/p}.$$

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Further we shall use the following two lemmas:

Lemma 1 (well-known Whitney's theorem [7]). *For each integer $n \geq 1$ there is a constant $c(n)$ such that for each bounded function f in the interval $[a, b]$ the inequality*

$$\max_{a \leq x \leq b} |f(x) - P(x)| \leq c(n)\omega_{n+1}\left(f, \frac{a+b}{2}; \frac{b-a}{n+1}\right)$$

holds, where P is a Lagrange polynomial for f of degree n, constructed with respect to equidistant knots in $[a, b]$. In the case $n=1$, $c(1)=1$.

Lemma 2. *Let L be a linear bounded functional in the space $M[a, b]$ of all bounded functions on the interval $[a, b]$ and for every polynomial P of degree $\leq n$ the condition $L(P)=0$ is fulfilled. Then for each function $f \in M[a, b]$ the following estimate*

$$|L(f)| \leq \| L \|_M \cdot c(n)\, \omega_{n+1}\left(f, \frac{a+b}{2}; \frac{b-a}{2}\right)$$

holds, where $\|L\|_M$ is the norm of the functional L and $c(n)$ is the constant from Lemma 1.

Proof. By lemma 1 there exists a polynomial P of degree n for which $\|f-P\|_M = \sup_{a \leq x \leq b} |f(x) - P(x) \leq c(n)\omega_{n+1}\left(f, \frac{a+b}{2}; \frac{b-a}{n+1}\right)$ and using the fact $L(P) = 0$ and a linearity of L we get

$$|L(f)| \leq |L(P)| + |L(f-P)| \leq \|L\|_M \|f-P\|_M \leq \|L\|_M\, c(n)\omega_{n+1}\left(f, \frac{a+b}{2}; \frac{b-a}{n+1}\right).$$

1. Runge-Kutta's methods. We shall consider the two simplest methods— Euler's method [1]

$$u_{i+1} = u_i + hf(x_i, u_i), \quad h = A/n, \quad x_i = x_0 + ih, \quad i = 0, 1, \ldots, n-1, \quad u_0 = y_0 \tag{4}$$

and the method with local error $O(h^3)$ [1]

$$u_{i+1} = u_i + phf(x_i, u_i) + qhf(x_i + \alpha h, u_i + \beta h f(x_i, u_i)), \quad i = 0, 1, \ldots, n-1, \quad u_0 = y_0, \tag{5}$$

where the constants p, q, α, β satisfy the conditions

$$p + q = 1, \quad \alpha q = \beta q = 1/2. \tag{6}$$

A. Euler's method. From (1) and (4) we receive $|y_{i+1} - u_{i+1}| = |y_{i+1} - u_i - hf(x_i, u_i) + hf(x_i, y_i) - hf(x_i, y_i)| \leq |y_{i+1} - y_i + y_i - u_i - hf(x_i, y_i)| + h|f(x_i y_i) - f(x_i, u_i)|$ and using (2) we have

$$|y_{i+1} - u_{i+1}| \leq (1 + hK)|y_i - u_i| + h|(y_{i+1} - y_i)/h - y'_i|.$$

As $|(y_{i+1} - y_i)/h - y'_i| \leq \omega(y', x_{i+1/2}; h)$ $(x_{i+1/2} = x_i + h/2)$ from above inequality it follows

$$|y_{i+1} - u_{i+1}| \leq (1 + hK)|y_i - u_i| + h\omega(y', x_{i+1/2}; h). \tag{7}$$

Applying recursively (7) we obtain $|y_{i+1} - u_{i+1}| \leq (1 + hK)|y_i - u_i| + h\omega(y', x_{i+1/2}; h) \leq (1 + hK)^2 |y_{i-1} - u_{i-1}| + (1 + hK) h\omega(y', x_{i-1/2}; h) + h\omega(y', x_{i+1/2}; h) \leq \ldots \leq (1 + hK)^i h\omega(y', x_{1/2}; h) + (1 + hK)^{i-1} h.$

$\omega(y', x_{3/2}; h) + \dots + h\omega(y', x_{i+1/2}; h)$ and from the last inequality it follows

$$|y_{i+1} - u_{i+1}| \leq (1+Kh)^n \sum_{k=0}^{n-1} h\omega(y', x_{i+1/2}; h)$$

$$= (1+\frac{KA}{n})^n \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \omega(y', x_{i+1/2}; h)dx \leq e^{KA} \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \omega(y', x; 2h)dx$$

$$= e^{KA} \int_{x_0}^{x_0+A} \omega(y', x; 2h)dx = e^{KA}\tau(y'; 2h)_L = 2e^{KA}\tau(y'; h)_L.$$

So we obtain

Theorem 1. *If we solve numerically the problem* (1), (2) *using Euler's method* (4) *the following estimate*

$$\max_{0 \leq i \leq n} |y_i - u_i| \leq 2e^{KA}\tau(y'; h)_L$$

holds.

B. The method (5), (6) *with a local error* $0(h^3)$. It follows from (6) that this system has one parametric solution

(8) $$p = 1 - s, \quad q = s, \quad \alpha = \beta = 1/2s.$$

From (5), (6) and (8) we get

(9) $$|y_{i+1} - u_{i+1}| = |y_{i+1} - u_i - phf(x_i, u_i) - qhf(x_i + \alpha h, \ u_i + \beta hf(x_i, u_i))|$$

$$= |y_{i+1} - u_i - (1-s)hf(x_i, u_i) + (1-s)hf(x_i, y_i) - (1-s)hf(x_i, y_i)$$

$$- shf(x_i + h/2s, u_i + \frac{h}{2s} f(x_i, u_i)) + shf(x_i + h/2s, y_i + \frac{h}{2s} f(x_i, y_i))$$

$$- shf(x_i + \frac{h}{2s}, y_i + \frac{h}{2s} f(x_i, y_i)| \leq K|1-s|h|y_i - u_i| + K|s|h|u_i$$

$$+ \frac{h}{2s} f(x_i, u_i) - y_i - \frac{h}{2s} f(x_i, y_i)| + |y_{i+1} - u_i - y_i + y_i - (1-s)hy'_i$$

$$- shf(x_i + \frac{h}{2s}, y_i + \frac{h}{2s} f(x_i, y_i))| \leq K|1-s|h|u_i - y_i| + K|s|h|u_i - y_i|$$

$$+ \frac{K^2h^2}{2}|u_i - y_i| + |u_i - y_i| + |y_{i+1} - y_i + (1-s)hy'_i - shf(x_i + h/2s, y_i$$

$$+ \frac{h}{2s} f(x_i, y_i))| \leq C_1|y_i - u_i| + C_2,$$

where $C_1 = 1 + Kh(|1-s| + |s|) + K^2h^2/2$, $C_2 = |x_{i+1} - y_i - (1-s)hy'_i - shf(x_i + \frac{h}{2s}, y_i + \frac{h}{2s} f(x_i, y_i))|$. Let's estimate C_2. Successively we get

$$(10)\quad C_2=|y_{i+1}-y_i-(1-s)hy_i'-shf(x_i+\frac{h}{2s},\ y_i+\frac{h}{2s}f(x_i,\ y_i))-hy'_{i+1/2}$$

$$+hy'_{i+1/2}-shf(x_i+\frac{h}{2s},\ y_{i+1/2s})+shf(x_i+\frac{h}{2s},\ y_{i+1/2s})|$$

$$\leqq|y_{i+1}-y_i-hy'_{i+1/2}|+h|y'_{i+1/2}-(1-s)y_i'-sy'_{i+1/2s}|+|s|h|f(x_{i+1/2s},$$

$$y_{i+1/2s})-f(x_{i+1/2s},\ y_i+\frac{h}{2s}f(x_i,\ y_i))|,$$

where $x_{i+1/2s}=x_i+h/2s$, $y_{i+1/2s}=y(x_{i+1/2s})$, $y'_{i+1/2s}=y'(x_{i+1/2s})$. We shall estimate the first term in the right side in (10)

$$(11)\quad |y_{i+1}-y_i-hy'_{i+1/2}|=h|(y_{i+1}-y_i)/h-y'_{i+1/2}|=|\int_{x_i}^{x_{i+1}}(y'(t)-y'_{i+1/2})\,dt|$$

$$=h|\int_{-1/2}^{1/2}(y'(x_{i+1/2}+th)-y'_{i+1/2})dt|=h|\int_0^{1/2}(y'(x_{i+1/2}+th)-2y'_{i+1/2}$$

$$+y'(x_{i+1/2}-th))dt|\leqq h\int_0^{1/2}\omega_2(y',\ x_{i+1/2};\ h/2)dt=\frac{h}{2}\omega_2(y',\ x_{i+1/2};\ h/2).$$

On the other hand

$$|f(x_{i+1/2s},\ y_{i+1/2s})-f(x_{i+1/2s},\ y_i+\frac{h}{2s}f(x_i,\ y_i))|\leqq K|y_{i+1/2s}-y_i-\frac{h}{2s}y_i'|$$

$$=\frac{Kh}{2|s|}\left|\frac{y_{i+1/2s}-y_i}{h/2s}-y_i'\right|=\frac{Kh}{2|s|}\left|\frac{2s}{h}\int_{x_i}^{x_{i+1/2s}}(y'(t)-y_i')dt\right|$$

$$\leqq\frac{Kh}{2|s|}\omega(y',\ x_{i+1/4s};\ h/2|s|)\leqq\frac{Kh}{2|s|}\omega(y',\ x_{i+1/2};\ (1+\frac{1}{|s|})h).$$

At last we have to estimate the term $|y'_{i+1/2}-(1-s)y_i'-sy'_{i+1/2s}|$. To this end let us denote

$$(12)\qquad a=\min(x_i,\ x_{i+1/2s}),\quad b=\min(x_{i+1/2},\ x_{i+1/2s}).$$

Let P be an algebraical polynomial of first degree, which interpolates y' at the points a and b. By Lemma 1 we have

$$(13)\qquad \max_{a\leqq x\leqq b}|y'(x)-P(x)|\leqq\omega_2(y',\ \frac{a+b}{2};\ \frac{b-a}{2}).$$

From (12) it follows $|y'_{i+1/2}-(1-s)y_i'-sy'_{i+1/2s}|\leqq|y'_{i+1/2}-P(x_{i+1/2})|+|1-s||y_i'-P(x_i)|+|s||y'_{i+1/2s}-P(x'_{i+1/2s})|+|P_{i+1/2}-(1-s)P_i-sP_{i+1/2s}|$ and in view of $P_{i+1/2}-(1-s)P_i-sP_{i+1/2s}=0$ (this equality holds for every polynomial of degree at most 1) we get using (13)

$$(14)\qquad |y'_{i+1/2}-(1-s)y_i'-sy'_{i+1/2s}|\leqq(1+|s|+|1-s|)\,\omega_2(y',\ \frac{a+b}{2};\frac{b-a}{2}).$$

From (12) it follows that there exist constants $c_3(s)$ and $c_4(s)$ such that

$$(15)\qquad (b-a)/2=c_3(s)h,\quad (a+b)/2=x_{i+1/2}+c_4(s)h$$

and finally from (14) and (15) we obtain

(16) $$|y'_{i+1/2}-(1-s)y'_i-sy'_{i+1/2s}|\leq(1+|s|+(1-s|)\omega_2(y',\ x_{i+1/2}+c_4(s)h;$$
$$c_3(s)h)\leq(1+|s|+|1-s|)\omega_2(y',\ x_{i+1/2};\ c_5(s)h),$$

where $c_5(s)=|c_4(s)|+c_3(s)$. From (9), (10), (11), (16) it follows

$$|y_{i+1}-u_{i+1}|\leq(1+Kh(|1-s|+|s|+Kh/2))|y_i-u_i|+\frac{h}{2}\omega_2(y',x_{i+1/2};\ h/2)$$
$$+\frac{Kh^2}{2}\omega(y',\ x_{i+1/2};\ (1+\frac{1}{|s|})h)+(1+|s|+|1-s|)h\omega_2(y',\ x_{i+1/2};\ c_5(s)h).$$

Applying recursively the last inequality we obtain

(17) $$|y_{i+1}-u_{i+1}|\leq\sum_{k=0}^{i}(1+Kh(|1-s|+|s|+\frac{Kh}{2})^{i-k}\{\frac{h}{2}\ \omega_2(y',\ x_{k+1/2};\ \frac{h}{2})$$
$$+\frac{Kh^2}{2}\omega(y',\ x_{k+1/2};\ (1+\frac{1}{|s|})h)+(1+|s|+|1-s|)h\omega_2(y',\ x_{k+1/2};\ c_5(s)h)\}.$$

Let us denote $|s|+|1-s|+Kh/2=|s|+|1-s|+KA/2n\leq|s|+|1-s|+KA/2=c_6(s)$, $1+|s|+|1-s|=c_7(s)$. Then from (17) we get

$$\max_{0\leq i\leq n}|y_i-u_i|\leq(1+c_6(s)KA/n)^n\sum_{k=0}^{n-1}\{\frac{h}{2}\omega_2(y',\ x_{k+1/2};\ h/2)+\frac{Kh^2}{2}\omega(y',\ x_{k+1/2};$$
$$(1+\frac{1}{|s|})h)+c_7(s)h\omega_2(y';\ x_{k+1/2};\ c_5(s)h)\}\leq\exp(c_6(s)KA)\sum_{k=0}^{n-1}\{\frac{1}{2}\int_{x_k}^{x_{k+1}}\omega_2(y',$$
$$x_{k+1/2};\ h/2)\,dx+\frac{Kh}{2}\int_{x_k}^{x_{k+1}}\omega(y',\ x_{k+1/2};\ (1+\frac{1}{|s|})h)dx+c_7(s)\int_{x_k}^{x_{k+1}}\omega_2(y',$$
$$x_{k+1/2};\ c_5(s)h)dx\}\leq\exp(c_6(s)KA)\sum_{k=0}^{n-1}\{\frac{1}{2}\int_{x_k}^{x_{k+1}}\omega_2(y',\ x;\ h)dx$$
$$+\frac{Kh}{2}\int_{x_k}^{x_{k+1}}\omega(y',\ x;\ (3+\frac{1}{|s|})h)dx+c_7(s)\int_{x_k}^{x_{k+1}}\omega_2(y',\ x;\ (c_5(s)+1)h)\,dx\}$$
$$\leq\exp(c_6(s)KA)\int_{x_0}^{x_0+A}\{\frac{1}{2}\omega_2(y',\ x;\ h)+\frac{Kh}{2}\omega(y',\ x;\ (3+\frac{1}{|s|})h)$$
$$+c_7(s)\omega_2(y',\ x;\ (c_5(s)+1)h)\}dx=\exp(c_6(s)KA)\ \{\frac{1}{2}\tau_2(y';\ h)_L$$
$$+\frac{Kh}{2}\tau(y',(3+\frac{1}{|s|})h)_L+c_7(s)\tau_2(y';\ (c_5(s)+1)h)_L\}.$$

From the last inequality we get (using the properties (3))

Theorem 2. *If we solve numerically the problem* (1), (2) *using the methods* (5), (6) *the following estimate*

$$\max_{0\leq i\leq n} |y_i - u_i| \leq c(s)\{\tau_2(y'\,;\,h)_L + h\tau(y'\,;\,h)_L\}$$

holds.

From the proof of the theorem 2 it is clear that the constant $c(s)$ can be estimated successfully for a given s.

2. Adams' methods. *A. Extrapolation formulas.* Let us take that we know the approximate values $u_0, u_1, \ldots, u_i$, of the solution y of (1) at the points $x_j = x_0 + jh$, $j = 0, 1, \ldots, i$. So we know the approximate values $u_0', u_1', \ldots, u_i'$ of y' at these points. Using Newton's backwards interpolated formula at the points $x_i, x_{i-1}, \ldots, x_{i-k}$ we have [1]

$$\text{(18)}\qquad \begin{aligned} u'(x_i+th) &= u_i' + \frac{t}{1!}\Delta_h u_{i-1}' + \frac{t(t+1)}{2!}\cdot\Delta_h^2 u_{i-2}' + \cdots \\ &+ \frac{t(t+1)\ldots(t+k-1)}{k!}\Delta_h^k u_{i-k}' + R_k(t),\ (x-x_i)/h = t, \end{aligned}$$

where $R_k(t)$ is the error. The term $R_k(t)$ in (18) can be neglected and integrating we have

$$\int_{-j}^{1} u'(x_i+ht)dt = \int_{-j}^{1}\{u_i' + \frac{t}{1!}\Delta_h u_{i-1}' + \cdots + \frac{t(t+1)\ldots(t+k-1)}{k!}\Delta_h^k u_{i-k}'\}\,dt$$

The last equality can be written in the form

$$\text{(19)}\qquad u_{i+1} = u_{i-j} + h\,\{(1+j)u_i' + \sum_{s=1}^{k} \alpha_{sj}\Delta_h^s u_{i-s}'\},$$

where

$$a_{sj} = \int_{-j}^{1}\frac{t(t+1)\ldots(t+s-1)}{s!}\,dt.$$

If we expand the finite differences in (19) then (19) can be written in the form

$$\text{(20)}\qquad u_{i+1} = u_{i-j} + h\sum_{m=0}^{k}\beta_{jm}\,u_{i-m}'.$$

Let us mention that for every polynomial P of degree at most k the following equality

$$\int_{-j}^{1} P(x_i+ht)dt = \int_{-j}^{1}\{P_i + \frac{t}{1!}\Delta_h P_{i-1} + \cdots + \frac{t(t+1)\ldots(t+k-1)}{k!}\Delta_h^k P_{i-k}\}dt$$

holds ($R_k(t) = 0$ in (18)) and therefore

$$\text{(21)}\qquad \int_{-j}^{1} P(x_i+ht)dt = \sum_{m=0}^{k}\beta_{jm}P_{i-m}.$$

We shall estimate the error $\varepsilon_i = u_i - y_i$, $i = 0, 1, \ldots, n$, $h = A/n$. The exact solution of (1), (2) satisfies equality of the type

$$y_{i+1} = y_{i-j} + h \sum_{m=0}^{k} \beta_{jm} y'_{i-m} + r_i. \tag{22}$$

First we shall estimate $|r_i|$. In this conection we consider the linear functional

$$L(g) = \int_{-j}^{1} g(x_i + ht)dt - \sum_{m=0}^{k} \beta_{jm} g_{i-m}.$$

From (21) it follows that $L(P) = 0$ if P is an algebraical polynomial of degree at most k. In the space $M[c, d]$ of all bounded in the interval $[c, d] = [\min(x_{i-k}, x_{i-j}), x_{i+1}]$ functions with the norm $\| g \|_{M[c,d]} = \sup_{c \leq x \leq d} |g(x)|$ we have

$$|L(g)| \leq (1+j) \| g \|_{M[c,d]} + \sum_{m=0}^{k} |\beta_{jm}| \| g \|_{M[c,d]}$$

$$= (1 + j + \sum_{m=0}^{k} |\beta_{jm}|) \| g \|_{M[c,d]}.$$

The last inequality shows that

$$\| L \|_{M[c,d]} \leq 1 + j + \sum_{m=0}^{k} |\beta_{jm}| = c(k, j). \tag{23}$$

The exact solution of (1) satisfies $L(y') = r_i/h$ and in view of (23), applying lemma 2 we get

$$|r_i|/h = |L(y')| \leq c(k, j)\, c(k)\, \omega_{k+1}(y', x_{i-(k-1)/2}; h) \tag{24}$$

(we presuppose that $j \leq k$, usually $j = 0$ or $j = 1$). From (2), (20) and (22) it follows

$$\varepsilon_{i+1} = y_{i+1} - u_{i+1} = y_{i-j} - u_{i-j} + h \sum_{m=0}^{k} \beta_{jm}(y'_{i-m} - u'_{i-m}) + r_i$$

$$= \varepsilon_{i-j} + h \sum_{m=0}^{k} \beta_{jm}(f(x_{i-m}, y_{i-m}) - f(x_{i-m}, u'_{i-m})) + r_i,$$

i. e.

$$|\varepsilon_{i+1}| \leq |\varepsilon_{i-j}| + hK \sum_{m=0}^{k} |\beta_{jm}| \, |\varepsilon_{i-m}| + |r_i|. \tag{25}$$

If we know the upper bound for the errors $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k$,

$$|\varepsilon_i| \leq \delta, \quad i - 0, 1, \ldots, k, \tag{26}$$

then under the notation

$$B = \sum_{m=0}^{k} |\beta_{jm}| \tag{27}$$

we get from (25) and (26)

$$|\varepsilon_{k+1}|\leqq(1+hKB)\delta+|r_k|$$

$$|\varepsilon_{k+2}|\leqq(1+hKB)^2\delta+(1+hKB)|r_k|+|r_{k+1}|$$

$$\cdots\cdots$$

$$|\varepsilon_n|\leqq(1+hKB)^{n-k}\delta+\sum_{m=k}^{n-1}(1+hKB)^{n-m-1}|r_m|.$$

From the above inequalities it follows

$$\max_{0\leqq i\leqq n}|\varepsilon_i|\leqq(1+AKB/n)^n(\delta+\sum_{m=k}^{n-1}|r_m|)\leqq\exp(KAB)(\delta+\sum_{m=k}^{n-1}|r_m|) \tag{28}$$

and from (24) we obtain

$$\sum_{m=k}^{n-1}|r_m|\leqq c(k,\ j)c(k)h\sum_{m=k}^{n-1}\omega_{k+1}(y',\ x_{m-(k-1)/2};\ h)$$

$$=c(k,\ j)\,c(k)\sum_{m=k}^{n-1}\int_{x_m}^{x_{m+1}}\omega_{k+1}(y',\ x_{m-(k-1)/2};\ h)dx$$

$$\leqq c(k,\ j)\,c(k)\sum_{m=k}^{n-1}\int_{x_m}^{x_{m+1}}\omega_{k+1}(y',\ x;\ c'(k)h)dx$$

$$=c(k,\ j)\,c(k)\int_{x_0}^{x_0+A}\omega_{k+1}(y',\ x;\ c''(k)h)dx=c(k,\ j)c(k)\tau_{k+1}(y';\ c''(k)h)_L$$

$$\leqq c_1'(k,\ j)\,\tau_{k+1}(y';\ h)_L.$$

So (28) and (29) prove

Theorem 3. *If we solve numerically the problem* (1), (2) *using Adams' extrapolated formula* (20) *the following estimate*

$$\max_{0\leqq i\leqq n}|y_i-u_i|\leqq\exp(KBA)(\delta+c_1'(k,\ j)\,\tau_{k+1}(y';\ h)_L)$$

holds, where the constants δ, B *and* $c_1'(k, j)$ *are given by* (26), (27), (29).

B. Interpolation formulas. Adams' interpolated formulas can be obtained if we integrate (18) in the interval $[-j, 0]$. Then

$$u_i=u_{i-j}+h\sum_{m=0}^{k}\alpha_{mj}u'_{i-m}, \tag{30}$$

where the constants α_{mj} are obtained in the same way as β_{jm} in (20) and for every polynomial P of degree at most k we have similarly to (21)

$$\int_{-j}^{0}P(x_i+ht)dt=\sum_{m=0}^{k}\alpha_{mj}P_{i-m}. \tag{31}$$

The exact solution y of (1) satisfies the equality

$$y_i=y_{i-j}+h\sum_{m=0}^{k}\alpha_{mj}y'_{i-m}+r_i, \tag{32}$$

If we consider the linear functional $L(g)=\int_{-j}^{0} g(x_i+ht)dt-\Sigma_{m=0}^{k}\alpha_{mj}g_{i-m}$, then in view of (31) it follows $L(P)=0$ for every polynomial P of degree $\leq k$ and analogously as in (23)

$$(33)\qquad \|L\|_{M[c,\ d]}\leq j+\sum_{m=0}^{k}|\alpha_{mj}|=c_1(k,\ j),\ [c,\ d]=[x_{i-k},\ x_i].$$

From (30) and (32) applying Lemma 2 we get

$$(34)\qquad |r_i|/h=|L(y')|\leq c_2(k,\ j)\omega_{k+1}(y',\ x_{i-k/2}\ ;\ h)$$

and

$$\varepsilon_i=y_i-u_i=y_{i-j}-u_{i-j}+h\sum_{m=0}^{k}\alpha_{mj}(y'_{i-m}-u'_{i-m})+r_i$$

$$=\varepsilon_{i-j}+h\sum_{m=0}^{k}\alpha_{mj}(f(x_{i-m},\ y_{i-m})-f(x_{i-m},\ u_{i-m}))+r_i.$$

The last inequality and (2) give

$$|\varepsilon_i|\leq|\varepsilon_{i-j}|+hK\sum_{m=0}^{k}|\alpha_{mj}|\ |\varepsilon_{i-m}|+|r_i|$$

and if the condition $1-hK|\alpha_{oj}|=1/\rho>0$ is fulfilled then

$$(35)\qquad |\varepsilon_i|\leq\rho(|\varepsilon_{i-j}|+hK\sum_{m=1}^{k}|\alpha_{mj}|\ |\varepsilon_{i-m}|+|r_i|).$$

Let us denote $\Sigma_{m=1}^{k}|\alpha_{mj}|=B_1$ and assume that $|\varepsilon_i|\leq\delta,\ i=0,\ 1,\ldots,\ k-1$. Then from (35) we obtain

$$|\varepsilon_k|\leq\rho\ [(1+hKB_1)\delta+|r_k|]$$

$$|\varepsilon_{k+1}|\leq\rho^2(1+hKB_1)^2\delta+\rho^2(1+hKB_1)|r_k|+\rho|r_{k+1}|$$

$$\cdots\cdots$$

$$|\varepsilon_n|\leq\rho^{n-k+1}(1+hKB_1)^{n-k+1}\delta+\sum_{m=k}^{n}\rho^{n-m+1}(1+hKB_1)^{n-m}|r_m|$$

and

$$\max_{0\leq i\leq n}|\varepsilon_i|\leq\rho^n(1+hKB_1)^n[\delta+\sum_{m=k}^{n}|r_m|].$$

Repeating now (29) and using (33) we prove

Theorem 4. *If we solve numerically the problem* (1), (2) *using Adams' interpolated formula* (30) *then under the condition* $1-hK|\alpha_{oj}|>0$ *the following estimate*

$$\max_{0\leq i\leq n}|y_i-u_i|\leq c_3(k,\ j)\ [\delta+\tau_{k+1}(y'\ ;\ h)_L]$$

holds.

Using the properties (3) of the modulus $\tau_k(f;\delta)_{L_p}$ series of corollaries can be obtained from Theorems 1—4. Also the well known order for convergence of Runge--Kutta's and Adems' methods come out under weaker conditions of smoothness of the solution of the equation (1). For example let us mention that for order of convergence $O(h)$ of Euler's method (4) a bounded variation of y' is enaugh, not the boundedness of y''.

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